Succinctness of Epistemic Languages

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Abstract

Proving that one language is more succinct than another becomes harder when the underlying semantics is stronger. We propose to use *Formula-Size Games* (as put forward by Adler and Immerman, 2003), games that are played on two *sets* of models, and that directly link the *length* of play with the *size* of the formula. Using FSGs, we prove three succinctness results for m-dimensional modal logic: (1) In system $\mathcal{K}_m$, a notion of 'everybody knows' makes the resulting language exponentially more succinct for $m > 1$ (2) In $\mathcal{S}5_m$, the same language becomes more succinct for $m > 3$ and (3) Public Announcement Logic is exponentially more succinct than $\mathcal{S}5_m$, if $m > 3$. The latter settles an open problem raised by Lutz, 2006.

1 Introduction

In Knowledge Representation, one of the key issues is to design logical languages that are *useful* to reason about a given set of structures, or models. Prominent examples are first-order languages to reason about structured domains, temporal languages to reason about time structures, and modal logics to reason about Kripke models. It may well be that two languages L_1 and L_2 are *equally expressive* on a set of models under consideration: any difference between two models can be expressed in L_1 iff it can be expressed in L_2. However, even if the two languages are equally expressive, L_1 may be preferred over L_2, if the two languages have the same *computational* complexity, but L_1 is *more succinct* than L_2: what one can say in both languages can be done shorter in L_1 than in L_2 (formal definition to follow). Let us abbreviate these three criteria to saying that L_1 is strictly preferred over L_2. In this paper we show for two epistemic languages, that they are strictly preferred over the 'standard epistemic language' [Fagin *et al.*, 1995]. The first example is conceptually simple: in epistemic logic for m agents, rather than writing a conjunction $K_1\varphi \wedge K_2\varphi \wedge \cdots \wedge K_m\varphi$, one can introduce an operator E ('everybody knows'), where $E\varphi$ is defined as this conjunction (this definition implies that there is no added expressivity, and equal complexity is established in [Lutz, 2006]). The E operator acts as an approximation for common knowledge (but without its computational complexity) in defining levels of consensus in policies and protocols in multi-agent systems. Our result is relevant for e.g., query languages: in epistemic logic, one can query a scenario ('who knows what?') more succinctly using the E operator.

The second language that we demonstrate to be strictly preferred over standard epistemic logic is that of Public Announcement Logic (PAL, [Plaza, 1989]). PAL enriches multi-agent epistemic logic with a construct $[\varphi]\psi$: 'after the announcement of φ, formula ψ is true'. Surprisingly, the language of public announcements is equally expressive as that of multi-agent epistemic logic. However, as was proven in [Lutz, 2006], the language of public announcement logic is computationally equally complex as, but more succinct than that of epistemic logic. But the succinctness result was only proven under the general, weak semantics $\mathcal{K}_m$ of modal logic. An issue left as an open problem in [Lutz, 2006, p. 140] is whether public announcement logic is also more succinct than epistemic logic under the semantics $\mathcal{S}5_m$: Kripke models where each relation is an equivalence relation, which is the class generally used for epistemic logics. One of the results in our paper is that we settle this question positively: the result holds also under $\mathcal{S}5_m$.

Although the paper focuses on the two epistemic languages described, we hope our techniques are applicable to various modal languages. Let us briefly explain why succinctness may fail when moving to a stronger semantics. To say that L_1 is more succinct than L_2 under semantics $\mathcal{S}$ roughly means that there are formulas $\varphi_1, \varphi_2, \ldots$ in L_1 such that any formulas $\psi_1, \psi_2, \ldots$ in L_2 that are equivalent to them in $\mathcal{S}$ must be significantly longer. In a stronger semantics $\mathcal{S}'$ there are generally more formulas that are equivalent to φ_i than in $\mathcal{S}$. So, under $\mathcal{S}'$, we may find more candidates ψ_i in L_2 that are equivalent to φ_i, and some of them might be shorter than those found under $\mathcal{S}$.

How to demonstrate that a formula equivalent to $\psi \in L_2$ must have at least a certain length? We propose to use *Formula Size Games* (FSGs), introduced in [Adler and Immerman, 2003]: which establish a direct link between the number of *moves* needed for one player to win, and the *size* of formulas associated with the game. Hence, reasoning about the shortest formulas equivalent to ψ amounts to reasoning about the shortest strategies to win an FSG related to ψ. In our formulation, FSG's, become essentially one-player games, which makes it easier to reason about winning strategies.

2 Preliminaries

In this paper, for each $m \in \mathbb{N}$ we fix a set of agents $Ag = \{a_1, \dots a_m\}$ and atoms $At = \{p_1, p_2, \dots\}$.

Definition 1 (Formulas) *The formulas ψ of the language of public announcement logic L_{PAL} are :*

$$\bot \mid \top \mid p \mid \neg\psi \mid \psi \vee \psi \mid \psi \wedge \psi \mid K_a\psi \mid M_a\psi \mid E\psi \mid [\psi]\psi$$

where $p \in At$ and $a \in Ag$. We denote the language not containing any formula of the form $[\psi_1]\psi_2$ or $E\psi$ with L_{EL}, the language of basic epistemic logic. The language not containing any formula of the form $[\psi_1]\psi_2$ is denoted L_{ELE}.

For a modal operator X and $n \in \mathbb{N}$, the formula $X^n\varphi$ is defined as usual: $X^0\varphi = \varphi$ and $X^{n+1}\varphi = X^nX\varphi$. The *length* of a formula $|\psi|$ is defined in the obvious way (note that we do not count parentheses): $|\bot| = |\top| = |p| = 1$, $|\psi_1 \wedge \psi_2| = |\psi_1 \vee \psi_2| = |\psi_1| + |\psi_2| + 1$ and $|\neg\psi| = |K_a\psi| = |M_a\psi| = |E\psi| = 1 + |\psi|$, $|[\psi_1]\psi_2| = |\psi_1| + |\psi_2|$.

Definition 2 (Kripke Model) *A Kripke model for Ag and At is a tuple $\langle S, R, V\rangle$ where S is a set of states, $R : Ag \to \mathcal{P}(S \times S)$, for which we write sR_at rather than $(s,t) \in R(a)$, and $V : P \to 2^S$ determines for every $p \in P$ the set of states $V(p) \subseteq S$ where p is true.*

Given a model $M = \langle S, R, V\rangle$, a pointed model is a pair (M, s) (sometimes written M, s) where $s \in S$. We will also write such a pair as M, s. Sets of pointed models are denoted $\mathbb{M}, \mathbb{M}', \mathbb{M}_1, \mathbb{M}_2, \dots$.

Definition 3 (Satisfaction) *$M, s \models \varphi$ is defined as usual: we only give the modal clauses:*

$$\begin{array}{lll} M, s \models M_a\varphi & \text{iff} & \text{for some } v, sR_av \text{ and } M, v \models \varphi; \\ M, s \models K_a\varphi & \text{iff} & \text{for all } v, sR_av \text{ implies } M, v \models \varphi; \\ M, s \models [\varphi]\psi & \text{iff} & \text{if } M, s \models \varphi, \text{ then } M|\varphi, s \models \psi. \end{array}$$

where $M|\varphi$ is the submodel of M restricted to the set of worlds where φ is true. We define

$$E\varphi = K_{a_1}\varphi \wedge \dots \wedge K_{a_m}\varphi, \text{ where } Ag = \{a_1, \dots, a_m\}$$

The fact that the operator E is defined using the operator K shows that L_{ELE} and L_{EL} are equally expressive. Using the reduction axioms below (cf. [Plaza, 1989]), one proves that L_{PAL} and L_{EL} are equally expressive, too.

$$\begin{array}{lll} [\varphi](\psi_1 \wedge \psi_2) & \leftrightarrow & [\varphi]\psi_1 \wedge [\varphi]\psi_2 \\ [\varphi]\neg\psi & \leftrightarrow & \varphi \to \neg[\varphi]\psi \\ [\varphi]K_a\psi & \leftrightarrow & \varphi \to K_a[\varphi]\psi \\ [\varphi_1][\varphi_2]\psi & \leftrightarrow & [\varphi_1 \wedge [\varphi_1]\varphi_2]\psi \end{array}$$

Definition 4 (Formula Size Game) *The formula size game (FSG) between Spoiler and Duplicator is played on a tree, where each node is labeled with a pair $\langle \mathbb{M} \circ \mathbb{M}'\rangle$ such that $\mathbb{M}$ and $\mathbb{M}'$ are sets of pointed models. At each step of the game, a node gets labeled with one of the symbols from the set $\mathbb{S} = At \cup \{\top, \bot, \neg, \vee, \wedge\} \cup \{M_a, K_a \mid a \in Ag\}$ and either it is closed or at most two new nodes are added. Given a node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$, Spoiler can make the following moves at this node:*

$\bot$-move *This can be played only if $\mathbb{M} = \emptyset$. When Spoiler plays this move, the leaf $\langle \mathbb{M} \circ \mathbb{M}'\rangle$ is closed and labeled with the symbol $\bot$.*

$\top$-move *This can be played only if $\mathbb{M}' = \emptyset$. When Spoiler plays this move, the node is closed and labeled with the symbol $\top$.*

atomic-move *Spoiler chooses a propositional variable p such that every pointed model in $\mathbb{M}$ satisfies p, and no pointed model in $\mathbb{M}'$ does. After this move, this node is closed and labeled with the symbol p.*

not-move *Spoiler labels the node with the symbol $\neg$ and adds one new node labeled $\langle \mathbb{M}' \circ \mathbb{M}\rangle$ as a successor to the node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$.*

and-move *Spoiler labels the node with the symbol $\wedge$ and splits $\mathbb{M}'$ in two (not necessarily disjoint) sets $\mathbb{M}' = \mathbb{M'}_1 \cup \mathbb{M'}_2$. Two new nodes are added to the tree as successors to $\langle \mathbb{M} \circ \mathbb{M}'\rangle$, namely $\langle \mathbb{M} \circ \mathbb{M}'_1\rangle$ and $\langle \mathbb{M} \circ \mathbb{M}'_2\rangle$.*

or-move *Spoiler labels the node with the symbol $\vee$ and splits $\mathbb{M}$ in two (not necessarily disjoint) sets $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$. Two new nodes are added to the tree as successors to the node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$, namely $\langle \mathbb{M}_1 \circ \mathbb{M}'\rangle$ and $\langle \mathbb{M}_2 \circ \mathbb{M}'\rangle$.*

K_a-move *Spoiler initiates this move by labeling the node with K_a and, for each pointed model $(M', w') \in \mathbb{M}'$ he chooses a pointed model (M', v') such that $w'R'_av'$. Then, Duplicator responds by choosing for each pointed model $(M, w) \in \mathbb{M}$ all the possible pointed models (M, v) such that wR_av. If for some (M, w) w does not have an R_a successor, Duplicator chooses no (M, v) for the pointed model (M, w). A new node consisting of the sets of pointed models that the players have chosen is added as a successor to the node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$.*

M_a-move *Spoiler labels the node with the symbol M_a and for each pointed model $(M, w) \in \mathbb{M}$, he chooses a pointed model (M, v) such that wR_av. Duplicator responds by choosing for each pointed model $(M', w') \in \mathbb{M}'$ all the possible pointed models (M', v') such that $w'R'_av'$. If for some (M', w') w'does not have an R'_a successor, Duplicator chooses no (M', v') for the pointed model (M', w'). A new node consisting of the sets of pointed models that the players have chosen is added as a successor to the node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$.*

and-moves and *or*-moves are collectively called *splitting* moves. M_a-moves and K_a-moves are called *agent* moves. The game is defined by specifiying a top-node $\langle \mathbb{M} \circ \mathbb{M}'\rangle$.

FSG's were first introduced in [Adler and Immerman, 2003] for first-order logic with two variables and the transitive closure operator for CTL. We added the and-move, $\top$-move and $\bot$-move. Moreover, our M_a-move is the modal translation of their **EX**-move, for which we also cater for the possibility that a world has no successors (and our K_a-move is a dual of it). However, in a direct translation of the **EX**-move to our M_a-move, the response of Duplicator would have prescribed that she chooses for every $M', w' \in \mathbb{M}'$ *some* (rather than *all*) possible models $M', v' \in \mathbb{M}'$ such that $w'R_av'$. Our formulation models a 'strongest' strategy of Duplicator: she wins the game using our strategy iff she has a winning strategy in the game as defined in [Adler and Immerman, 2003]. So, our formulation makes the game essentially a one-player game, in which Duplicator is not left with any choice during the game— our formulation plays her optimal strategy.

Definition 5 *Given a game tree T with n nodes that is the result of an* FSG*, we say that Spoiler has won the game in n moves iff every leaf of T is closed. Otherwise Duplicator has won.*

The next theorem connects FSGs with the length of formulas in the language of basic epistemic logic L_{EL}.

Theorem 1 *Spoiler can win the* FSG *starting with $\langle \mathbb{M} \circ \mathbb{M}' \rangle$ in n moves iff there is a formula φ in L_{EL} of size n true in all models of $\mathbb{M}$ but false in all of $\mathbb{M}'$.*

Proof (If) Suppose that there is a formula φ of size n such that every pointed model in $\mathbb{M}$ satisfies φ and no pointed model in $\mathbb{M}'$ does. We prove by induction on the structure of φ that Spoiler can win the game starting in $\langle \mathbb{M} \circ \mathbb{M}' \rangle$ in n moves by playing according to φ.

Base case: If φ is the formula $\bot$, and every model in $\mathbb{M}$ satisfies φ, it is obvious that $\mathbb{M} = \emptyset$ and Spoiler can win the game by playing a $\bot$-move. Similarly for $\top$. If φ is the propositional variable p, then Spoiler plays the atomic-move and the tree is closed, as required. It is obvious that the tree has just one node, i.e. Spoiler can win the game in $|\varphi|$ moves by playing according to φ.

Induction step: We only present the step for the K_a-move. If φ is a formula of the form $K_a\psi$, then for each model $(M', w') \in \mathbb{M}'$ Spoiler chooses a model (M', v') such that $w' R'_a v'$ and ψ is false in (M', v'). Let $\mathbb{N}'$ be the set of models Spoiler has chosen. For each pointed model $(M, w) \in \mathbb{M}$, Duplicator chooses all the possible pointed models (M, v) such that $wR_a v$. Let $\mathbb{N}$ be the set of models Duplicator has chosen. A new node $\langle \mathbb{N} \circ \mathbb{N}' \rangle$ is added as a successor of $\langle \mathbb{M} \circ \mathbb{M}' \rangle$. Clearly, ψ must be true in all the new pointed models Duplicator has chosen and false in all of those that Spoiler has chosen. Applying the induction hypothesis, we see that Spoiler can win the subgame starting at $\langle \mathbb{N} \circ \mathbb{N}' \rangle$ in $|\psi|$ moves. Therefore, Spoiler can win the FSG starting at $\langle \mathbb{M} \circ \mathbb{M}' \rangle$ in $|\varphi| = |K_a\psi| = |\psi| + 1 = n$ moves by playing according to φ.

(Only if) Suppose that Spoiler has won the n-round formula size game starting at $\langle \mathbb{M} \circ \mathbb{M}' \rangle$. We claim that the game tree is a parse tree of a formula of length n that is true in all the models in $\mathbb{M}$ and false in all the models in $\mathbb{M}'$. In order to prove this, we label the nodes of the tree step by step starting with the leaves. These were labeled during the game with the propositional variables p, $\top$ and $\bot$ that Spoiler used to close them. Then the rest of the nodes are labeled successively. If a node has a $\neg$ label and its successor is labeled with ψ, then that node is labeled with $\neg\psi$, etc. By a straightforward backward induction on the tree we can see that for each node we have the following.

- The string of symbols labeling the node is indeed a well-formed formula of the language L_{EL}.
- The formula labeling the node is true in all the models on the left and false in all the models on the right. Therefore, the formula labeling the root of the tree is true in all the pointed models in $\mathbb{M}$ and false in all the pointed models in $\mathbb{M}'$.

It is obvious that the game tree is a parse tree for the formula labeling the root. □

Definition 6 (Succinctness) *Let L_1 and L_2 be two logical languages and let $\mathcal{S}$ be a class of models (or 'semantics'). We say that L_1 is exponentially more succinct than L_2 on $\mathcal{S}$, and we write $L_1 \prec_{\mathcal{S}} L_2$, if there exists a sequence of formulas $\varphi_0, \varphi_1, \dots$ in L_1 such that*

1. *for every n, $|\varphi_{n+1}| > |\varphi_n|$, and*
2. *there is a polynomial $f > 0$ of degree at least 1 such that for every sequence $\psi_0, \psi_1, \dots \in L_2$ with $\psi_n \equiv_{\mathcal{S}} \varphi_n$, for all n, we have $|\psi_n| \geq 2^{f(|\varphi_n|)}$.*

Note that it is possible to have both $L_1 \prec_{\mathcal{S}} L_2$ and $L_2 \prec_{\mathcal{S}} L_1$ at the same time. However, if $L_1 \subseteq L_2$, we can only have $L_2 \prec_{\mathcal{S}} L_1$.

Proposition 1 *Let $\mathcal{S}$ and $\mathcal{T}$, be two classes of models such that $\mathcal{S} \subseteq \mathcal{T}$ and let L_1, L_2 be two languages. If $L_1 \prec_{\mathcal{S}} L_2$, then $L_1 \prec_{\mathcal{T}} L_2$.*

Our strategy to prove that $L_1 \prec L_2$ under semantics $\mathcal{S}$ consists of defining an infinite sequence of formulas $\varphi_0, \varphi_1, \dots$, in L_1 such that:

1. for some d, for every $n > 0$, $|\varphi_{n+1}| = |\varphi_n| + d$;
2. for every n, we construct two sets of pointed models $\mathbb{M}_n$ and $\mathbb{M}'_n$ from $\mathcal{S}$ such that φ_n is true in all $\mathbb{M}_n$, but false in all of $\mathbb{M}'_n$, and the number of moves for Spoiler to win the game starting in $\langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$ is at least 2^n.

Lemma 1 *If L_1 and L_2 are such that there is a sequence of formulas $\varphi_0, \varphi_1, \dots$ in L_1 satisfying the two requirements above, then $L_1 \prec L_2$ under $\mathcal{S}$.*

In this paper, $L_2 = L_{EL}$. In Section 3, $L_1 = L_{ELE}$, and in Section 4, $L_1 = L_{PAL}$. In Section 3.1, the semantics is $\mathcal{K}_m$, and in Sections 3.2 and 4, the semantics is $\mathcal{S}5_m$.

3 Succinctness of L_{ELE}

In this section we prove that L_{ELE} is exponentially more succinct than L_{EL} on the class of $\mathcal{S}5_m$ models. However, we first look at the simpler case for the semantics $\mathcal{K}_m$.

3.1 Succinctness of L_{ELE} on $\mathcal{K}_2$

Let the formulas $\varphi_n \in L_{ELE}$ be:

$$\varphi_n = \neg E^n \neg p \tag{1}$$

If we define $\psi_0 = p$ and $\psi_{n+1} = M_a\psi_n \vee M_b\psi_n$, It is obvious that $\varphi_n \in L_{ELE}$ is equivalent to $\psi_n \in L_{ELE}$. Formula φ_n expresses that there is some p-world at most n steps away.

We now define two sets of pointed models $\mathbb{M}_n$ and $\mathbb{M}'_n$ for each n whose underlying frame is a binary tree of depth n. The vertices of the tree are labelled with a and b and the nodes are labeled with words over $\{a, b\}$. $\mathbb{M}'_n$ contains only one model: the binary tree of depth n with no atom true anywhere. $\mathbb{M}_n$ consists of all the models based on binary trees of depth n containing just one world satisfying the proposition p and this world is a leaf.

Definition 7 (Words) *For a set of Ag we define the set of words of length at most n, $W_n(Ag)$ as follows. $W_0 = \{\epsilon\}$ and $W_{n+1} = W_n \cup \{wx \mid w \in W_n \,\&\, x \in Ag\}$. The length of w and concatenation of words is defined in the standard way. For a word w, w_1 denotes its first component, w_2 its second, etc. If Ag is clear, we will write W_n rather than $W_n(Ag)$. For $k \in \mathbb{N}$ with $k \geq |Ag|$, let $W_n^{\neq k}(Ag) \subseteq W_n(Ag)$ be the set of k-diff words, that is, words w for which every subsequence of length k is such that all elements in that subsequence are different.*

Definition 8 ($\mathcal{K}_2$-models for E) *For each $n > 0$ and each $w \in W_n(\{a, b\})$, we define a model $M^n_w = \langle W, R, V\rangle$, where*

- *$W = W_n$ is as defined earlier;*
- *$V(p) = \{w\}$*
- *$R_i uv$ iff $v = ui$ ($i \in \{a, b\}$).*

Let $\mathbb{M}_n$ consist of all the models M^n_w, ϵ. Furthermore, $\mathbb{M}'_n$ contains the binary tree (B^n, ϵ) with $B^n = \langle W_n, R, V\rangle$ where $V(p) = \emptyset$. Let us say that (M^n_w, v) is a-imperfect if the p world can only be reached from v by first taking a step along the relation R_a. Similarly for b-imperfect models.

It is clear that every model in $\mathbb{M}_n$ satisfies φ_n, while the only model in $\mathbb{M}'_n$ falsifies φ_n. Therefore, Spoiler can win the game starting at $\langle \mathbb{M}_\ltimes \circ \mathbb{M}'_n\rangle$.

Lemma 2 *Fix $n \in \mathbb{N}$. Consider the* FSG *game with initial node $\langle \mathbb{M}_n \circ \mathbb{M}'_n\rangle$ and suppose Spoiler plays a winning strategy. Then, in the tree generated by the play, for every different words x and y over Ag, there are two different closed branches.*

Proof First note that in any node of the game tree, the state of all pointed models will be at the same distance from ϵ. For the first n agent-moves that Spoiler plays, and for $\alpha = a, b$, the effect of an M_α-move and that of a K_α-move are the same (since at depth $\leq n$, every model has exactly one a and one b-successor). Since the only state where M^n_w, ϵ and B^n, ϵ differ in an atom is at state w, Spoiler needs to reach state w in both models. As long as there are an a-imperfect and a b-imperfect model on the same side of the node, Spoiler cannot play an agent-move, since in one of the models it would lead to a successor from which only $\neg p$ worlds are reachable (which is also the case in the model B^n on the other side of the node, implying that Spoiler would lose the game). But this proves the lemma: if $x \neq y$, at some point in the game one of the models M^n_x and M^n_y will be a-imperfect while the other is b-imperfect. □

Theorem 2 *On $\mathcal{K}_2$, the language L_{ELE} is exponentially more succinct than L_{EL}.*

Proof Note that $|\varphi_n| = n + 3$. Now let ψ_n be an arbitrary formula in L_{EL} that is equivalent to φ_n. Spoiler is able to win the FSG starting in $\langle \mathbb{M}_n \circ \mathbb{M}'_n\rangle$ as defined above, but any such game has a distinct branch for any word $w \in W_n(Ag)$. There are at least 2^n such branches. By Theorem 1, ψ_n is at least 2^n long. □

The proof of Theorem 2 also shows that on the class of binary trees, L_{ELE} is exponentially more succinct than L_{EL} and that L_{ELE} is exponentially more succinct than L_{EL} for any $\mathcal{K}_m$ with $m \geq 2$.

3.2 Succinctness of L_{ELE} in $\mathcal{S}5_m$

Modal logics for knowledge typically assume some additional properties of the K-operator. In particular, it is often assumed that knowledge is veridical (for all φ, we have $K_a\varphi \to \varphi$), and that agents have positive ($K_a\varphi \to K_aK_a\varphi$) and negative ($\neg K_a\varphi \to K_a\neg K_a\varphi$) introspection. The semantics $\mathcal{S}5_m$ is obtained by requiring that in a model $M = \langle S, R_{Ag}, V\rangle$, the accessibility relations R_a are equivalence relations. Inspecting the proof of Lemma 2, it is not difficult to see that building a similar proof for $\mathcal{S}5_m$ is harder: in the proof of Lemma 2 we for instance relied on the fact that once Spoiler takes an M_a-move in a model that is b-imperfect, the p-world can never be reached again. In $\mathcal{S}5_m$, such an argument is not valid, because Spoiler and Duplicator can always go back to the initial state after playing an M_a-move, or a K_a-move. We will see that in order to cater for this, our models in this section will be a bit more involved than those in the previous section, and moreover, we present our result only for the case in which $m \geq 4$: we come back to this in Remark 1.

However, for games on $\mathcal{S}5_m$ models, we may assume that Spoiler never plays two moves in succession that involve the same agent. One way to see this is to first recall [Meyer and van der Hoek, 1995] the following equivalences as validities of $\mathcal{S}5_m$, where a is an arbitrary agent:

$$K_aK_a\varphi \leftrightarrow K_a\varphi \qquad K_aM_a\varphi \leftrightarrow M_a\varphi$$
$$M_aM_a\varphi \leftrightarrow M_a\varphi \qquad M_aK_a\varphi \leftrightarrow K_a\varphi$$

This can even be strengthened to the following equivalences in $\mathcal{S}5_m$, where we assume that $X_a, Y_a \in \{K_a, M_a\}$ and $\circ \in \{\wedge, \vee\}$ (cf. [Meyer and van der Hoek, 1995, Section 1.7.6]:

$$X_a(Y_a\varphi \circ \psi) \leftrightarrow (Y_a\varphi \circ X_a\psi) \tag{2}$$

Definition 9 (Simple $\mathcal{S}5_m$-strategies) *A strategy is a simple $\mathcal{S}5_m$-strategy for Spoiler if for any path in the game tree, two consecutive agent-moves (even when separated by moves different from agent-moves) concern different agents.*

Lemma 3 (Simple $\mathcal{S}5_m$-strategies Lemma) *Let $\mathbb{M}$ and $\mathbb{M}'$ be sets of pointed $\mathcal{S}5_m$ models. If Spoiler can win the* FSG *starting in $\langle \mathbb{M} \circ \mathbb{M}'\rangle$ and the resulting closed game tree contains n nodes and b branches, then he can also win this* FSG *using a simple $\mathcal{S}5_m$-strategy, resulting in a game tree with at most n nodes and at most b branches.*

Definition 10 ($\mathcal{S}5_m$-models for E) *Let $m \geq 3$. Given $n \in \mathbb{N}$ and 3-diff word w of length n, the model $M^n_w = \langle S, R, V\rangle$ is determined as follows (see also Figure 1, left):*

$S = \{u_i, v_i \mid 0 \leq i \leq n\}$ *and* $V(p) = \{u_n\}$

$R_a =$ *the reflexive, transitive symmetric closure of*
$\{(u_i, u_{i+1}), (v_i, v_{i+1})) \mid i < n \,\&\, w_{i+1} = a\} \cup$
$\{(u_0, v_0) \mid w_1 \neq a\} \cup \{(u_n, v_n) \mid w_n \neq a\}$
$\{(u_i, v_i) \mid w_i \neq a \neq w_{i+1}, 0 < i < n\} \cup$

Note that from M^n_w, u_0 there is a path of length n ('down', in Figure 1, left), following the agents in w, to the only p-world, u_n. The shortest path from v_0 to a p-world takes $n+1$ steps. The states u_i and u_{i+1} are connected through agent w_{i+1}, similarly for v_i and v_{i+1}. Also, u_i and v_i are connected

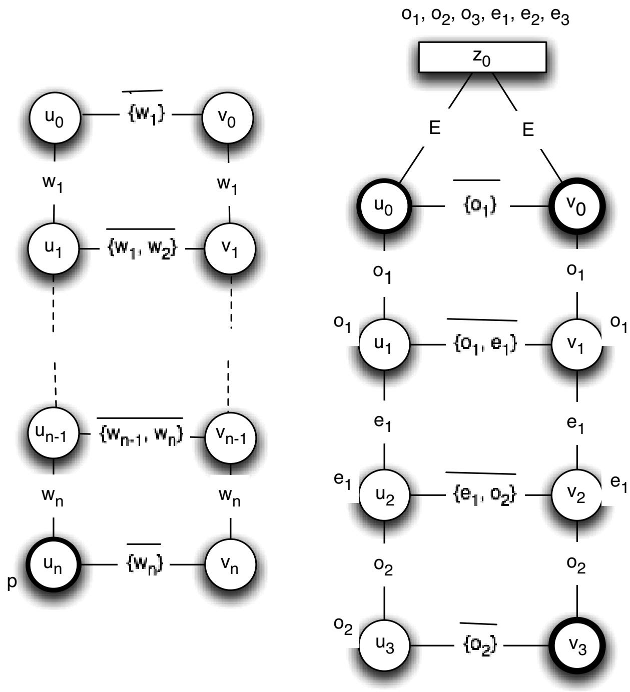

Figure 1: Model M_w^n with $w = w_1 w_2 \dots w_n$ (left) and M_w^3 with $w = o_1 e_2 o_3$ (right). $\overline{X}$ means $Ag \setminus X$.

through agent a iff there is no other a-arrow leaving u_i or v_i: neither 'up' (through w_i) nor 'down' (through w_{i+1}). Finally, for fixed n, let $\mathbb{M}_n$ be the set of models M_w^n, u_0 and let $\mathbb{M}'_n$ be the set of models of the form M_w^n, v_0, where w ranges over all 3-diff words over Ag.

Observation 1 *Define φ_n again as in* (1). *It is easy to check that φ_n is true in $\mathbb{M}_n$ and false in $\mathbb{M}'_n$, and hence he can win the game starting in $\langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$. Consider two models M_w^n, u_0 and M_w^n, v_0 appearing at node $\eta = \langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$.*

1. *In order to win the game, Spoiler needs to reach states that disagree on an atom, i.e., he has to reach the models M_w^n, u_n and M_w^n, v_n*
2. *If in some node M_w^n, u_i and M_w^n, v_i appear, no matter what Spoiler's play is, there will be a branch such that in the next node M_w^n, u_j and M_w^n, v_j both appear ($j \in \{i-1, i, i+1\}$*
3. *If in some node M_w^n, u_i and M_w^n, v_i appear and Spoiler is going to play an agent a-move, then (1) if $i = 0$, a has to be $w+1$, (2) if $0 < i < n$, a has to be w_{i-1} or w_i, and (3) if $i = n$, a has to be w_n. This is because in all other cases, u_i and v_i have the same a-successors, so that Duplicator's response in that case would lead to a node where some model appears on both sides of the node, which is a loss for Spoiler. In the case of (3), if the last agent-move was an w_n-move, if Spoiler plays a simple $\mathcal{S}5_m$-strategy, no agent moves will be played any further.*

Lemma 4 *Let n be given. Consider the game starting in node $\eta = \langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$, and suppose Spoiler plays a winning simple $\mathcal{S}5_m$-strategy. Then, in the tree generated by the play, for every two different 3-diff words x and y over Ag, there are two different closed branches.*

Proof By Observation 1, items 1 and 2, from η, at some point Spoiler has to play an agent-move. By Observation 1.3, this implies that if $x_1 \neq y_1$, in order to play an agent-move, Spoiler has to first split the node such that M_x^n, u_0 and M_y^n, u_0 appear at different nodes, which gives rise to two different branches. If $x_1 = y_1$, Spoiler can play an agent x_1-move, generating a node which includes the models M_x^n, u_1 and M^n, y, u_1 on one side of the node, and the models M_x^n, v_1 and M_y^n, v_1 on the other. We can repeat this argument, Spoiler playing some splitting moves and not-moves, and some agent $x_{i+1} = y_{i+1}$-moves visiting nodes with models M_x^n, u_i and M_y^n, u_i on one side of the node, and the models M_x^n, v_i and M_y^n, v_i on the other side, until either $x_{i+1} \neq y_{i+1}$, (reaching a node where Spoiler has to create the two branches we were looking for), or reaching a node where $x_{i+1} = y_{i+1}$ but where Spoiler decides to play 'down' in one model, say M_x^n but stays at the 'same level' with M_y^n. Thus, the resulting node of such a move would contain M_x^n, z_{i+1} and M_y^n, z_i on one side of the node, and M_x^n, r_{i+1} and M_y^n, r_i on the other for some $z \neq r \in \{u, v\}$. .

If $i + 1 = n$, by Observation 1.3, no agent-moves need to be played for M_x^n, z_{i+1}, while still some have to be played for M_y^n, z_i, so Spoiler will need to create two separate branches. If $i + 1 \neq n$, for M_x^n, z_{i+1}, Spoiler's next agent-move is an x_{i+2}-move, and for M_y^n, z_i, it needs to be an y_i-move. Since $x_{i+2} \neq x_i = y_i$, Spoiler needs to first separate the M_x^n, z_{i+1} models from the M_y^n, z_i models. □

Theorem 3 (Succinctness of Everybody Knows in $\mathcal{S}5_m$) *For $m \geq 4$, $L_{ELE} \prec_{\mathcal{S}5_m} L_{EL}$.*

Proof Note that $|\varphi_n| = n + 3$. Any closed game tree with root $\langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$ contains a different branch for every $w \in W_n^{\neq 3}(Ag)$. There are at least $(m-2)^n$ such different paths. So if $m \geq 4$, we apply Lemma 1. □

Corollary 1 *In any system weaker than $\mathcal{S}5_m$ ($m \geq 4$), $L_{ELE} \prec_{\mathcal{S}5_m} L_{EL}$.*

Remark 1 *The proof of Lemma 4 would not hold if we had not restricted our words to be 3-diff word. To see this, consider the two different paths $x = ababcab$ and $y = abcabab$. Spoiler might well be playing agent-moves in the order $ababcabab$ in a node where the models M_x^7, u_0 and M_y^7, u_0 both appear in the starting node: instead of having two different branches, this would generate one (slightly longer) branch that leads to a closing node. Now, for $|Ag| = m = 2$, there are only two 3-diff words, and if $m = 3$, there are only 6 such words. Only when $m \geq 4$, the number of 3-words grows exponentially. Obviously, Theorem 3 does not hold for $m = 1$, which leaves us the open problem whether it holds for $m = 2, 3$.*

4 Public Announcement Logic

For L_{PAL}, in our presentation we will assume to have at least 6 agents, we will later argue this can be brought down to 4.

For any $n \in \mathbb{N}$, let $2|n$ denote that n is even. In the sequel assume that $Ag = O \cup E$, with $O = \{o_1, o_2, o_3\}$ and $E = \{e_1, e_2, e_3\}$. We will also use $O \cup E$ as our set of atomic propositions. Define, for a set $A \subseteq Ag$ the formula

$$\Delta(A, \psi) = \bigvee_{a \in A} M_a(a \wedge \psi)$$

Definition 11 *Consider the following sequence of formulas* $\varphi_n \in L_{PAL}, n \in \mathbb{N}$.

$$\varphi_0 = \top$$
$$\varphi_{n+1} = \begin{cases} \langle\varphi_n\rangle(M_{o_1}o_1 \vee M_{o_2}o_2 \vee M_{o_3}o_3) & \text{if } 2|n \\ \langle\varphi_n\rangle(M_{e_1}e_1 \vee M_{e_2}e_2 \vee M_{e_3}e_3) & \text{else} \end{cases}$$

Definition 12 *Consider the following sequence of formulas* $\psi_n \in L_{EL}, n \in \mathbb{N}$.

$$\psi_0 = \top$$
$$\psi_1 = M_{o_1}o_1 \vee M_{o_2}o_2 \vee M_{o_3}o_3$$
$$\psi_n = \begin{cases} \Delta(O, \psi_{n-2}) \wedge \Delta(E, \psi_{n-1}) & \text{if } 2|n \\ \Delta(E, \psi_{n-2}) \wedge \Delta(O, \psi_{n-1}) & \text{else} \end{cases}$$

Lemma 5 *For all n,* $\mathcal{S}5_m \models \varphi_n \leftrightarrow \psi_n$.

The models M^n_w, u_0 and M^n_w, v_0 that we will define will agree on the first conjunct of ψ_n, but they will disagree on the second conjunct ($n > 1$).

Definition 13 *Define the set of coloured alternating words (ca-words) w over O and E of length n, notation* $\Pi^n(O, E)$, *or shortly* Π^n, *as follows: (1)* $w \in W_n^{\neq 3}$, *(2) for* $i < n : w_i \in O \Rightarrow w_{i+1} \in E$ *and* $w_i \in E \Rightarrow w_{i+1} \in O$ *and (3)* $w_1 \in E$ *if* $2|n$, $w_1 \in O$ *else. So a ca-word w in* Π^n *has no repetition of an agent in any sub-word of length 3, it alternates between agents from O and E and the first agent is from E iff n is even. If n is even, each model* M^n_w *is determined by an ca-word w of even length, otherwise w has an odd length. Model* $M^n_w = \langle S, R, V\rangle$ *is constructed as follows (see Figure 1 (right) for a model* M^3_w*). Let* $A(a) = O$ *if* $a \in O$ *and* $A(a) = E$ *if* $a \in E$.

$$S = \{z_0\} \cup \{u_i, v_i \mid i \leq n\}$$
$$V(p) = \{z_0, u_i, v_i \mid 0 < i \ \&\ w_i = p\} \setminus \{v_n\}$$
$$R_a = \text{the reflexive transitive symmetric closure of}$$
$$\{(z_0, u_0), (z_0, v_0) \mid a \notin A(w_1)\} \cup$$
$$\{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid w_i = a\} \cup$$
$$\{u_0, v_0)\} \mid w_1 \neq a\} \cup$$
$$\{(u_i, v_i) \mid 0 < i < n \ \&\ w_i \neq a \neq w_{i+1}\} \cup$$
$$\{(u_n, v_n)\} \mid w_n \neq a\}$$

So, we have a state z_0 *in S, in which all the atoms are true. Moreover, no atom is true in* u_0, v_0, *and the states* u_0, v_0 *and* z_0 *are all i-accessible from each other for all* $i \in Ag \setminus A(w_1)$, *that is, if n is odd, all states* u_0, v_0, z_0 *are E-accessible from each other, and if n is even, they are O-accessible.*

The other states in M^n_w *form again a ladder: we have states* $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots u_n$ *in S and stipulate that atom p* ($p \in \{o_i, e_i \mid 1 \leq i \leq 3\}$) *is true in* v_i *and* u_i ($i < n$) *iff* $w_i = p$. *The only state for which we make an exception is* v_n: *here, no atom is true. In terms of access, we state that* $R_g v_i v_{i+1}$ *iff* $R_g u_i u_{i+1}$ *iff* $w_{i+1} = g$ ($i < n$). *Moreover, access between* $u_i, u_{i+1}, v_i, v_{i+1}$ *is defined as in Definition 10. The class of models* M^n, u_0 *thus obtained will be* $\mathbb{M}(n)$ *and the class of models* M^n_w, v_0 *is denoted* $\mathbb{M}'(n)$.

Lemma 6 *Let* $n \in \mathbb{N}$, $w \in \Pi^n$, *and* ψ_n *be as defined earlier.* $M^n_w, u_0 \models \psi_n$ *but* $M^n_w, v_0 \not\models \psi_n$.

The following Lemma and Theorem are proven as in the previous section.

Lemma 7 *Let n be given. Consider the game starting in node* $\eta = \langle \mathbb{M}_n \circ \mathbb{M}'_n \rangle$, *and suppose Spoiler plays a winning simple* $\mathcal{S}5_m$*-strategy. Then, in the tree generated by the play, for every two different x and y from* Π^n, *there are two different closed branches.*

Theorem 4 (Succinctness of L_{PAL}) *For* $m \geq 6$, *we have* $L_{PAL} \prec_{\mathcal{S}5_m} L_{EL}$.

Corollary 2 *In any system weaker than* $\mathcal{S}5_m$ ($m \geq 6$) *the language* L_{PAL} *is exponentially more succinct than* L_{EL}.

Consider four agents $Ag = \{a_1, a_2, b, c\}$ and define $\varphi_0 = \top$, $\varphi_{3i+1} = \langle\varphi_i\rangle(M_{a_1}a_1 \vee M_{a_2}a_2)$, $\varphi_{3i+2} = \langle\varphi_i\rangle M_b b$, and $\varphi_{3i+3} = \langle\varphi_i\rangle M_c c$. Words are now built by gluing sequences a_1bc and a_2bc together, and Lemma 7 would go through in this setting, but this time, for $n \in \mathbb{N}$, there will be $2^{n/3}$ different words, which is still sufficient to prove Theorem 4.

Proposition 2 *For* $m \geq 4$, $L_{PAL} \prec_{\mathcal{S}5_m} L_{ELE}$.

5 Conclusion

We used FSGs to establish three succinctness results for epistemic logics under various semantics. It is interesting to explore what kind of assumptions about Spoiler's strategy one is allowed to make under other semantics. This could lead to a 'toolkit' for FSGs to be used for a suite of logics,.

Apart from settling the question of succinctness of L_{ELE} and L_{PAL} for $m = 3$, future work would explore the use of FSGs for other logics, like for instance for Coalition Logic with its non-standard modal semantics. Finally, we'd like to make a precise connection between FSGs on the one hand, and other kind of games that are used in modal logics property checking or model checking, like games based on μ-calculus (cf. [Stirling, 1997]) or bisimulation games (where the emphasis is on *modal depth* rather than *formula size*).

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