A Matroid Approach to the Worst Case Allocation of Indivisible Goods*

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Abstract

We consider the problem of equitably allocating a set of indivisible goods to n agents so as to maximize the utility of the least happy agent. [Demko and Hill, 1988] showed the existence of an allocation where every agent values his share at least $V_n(\alpha)$, which is a family of nonincreasing functions in a parameter α, defined as the maximum value assigned by an agent to a single good. A deterministic algorithm returning such an allocation in polynomial time was proposed [Markakis and Psomas, 2011]. Interestingly, $V_n(\alpha)$ is tight for some values of α, i.e. it is the best lower bound on the valuation of the least happy agent. However, it is not true for all values of α. We propose a family of functions W_n such that $W_n(x) \geq V_n(x)$ for all x, and $W_n(x) > V_n(x)$ for values of x where $V_n(x)$ is not tight. The new functions W_n apply on a problem which generalizes the allocation of indivisible goods. It is to find a solution (base) in a matroid which is common to n agents. Our results are constructive, they are achieved by analyzing an extension of the algorithm of Markakis and Psomas.

1 Introduction

[Demko and Hill, 1988] addressed the problem of equitably allocating a set of indivisible goods to n agents ($n \geq 2$). The agents have possibly different utilities for the individual goods, and an agent's utility for a bundle B is defined as the sum of individual utilities for the goods in B. After a normalization, it is assumed that everyone has utility 1 for the whole set of goods.

Demko and Hill's work focused on how good the least happy agent can value his share. It is to find a threshold $t_n \in [0, 1]$ such that, in any case, there exists an allocation where each of the n agents has utility at least t_n for his share. To be complete, t_n should come with a family of instances without any feasible allocation where everyone values his share $t_n + \epsilon$ (or more) for some positive ϵ.

In the context of sharing *divisible* goods, it is long known that $t_n = 1/n$ [Steinhaus, 1948]. Dealing with indivisible goods leads to a trickier situation. As a devastating example, imagine n agents having utility 1 for the same item, say i_1, and utility 0 for any other item. The agents who do not receive i_1 have a global utility of 0, meaning that t_n equals to 0. Meanwhile, if the maximum utility for a good was upper bounded by a quantity tending to 0, the indivisible model would gradually tend to the divisible model, where $t_n = 1/n$. Then, what is t_n between these two extremal cases?

In fact, the maximum value for a single element appears to significantly influence t_n, as in the pioneering work of [Hill, 1987] who defined a family of nonincreasing functions $V_n : [0, 1] \to [0, n^{-1}]$ for any integer $n \geq 2$ (see Definition 1 and Figure 1). Following [Hill, 1987], [Demko and Hill, 1988] considered the parameter $\alpha \in [0, 1]$, defined as the maximum value assigned by an agent to a single good, and showed that it is possible to allocate the indivisible goods to n agents such that every agent's valuation for his share is at least $V_n(\alpha)$. In addition, they showed with some instances that V_n is exactly the best utility of the least happy agent for some values of α, but the instances do not cover the entire interval $[0, 1]$.

Defining α_i as agent i's maximum valuation for a single item, [Markakis and Psomas, 2011] have recently strengthened the results of Demko and Hill. Indeed, they show the existence of an allocation guaranteeing $V_n(\alpha_i)$ for every agent i. Since V_n is nonincreasing, $V_n(\alpha_i) \geq V_n(\alpha)$ holds and the vector $(V_n(\alpha_i))_{i\in[n]}$ weakly Pareto dominates $(V_n(\alpha))_{i\in[n]}$. The other contribution of Markakis and Psomas relies on the fact that, unlike the results in [Hill, 1987; Demko and Hill, 1988], the allocations are obtained with a deterministic algorithm which runs in polynomial time.

Our work deals with a problem which encompasses the allocation of indivisible goods. It is a problem on a *matroid* (defined in Section 3) where one has to find a common solution (base) to n agents. The agents have possibly different utilities for the elements of the matroid, and an agent's utility for a solution B is defined as the sum of individual utilities for the elements in B. After a normalization ensuring that the maximum utility for a solution of the matroid is exactly 1 for everyone, we define α_i as the maximum value that agent i assigns to a single element, and $\alpha = \max_{i\in[n]} \alpha_i$.

Interestingly, we show that V_n is still valid in this generalized context and we can even improve it. We propose a family

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of functions $W_n : [0,1] \to [0, n^{-1}]$ defined for any positive integer n (see Definition 2). We have $W_n(x) \geq V_n(x)$ for all $x \in [0,1]$, and $W_n(x) > V_n(x)$ for values of x where $V_n(x)$ is not tight (see Figure 1 for illustration). Like V_n, W_n is piecewise linear on $[0,1]$ but unlike V_n, W_n alternates decreasing and increasing phases. This gives a new insight in the particularity of handling indivisible objects.

We also propose a deterministic algorithm which is an extension of the one in [Markakis and Psomas, 2011] for the generalized problem on matroids. The algorithm returns a solution (base) where every agent i has utility at least $W_n(\alpha_i)$.

In all, our contribution consists of dealing with matroids which capture more situations than the basic model of allocating indivisible goods, the new functions W_n improve on V_n, and the solution is built by a polynomial time algorithm.

The rest of the article is organized as follows. In section 2, we discuss some related work. In order to be self-contained, we give basic notions on matroids in Section 3. Section 4 introduces the model that we deal with. In section 5, we present a polynomial time algorithm which provides a solution in which every agent i receives at least $W_n(\alpha_i)$. Due to lack of space, some parts of proofs are omitted. Further results and future directions are given in Section 6.

2 Related work

Sharing scarce resources like food, energy or jobs is a very old, but never outdated, issue. From the various fields where this problem is addressed, the recurrent challenge is obviously to reach fairness and efficiency.

Fair division is to allocate a set of goods to a set of n agents, having heterogeneous valuations, in a way that leaves every agent satisfied. It is extensively studied in economics (especially social choice theory), mathematics and political science. However, computer science (CS) comes naturally into play when the computational aspects of allocation procedures are investigated [Chevaleyre *et al.*, 2007]. During the last few years, the CS community (especially in Artificial Intelligence) has shown a growing interest in the topic [Bouveret and Lang, 2008; Procaccia, 2013].

The literature on fair division usually distinguishes the case of *divisible* goods (e.g. cakes) and the case of *indivisible* goods (e.g. houses) [Young, 1994; Brams and Taylor, 1996; Moulin, 2003]. Typically desired properties are *proportionality* (each of the n agents gets at least a proportion of $1/n$ of the goods) and *envy-freeness* (every agent weakly prefers his share to the share of any other agent).

Proportionality for n agents (Banach-Knaster [Steinhaus, 1948]) and envy-freeness can be reached in the divisible case [Brams and Taylor, 1995] but it is not true for indivisible goods. Instead, one can seek for allocations that minimize envy [Lipton *et al.*, 2004] or maximize the utility of the least happy agent [Asadpour and Saberi, 2010]. These approaches lead to computationally hard optimization problems.

Another approach for indivisible goods is to determine a value, and guarantee the existence of an allocation such that the relative utility of the poorest agent is at least this value. This is the path followed by [Demko and Hill, 1988] on the basis of [Hill, 1987]. Recently, [Markakis and Psomas, 2011] proposed a deterministic algorithm for computing such a solution in polynomial time. From a computational perspective, this task is less demanding than the maximization of the poorest agent's utility.

3 Matroids

Matroid theory is central in combinatorial optimization. In particular, it has permitted to unify apparently separated structures like trees and matchings in a graph. In order to be selfcontained, we give some basic notions on matroids. An interested reader may refer to [Schrijver, 2003; Korte and Vygen, 2007; Oxley, 1992] for more details.

A matroid $\mathcal{M} = (X, \mathcal{F})$ consists of a finite set of elements X and a collection $\mathcal{F}$ of subsets of X such that:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $F_2 \subseteq F_1$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$,

(iii) for every $F_1, F_2 \in \mathcal{F}$ where $|F_1| < |F_2|$, $\exists\ e \in F_2 \backslash F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called *independent* sets. Inclusion-wise maximal independent sets are called *bases*. All bases of a matroid $\mathcal{M}$ have the same cardinality $r(\mathcal{M})$, defined as the *rank* of $\mathcal{M}$. Given a matroid $\mathcal{M} = (X, \mathcal{F})$ and a subset $X' \subset X$, if $X' \in \mathcal{F}$, the *contraction* of $\mathcal{M}$ by X', denoted by $\mathcal{M}/X'$, is the structure $(X \setminus X', \mathcal{F}')$ where $\mathcal{F}' = \{F \subseteq X \setminus X' : F \cup X' \in \mathcal{F}\}$. It is well known that $\mathcal{M}/X'$ is a matroid.

A typical example of a matroid is the forests (acyclic set of edges) of a multigraph G, usually called the *graphic matroid*. The bases are the spanning trees if the graph G is connected.

Another example is the *partition matroid*: given k disjoint sets $P_1, \ldots, P_k$ which form a ground set $P = \cup_{i=1}^k P_i$ and k nonnegative integers b_i $(i = 1..k)$, the sets $F \subseteq P$ satisfying $|F \cap P_i| \leq b_i$ form a matroid. Notably (and it is crucial in the present work), allocating a set of m indivisible items to n agents can be seen as a partition matroid. Build m sets $P_i = \{i^1, i^2, \ldots, i^n\}$ and let $b_i = 1$ for $i \in [m]$. Taking i^k means allocating item i to agent k.

Last example of a well known matroid is the "matching" matroid defined over a graph $G = (V, E)$: take $X = V$ and define $\mathcal{F}$ as the subsets of V which can be covered by a matching of G.

When every element $e \in X$ has a weight $w(e) \in \mathbb{R}^+$, a classical optimization problem consists in computing a base $B \in \mathcal{F}$ that maximizes $\sum_{e \in B} w(e)$. This problem is solved in polynomial time by the famous GREEDY algorithm (also known as Kruskal's algorithm for the maximum weight spanning tree problem) described in Algorithm 1.

Let us give here a general lemma that we will find useful in Section 5.

Lemma 1 *Let $\mathcal{M} = (X, \mathcal{F})$ be a matroid and a function $w : X \to \mathbb{R}^+$ such that $w(X') = \sum_{e \in X'} w(e)$, $\forall X' \subseteq X$. Given $F_1, F_2 \in \mathcal{F}$ where $|F_1| < |F_2|$, suppose that $F_2 = \{e^1, \ldots, e^{|F_2|}\}$ with $w(e^1) \geq \cdots \geq w(e^{|F_2|})$. In the contracted matroid $\mathcal{M}/F_1$, $\exists E \subseteq F_2 \backslash F_1$ such that E is independent in $\mathcal{M}/F_1$ (i.e. $F_1 \cup E \in \mathcal{F}$) where $|E| = |F_2| - |F_1|$ and $w(E) \geq w\left(\{e^{|F_1|+1}, \ldots, e^{|F_2|}\}\right)$.*

Proof. By induction on $k = |F_2| - |F_1|$, we prove that there is an independent set E_k satisfying the lemma.

For $k = 1$, from property (iii) of matroids, $\exists e \in F_2 \backslash F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$. Since $w(e^1) \geq \cdots \geq w(e^{|F_2|})$, then $w(e) \geq w(e^{|F_2|})$. Hence, $E_1 = \{e\}$.

We assume that Lemma 1 is true for $k \geq 1$ and we show it for $k+1$. Let $F_1, F_2 \in \mathcal{F}$ such that $|F_2| - |F_1| = k+1$, $F_2 = \{e^1, \ldots, e^{|F_1|+k+1}\}$ and $w(e^1) \geq \ldots \geq w(e^{|F_1|+k+1})$. Let $F_2' = F_2 \backslash \{e^{|F_1|+k+1}\}$. Using the inductive hypothesis, $\exists E_k \subseteq F_2' \backslash F_1$ in the contracted matroid $\mathcal{M}/F_1$ such that $F_1 \cup E_k \in \mathcal{F}$, $|E_k| = |F_2'| - |F_1| = k$ and $w(E_k) \geq w\left(\{e^{|F_1|+1}, \ldots, e^{|F_1|+k}\}\right)$.
Consider the sets $F_1 \cup E_k$ and F_2 where $|F_1 \cup E_k| = |F_1| + k$ and $|F_2| = |F_1| + k + 1$. From property (iii) of matroids, $\exists e \in F_2 \backslash (F_1 \cup E_k)$ such that $F_1 \cup E_k \cup \{e\} \in \mathcal{F}$. In other words, $E_k \cup \{e\}$ is an independent set of $\mathcal{M}/F_1$. Since $w(e^1) \geq \cdots \geq w(e^{|F_1|+k+1})$ then, $w(e) \geq w(e^{|F_1|+k+1})$. Let $E_{k+1} = E_k \cup \{e\}$. Hence, $w(E_{k+1}) \geq w\left(\{e^{|F_1|+1}, \ldots, e^{|F_2|}\}\right)$ as claimed. □

Algorithm 1 GREEDY

Require: $\mathcal{M} = (X, \mathcal{F})$, $w : X \rightarrow \mathbb{R}^+$
Sort $X = \{e^1, \cdots, e^{|X|}\}$ such that $w(e^i) \geq w(e^{i+1})$, $i = 1..|X|-1$
$F \leftarrow \emptyset$
for $i = 1$ to $|X|$ **do**
  if $F \cup \{e^i\} \in \mathcal{F}$ **then**
    $F \leftarrow F \cup \{e^i\}$
  end if
end for
return F

The time complexity of matroid algorithms depends of the difficulty of testing if a set $F \in \mathcal{F}$. This is usually done by a dedicated subroutine called the *independence oracle*. We always assume that this subroutine runs in polynomial time.

4 The model

We are given a matroid $\mathcal{M} = (X, \mathcal{F})$, a set of agents $N = \{1, ..., n\}$ and $u_i(e) \in \mathbb{R}^+$ for every pair $(i, e) \in N \times X$. Actually, $u_i(e)$ is the utility of agent i for the element e. With a slight abuse of notation, the utility of an agent $i \in N$ for a subset X' of X is denoted by $u_i(X')$ and defined as $\sum_{x \in X'} u_i(x)$. As a convention, $u_i(\emptyset)$ is equal to 0.

Agent i prefers the solutions (bases) that maximize u_i. One of these solutions, denoted by B_i^*, can be built by the GREEDY algorithm. We can assume that B_i^* is a base of $\mathcal{M}$ because utilities are nonnegative. We suppose w.l.o.g. that every base B_i^* optimal for agent i satisfies $u_i(B_i^*) = 1$. This normalization allows to capture the agents' relative utilities instead of their absolute utilities. If $u_i(B_i^*) \neq 1$ for some agent i, it suffices to replace $u_i(e)$ by $u_i(e)/u_i(B_i^*)$ for all $e \in X$. This is trivially done in polynomial time.

Throughout the article, we use $\alpha_i = \max_{\{e\} \in \mathcal{F}} u_i(e)$, the maximal utility that agent i has for an element of the matroid and $\alpha = \max_{i \in N} \alpha_i$, the maximal utility assigned to an element of the matroid, over all agents.

Following [Demko and Hill, 1988], we are interested in determining a value t_n such that, in any case, there exists a solution $B \in \mathcal{F}$ satisfying $u_i(B) \geq t_n$, $\forall i \in N$. As done in [Markakis and Psomas, 2011], a strengthening of this result would be a vector $(t_{i,n})_{i \in N}$ such that, in any case, there exists a solution B of the matroid with $u_i(B) \geq t_{i,n} \geq t_n$, $\forall i \in N$.

Since our model extends the one in [Demko and Hill, 1988; Markakis and Psomas, 2011], parameters α and α_i play an important role in the determination of t_n and $(t_{i,n})_{i \in N}$.

In Section 3, we see that the allocation of indivisible goods is a particular case of our model. A possible application of the matroid model, that allocation of indivisible goods cannot cover, is the following: consider a bipartite graph $(A \cup T, E)$ whose node sets A and T correspond to a set of activities and a set of time slots, respectively. At most one activity can be scheduled during a time slot and there is an edge $(a, t) \in E$ iff activity a is available during time slot t. It is possible to schedule a subset of activities A' if there exists a matching covering A'. As mentioned in Section 3, the subsets of A for which a feasible schedule exists, form a matroid. In the presence of n agents having heterogeneous utilities for the activities, it is relevant to seek for a common set of activities that is feasible and fair.

5 Technical results

Let us begin by defining the function V_n of Hill.

Definition 1 *[Hill, 1987; Markakis and Psomas, 2011] Given any integer $n \geq 2$, let $V_n : [0,1] \rightarrow [0, n^{-1}]$ be the unique nonincreasing function satisfying $V_n(x) = 1/n$ for $x = 0$, whereas for $x > 0$:*

$$V_n(x) = \begin{cases} 1 - p(n-1)x, & x \in I(n,p) \\ 1 - \frac{(p+1)(n-1)}{(p+1)n-1}, & x \in NI(n,p) \end{cases}$$

where for any integer $p \geq 1$, $I(n,p) = \left[\frac{p+1}{p((p+1)n-1)}, \frac{1}{pn-1}\right]$ and $NI(n,p) = \left(\frac{1}{(p+1)n-1}, \frac{p+1}{p((p+1)n-1)}\right)$.

[Markakis and Psomas, 2011] produce an allocation such that each agent i receives at least $V_n(\alpha_i)$ which is tight in $I(n,p)$. We define a new function W_n.

Definition 2 *Given any integer $n \geq 1$, let $W_n : [0,1] \rightarrow [0, n^{-1}]$ be the function satisfying $W_1(x) = 1$, $\forall x \in [0, 1]$, whereas for any integer $n \geq 2$,*
$W_n(x) = 1/n$ for $x = 0$ and for $x > 0$,

$$W_n(x) = \begin{cases} 0, & x \in I_{(n,0)} \\ 1 - p(n-1)x, & x \in I^1_{(n,p)} \\ \frac{p(1-px)}{(p+1)(n-1)-1}, & x \in I^2_{(n,p)} \\ \frac{p(x+p-1)}{np^2-p-n+2}, & x \in I^3_{(n,p)} \end{cases}$$

where $I_{(n,0)} = \left[\frac{1}{n-1}, 1\right]$ and for any integer $p \geq 1$,

- $I^1_{(n,p)} = \left[\frac{p+1}{p((p+1)n-1)}, \frac{1}{pn-1}\right)$
- $I^2_{(n,p)} = \left[\frac{p^2}{np^3-p^2+p+n-2}, \frac{p+1}{p((p+1)n-1)}\right)$
- $I^3_{(n,p)} = \left[\frac{1}{(p+1)n-1}, \frac{p^2}{np^3-p^2+p+n-2}\right)$
- $I_{(n,p)} = I^1_{(n,p)} \cup I^2_{(n,p)} \cup I^3_{(n,p)} = \left[\frac{1}{(p+1)n-1}, \frac{1}{pn-1}\right)$

Note that $I_{(n,p)} \neq I(n,p)$. In the following, we use only the intervals defined in Definition 2. We see that $W_n(x) > V_n(x)$ for intervals $I^2_{(n,p)}$ and $I^3_{(n,p)}$, and $V_n(x) = W_n(x)$ elsewhere.

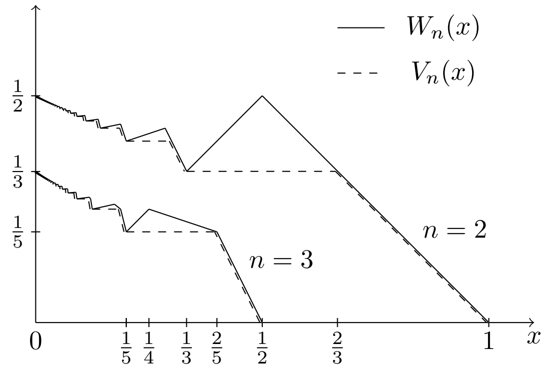

Figure 1: $W_n(\cdot)$ and $V_n(\cdot)$ for $n = 2$ and $n = 3$.

The following properties can easily be proved:

Property 1 *Given any integers $n \geq 2$ and $p \geq 1$,*

1. $V_n(x) \leq W_n(x) \leq \frac{1}{n}$, $\forall x \in [0,\, 1]$.
2. *If* $x \leq \frac{p+1}{p((p+1)n-1)}$ *then* $W_n(x) \geq \frac{p}{(p+1)n-1}$.
3. *If* $x \in I_{(n,p)}$ *then* $(p-1)x < W_n(x) \leq px$.

Now, we present Algorithm 2 which constructs a base (solution) by executing THRESHOLD($N, \mathcal{M}, (u_i)_{i\in N}, \emptyset$).

Algorithm 2 THRESHOLD

Require: $N, \mathcal{M} = (X,\, \mathcal{F}), (u_i)_{i\in N}, B$

```
for all i ∈ N do
    B*_i ← GREEDY(M, u_i)
    for all e ∈ X do
        ũ_i(e) ← u_i(e)/u_i(B*_i)
    end for
    α_i ← max_{{e}∈F} ũ_i(e)
    S_i ← ∅
    add in a greedy manner elements of B*_i to S_i by non-
    increasing order of ũ_i until ũ_i(S_i) ≥ W_{|N|}(α_i)
end for
pick i ∈ N such that |S_i| ≤ |S_k|, ∀k ∈ N
B_i ← S_i
B ← B ∪ B_i
if |N| = 1 then
    return B
else
    let M/B_i = (X\B_i, F̃) be the contraction of M to
    B_i where F ∈ F̃ iff F ∪ B_i ∈ F
    THRESHOLD(N\{i}, M/B_i, (ũ_k)_{k∈N\{i}}, B)
end if
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THRESHOLD is an adaptation of the algorithm of [Markakis and Psomas, 2011] on matroids. Our algorithm sorts the set X, this is done in $O(|X| \ln |X|)$, then tests the independence of adding elements of X to the solution, done in $O(\theta|X|)$, where θ is the complexity of the independence oracle. Since we repeat these steps n times (induction), the complexity of THRESHOLD is $O\left(n|X| \max\{\ln |X|,\, \theta\}\right)$. θ is not given explicitly, it depends on the matroid under consideration. In our study, we suppose that θ is a polynomial.

In the rest of the paper, we always assume that agent i has been selected during the i-th call of THRESHOLD. So, $B = B_1 \cup \cdots \cup B_i$ and the contracted matroid is $\mathcal{M}/(B_1 \cup \cdots \cup B_i)$ at the end of the i-th call. At the end of the n-th call of THRESHOLD, $B = B_1 \cup \cdots \cup B_n$ and it is a base of $\mathcal{M}$ because during the n-th call, the algorithm adds by construction an independent set which is a base of $\mathcal{M}/(B_1 \cup \cdots \cup B_{n-1})$.

Example 1 *Consider a set of agents $N = \{1, 2\}$ and the graphic matroid defined over a connected graph $G = (V,\, E)$, illustrated by Figure 2. The aim is to find a common base to 2 agents with a guarantee on the utility of each one. In the graphic matroid of G, a base corresponds to a spanning tree.*

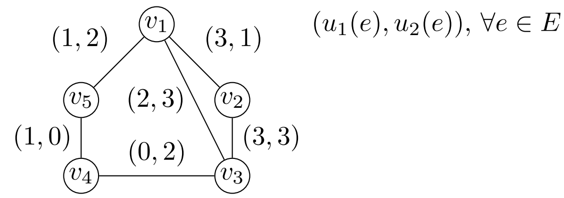

Figure 2: The graph G.

During the first call of THRESHOLD, *we have $B^*_1 = \{(v_1, v_2), (v_2, v_3), (v_4, v_5), (v_5, v_1)\}$ and $B^*_2 = \{(v_2, v_3), (v_1, v_3), (v_3, v_4), (v_5, v_1)\}$ with $u_1(B^*_1) = 8$ and $u_2(B^*_2) = 10$ where B^*_i corresponds to a maximum spanning tree in (G, u_i) (for instance, by applying* GREEDY *or Kruskal's algorithm). The algorithm normalizes utilities $u_i(e)$ to get $\tilde{u}_i(e) = u_i(e)/u_i(B^*_i)$, $\forall i \in \{1, 2\}$, $\forall e \in E$. Then, $\alpha_1 = 3/8 \in I^3_{(2,1)}$ and $\alpha_2 = 3/10 \in I^1_{(2,2)}$. So, $W_2(\alpha_1) = \alpha_1 = 3/8$ and $W_2(\alpha_2) = 1 - 2\alpha_2 = 4/10$. Now, each agent $i \in \{1, 2\}$ builds a forest S_i by adding the heaviest edges of B^*_i until $\tilde{u}_i(S_i) \geq W_2(\alpha_i)$. We find $S_1 = \{(v_1, v_2)\}$ and $S_2 = \{(v_2, v_3), (v_1, v_3)\}$. Since $|S_1| < |S_2|$, we have $i = 1$, $B_1 = S_1$ and $B = B_1$.*

Now, let G/B_1 be the new graph obtained by contracting edge (v_1, v_2) of B_1 into vertex $v_{1,2}$ as it is done in Figure 3.

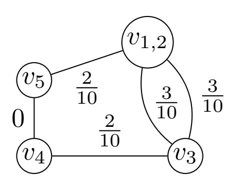

Figure 3: The contracted graph G/B_1.

The utilities of agent 1 are omitted because he has already chosen his edges. Recall that the second agent's utility has been normalized during the first call of THRESHOLD *$\tilde{u}_2(e) = u_2(e)/u_2(B^*_2)$. Let $\tilde{B}^*_2$ be an optimal base of the contracted matroid $(G/B_1, \tilde{u}_2)$. During the second call of* THRESHOLD, *the algorithm finds* $\tilde{B}_2^* = \{(v_{1,2}, v_3), (v_3, v_4), (v_5, v_{1,2})\}$ *with* $\tilde{u}_2(\tilde{B}_2^*) = 7/10$. *Now, the utilities of agent 2 are normalized* $\tilde{\tilde{u}}_2(e) = \tilde{u}_2(e)/\tilde{u}_2(\tilde{B}_2^*)$ *and* $\tilde{\alpha}_2 = 3/10 * 10/7 = 3/7$. *Then, agent 2 builds a forest* $\tilde{S}_2$ *by adding the heaviest edges of* $\tilde{B}_2^*$ *until the threshold* $\tilde{\tilde{u}}_2(\tilde{S}_2) \geq W_1(\tilde{\alpha}_2) = 1$ *is reached. So,* $B_2 = \tilde{B}_2^*$ *and finally* THRESHOLD *returns* $B = B_1 \cup B_2 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_5, v_1)\}$ *where* $u_1(B) = 7 \geq W_2(\alpha_1)u_1(B_1^*)$ *and* $u_2(B) = 8 \geq W_2(\alpha_2)u_2(B_2^*)$.

Theorem 1 THRESHOLD *returns a base* B *which satisfies* $u_i(B) \geq W_{|N|}(\alpha_i)u_i(B_i^*)$ *for all* $i \in N$.

Proof. Let $n = |N|$ and recall that the base returned by THRESHOLD is $B = B_1 \cup \cdots \cup B_n$.

Actually, we will prove, by induction on n, a stronger result: $u_i(B_i) \geq W_n(\alpha_i)u_i(B_i^*)$, $\forall i \in N$. Because the utilities are nonnegative, we conclude $u_i(B) \geq u_i(B_i)$.

For $n = 1$, line 8 of THRESHOLD is equivalent to applying GREEDY$(\mathcal{M}, u_1)$ to get the base $B = B_1 = B_1^*$ which satisfies: $\tilde{u}_1(B_1) \geq W_1(\alpha_1) = 1$ because $W_1(x) = 1$, $\forall x \in [0, 1]$. Thus, $u_1(B_1) = W_1(\alpha_1)u_1(B_1^*)$.

Let $n \geq 2$. We assume that $r(\mathcal{M}) \geq n$ because otherwise all bases have size at most $n - 1$ and then, $\forall i \in N$, $\alpha_i \geq 1/(n-1)$. In this case, $\forall i \in N$, $W_n(\alpha_i) = 0$ and the result is trivially satisfied.
For similar reasons, we assume that $\alpha_i < 1/(n-1)$, $\forall i \in N$ because otherwise, for agents i with $\alpha_i \geq 1/(n-1)$, we get $W_n(\alpha_i) = 0$ and the bounds of agents i are clearly satisfied. W.l.o.g., we assume that agent 1 has been selected first. So, $B_1 = S_1$ and

$$|B_1| \leq |S_i|, \forall i \in N \tag{1}$$

In order to avoid confusion between the notations during first and second calls of THRESHOLD, we add a tilde for the notations used during the second call of THRESHOLD (like it is done in Example 1). Hence, $\tilde{N} = N \setminus \{1\}$, $\tilde{u}_i(e) = u_i(e)/u_i(B_i^*)$, $\tilde{\mathcal{M}} = \mathcal{M}/B_1 = (X \backslash B_1, \tilde{\mathcal{F}})$ and $\tilde{B}_i^*$ is an optimal base of $(\tilde{\mathcal{M}}, \tilde{u}_i)$. Moreover for $i \in \tilde{N}$, $\tilde{\tilde{u}}_i(e) = \tilde{u}_i(e)/\tilde{u}_i(\tilde{B}_i^*)$ for the elements $e \in X \backslash B_1$ of $\tilde{\mathcal{M}}$ and $\tilde{\alpha}_i = \max_{\{e\} \in \tilde{\mathcal{F}}} \tilde{\tilde{u}}_i(e)$.

The inductive hypothesis affirms that

$$\tilde{u}_i(B_i) \geq W_{n-1}(\tilde{\alpha}_i)\tilde{u}_i(\tilde{B}_i^*), \forall i \in \tilde{N} \tag{2}$$

and we want to show that $u_i(B_i) \geq W_n(\alpha_i)u_i(B_i^*)$, $\forall i \in N$ or equivalently $\tilde{u}_i(B_i) \geq W_n(\alpha_i)$, $\forall i \in N$. By construction of THRESHOLD, during the first call we have (lines 3-8):

$$\tilde{u}_i(S_i) \geq W_n(\alpha_i), \forall i \in N \tag{3}$$

Since $B_1 = S_1$, then by (3), we get the expected result for agent 1. In addition, since the sets $S_i = \left\{e_i^1, \ldots, e_i^{|S_i|}\right\}$ are minimal for inclusion, we deduce:

$$\tilde{u}_i(S_i \backslash \{e_i^{|S_i|}\}) < W_n(\alpha_i), \forall i \in N \tag{4}$$

Let $k \in \tilde{N}$. We decompose $B_k^* = B_k^{*1} \cup B_k^{*2}$ such that B_k^{*1} contains the $|B_1|$ first largest elements of B_k^*. It is possible because of (1) and recalling that $S_k \subseteq B_k^*$. Hence, $B_k^{*1} \subseteq S_k$ and

$$\tilde{u}_k(B_k^{*1}) \leq \tilde{u}_k(S_k) \tag{5}$$

Since for $n \geq 2$, $W_n(x) \leq 1/n \leq 1/2$ (see 1. of Property 1) and $r(\mathcal{M}) \geq n \geq 2$, we must have $|B_1| < |B_1^*| = |B_k^*|$ (all the bases have the same size $r(\mathcal{M})$). We can prove it by contradiction: assume $|B_1| = |B_1^*|$. Since $S_1 = B_1$ and $S_1 \subseteq B_1^*$, we have $S_1 = B_1 = B_1^*$ so, $\tilde{u}_1(S_1) = \tilde{u}_1(B_1^*) = 1$. By (4), $\tilde{u}_1(S_1 \backslash \{e_1^{|S_1|}\}) = \tilde{u}_1(S_1) - \tilde{u}_1(e_1^{|S_1|}) = 1 - \tilde{u}_1(e_1^{|S_1|}) < W_n(\alpha_1) \leq 1/2$ so, $\tilde{u}_1(e_1^{|S_1|}) > 1/2$. Since $|S_1| = r(\mathcal{M}) \geq 2$ and S_1 is sorted by nonincreasing order of $\tilde{u}_1$, we get $\tilde{u}_1(S_1) \geq 2\tilde{u}_1(e_1^{|S_1|}) > 1$ which is a contradiction.

Hence, we can apply Lemma 1 for $\mathcal{M}$, $w = \tilde{u}_k$, $F_1 = B_1$ and $F_2 = B_k^*$. We deduce that there exists $E \subseteq B_k^* \backslash B_1$ such that $B_1 \cup E \in \mathcal{F}$ where $|E| = |B_k^*| - |B_1| = |B_k^{*2}|$ and $\tilde{u}_k(E) \geq \tilde{u}_k(B_k^{*2})$. E is an independent set of $\tilde{\mathcal{M}} = \mathcal{M}/B_1$ whereas $\tilde{B}_k^*$ is a base of $\tilde{\mathcal{M}}$ maximum for $\tilde{u}_k$. So, $\tilde{u}_k(\tilde{B}_k^*) \geq \tilde{u}_k(E) \geq \tilde{u}_k(B_k^{*2})$. Due to the normalization, we get $\tilde{u}_k(\tilde{B}_k^*) \geq 1 - \tilde{u}_k(B_k^{*1})$. From (5) and the previous inequality, it holds that:

$$\tilde{u}_k(\tilde{B}_k^*) \geq 1 - \tilde{u}_k(S_k) \tag{6}$$

and by (2) and (6),

$$\tilde{u}_k(B_k) \geq W_{n-1}(\tilde{\alpha}_k)\,(1 - \tilde{u}_k(S_k)) \tag{7}$$

Now, from line 6 of THRESHOLD, we know that:

$$\tilde{\alpha}_k = \max_{B_1 \cup \{e\} \in \mathcal{F}} \tilde{\tilde{u}}_k(e) = \frac{\max_{B_1 \cup \{e\} \in \mathcal{F}} \tilde{u}_k(e)}{\tilde{u}_k(\tilde{B}_k^*)} \leq \frac{\alpha_k}{\tilde{u}_k(\tilde{B}_k^*)}$$

By (6), the last inequality becomes

$$\tilde{\alpha}_k \leq \frac{\alpha_k}{1 - \tilde{u}_k(S_k)} \tag{8}$$

Let $p_k \in \mathbb{N}^*$ such that $\alpha_k \in I_{(n,p_k)}$ (see Definition 2).

We first show that $|S_k| \geq p_k$. By contradiction, suppose that $|S_k| \leq p_k - 1$. Then, by the definition of α_k, $\tilde{u}_k(S_k) \leq (p_k - 1)\alpha_k$. Using 3. of Property 1, we get $(p_k - 1)\alpha_k < W_n(\alpha_k)$ which leads to a contradiction with (3). Now, we distinguish two cases: $|S_k| = p_k$ and $|S_k| \geq p_k + 1$.

Case 1: $|S_k| = p_k$. Line 6 of THRESHOLD implies that $\tilde{u}_k(S_k) \leq p_k\alpha_k$. Inequality (7) becomes

$$\tilde{u}_k(B_k) \geq W_{n-1}(\tilde{\alpha}_k)\,(1 - p_k\alpha_k) \tag{9}$$

and by (8),

$$\begin{aligned} \tilde{\alpha}_k &\leq \frac{\alpha_k}{1 - p_k\alpha_k} \\ &< \frac{1}{p_k(n-1) - 1} \text{ for } \alpha_k \in I_{(n,p_k)} \end{aligned} \tag{10}$$

So, either $\tilde{\alpha}_k \in I^1_{(n-1,p_k)}$ or $\tilde{\alpha}_k \leq \frac{p_k+1}{p_k((p_k+1)(n-1)-1)}$.

(a) If $\tilde{\alpha}_k \in I^1_{(n-1,p_k)}$ then $W_{n-1}(\tilde{\alpha}_k) = 1 - p_k(n-2)\tilde{\alpha}_k$. From (9), we deduce that $\tilde{u}_k(B_k) \geq (1 - p_k(n-2)\tilde{\alpha}_k)\,(1 - p_k\alpha_k)$. Finally, using (10), we get that $\tilde{u}_k(B_k) \geq \left(1 - \frac{p_k(n-2)\alpha_k}{1 - p_k\alpha_k}\right)(1 - p_k\alpha_k) = 1 - p_k(n-1)\alpha_k \geq W_n(\alpha_k)$, which can easily be checked because $\alpha_k \in I_{(n,p_k)}$.

(b) Otherwise, $\tilde{\alpha}_k \leq \frac{p_k+1}{p_k((p_k+1)(n-1)-1)}$. Using 2. of Property 1 if $n \geq 3$ or $W_1(x) = 1$ if $n = 2$, we obtain

$W_{n-1}(\tilde{\alpha}_k) \geq \frac{p_k}{(p_k+1)(n-1)-1}$. By (9), $\tilde{u}_k(B_k) \geq \frac{p_k}{(p_k+1)(n-1)-1}(1 - p_k\alpha_k) \geq W_n(\alpha_k)$, which can easily be checked because $\alpha_k \in I_{(n,p_k)}$.

Case 2: $|S_k| \geq p_k + 1$. Then, we prove that $p_k \geq 2$. By contradiction, assume $p_k = 1$, then $|S_k| \geq 2$. Since, $n \geq 2$, 3. of Property 1 with $\alpha_k \in I_{(n,p_k)}$ gives $W_n(\alpha_k) \leq p_k\alpha_k = \alpha_k$ because $p_k = 1$. Thus, agent k reaches the threshold $\tilde{u}_k(S_k) = \alpha_k \geq W_n(\alpha_k)$ just by selecting the heaviest element of B_k^* which is a contradiction with $|S_k| \geq 2$.

Since $|S_k| \geq p_k + 1$, (4) implies that $W_n(\alpha_k) > \tilde{u}_k(S_k \backslash \{e_k^{|S_k|}\}) \geq \tilde{u}_k(e_k^1) + (p_k - 1)\tilde{u}_k(e_k^{p_k}) = \alpha_k + (p_k - 1)\tilde{u}_k(e_k^{p_k})$ because $|S_k| \geq p_k + 1$, the elements of S_k are sorted by nonincreasing order of $\tilde{u}_k$ and $\tilde{u}_k(e_k^1) = \alpha_k$. From this inequality, we deduce $\tilde{u}_k(e_k^{|S_k|}) \leq \tilde{u}_k(e_k^{p_k}) < \frac{W_n(\alpha_k)-\alpha_k}{p_k-1}$ because $p_k \geq 2$. Finally, by adding the last inequality of $\tilde{u}_k(e_k^{|S_k|})$ to (4), we get:

$$\tilde{u}_k(S_k) < \frac{p_k W_n(\alpha_k) - \alpha_k}{p_k - 1} \tag{11}$$

On the one hand, by (7) and (11) we have:

$$\tilde{u}_k(B_k) \geq W_{n-1}(\tilde{\alpha}_k)\left(1 - \frac{p_k W_n(\alpha_k) - \alpha_k}{p_k - 1}\right) \tag{12}$$

On the other hand, by (8) and (11) we get:

$$\tilde{\alpha}_k \leq \frac{\alpha_k(p_k - 1)}{p_k - 1 + \alpha_k - p_k W_n(\alpha_k)} \tag{13}$$

Let us analyze the different cases according to the values of α_k in $I_{(n,p_k)} = I^1_{(n,p_k)} \cup I^2_{(n,p_k)} \cup I^3_{(n,p_k)}$.

(a): $\alpha_k \in I^1_{(n,p_k)}$. Then by construction, $W_n(\alpha_k) = 1 - p_k(n-1)\alpha_k$. Using (12), we obtain:

$$\tilde{u}_k(B_k) \geq W_{n-1}(\tilde{\alpha}_k)\frac{((n-1)p_k^2 + 1)\alpha_k - 1}{p_k - 1} \tag{14}$$

On the other hand, by (13) we get:

$$\begin{aligned}\tilde{\alpha}_k &\leq \frac{\alpha_k(p_k - 1)}{((n-1)p_k^2 + 1)\alpha_k - 1} \\ &\leq \frac{p_k + 1}{p_k((p_k + 1)(n-1) - 1)} \text{ for } \alpha_k \in I^1_{(n,p_k)}\end{aligned}$$

Using 2. of Property 1 if $n \geq 3$ with this last inequality of $\tilde{\alpha}_k$ or $W_1(x) = 1$ if $n = 2$, we deduce: $W_{n-1}(\tilde{\alpha}_k) \geq \frac{p_k}{(p_k+1)(n-1)-1}$. Inequality (14) becomes $\tilde{u}_k(B_k) \geq \left(\frac{p_k}{(p_k+1)(n-1)-1}\right)\frac{((n-1)p_k^2+1)\alpha_k-1}{p_k-1} \geq 1 - p_k(n-1)\alpha_k = W_n(\alpha_k)$ which can easily be checked because $\alpha_k \in I^1_{(n,p_k)}$.

(b): $\alpha_k \in I^2_{(n,p_k)}$. Then by construction, $W_n(\alpha_k) = \frac{p_k(1-p_k\alpha)}{(p_k+1)(n-1)-1}$. On the one hand, by (12),

$$\begin{aligned}\tilde{u}_k(B_k) \geq \Bigg(&\frac{(p_k^3 + (n-1)p_k + n - 2)\alpha_k}{(p_k - 1)((p_k + 1)(n-1) - 1)} \\ &+ \frac{(n-2)p_k^2 - p_k - n + 2}{(p_k - 1)((p_k + 1)(n-1) - 1)}\Bigg) \\ &\times W_{n-1}(\tilde{\alpha}_k)\end{aligned} \tag{15}$$

On the other hand, by (13),

$$\begin{aligned}\tilde{\alpha}_k \leq &\Bigg[\frac{(p_k^3 + (n-1)p_k + n - 2)\alpha_k}{(p_k - 1)((p_k + 1)(n-1) - 1)} \\ &+ \frac{(n-2)p_k^2 - p_k - n + 2}{(p_k - 1)((p_k + 1)(n-1) - 1)}\Bigg]^{-1} \\ \leq &\frac{p_k + 1}{p_k((p_k + 1)(n-1) - 1)} \text{ for } \alpha_k \in I^2_{(n,p_k)}\end{aligned}$$

Using 2. of Property 1 if $n \geq 3$ with this last inequality of $\tilde{\alpha}_k$ or $W_1(x) = 1$ if $n = 2$, we deduce that $W_{n-1}(\tilde{\alpha}_k) \geq \frac{p_k}{(p_k+1)(n-1)-1}$. Inequality (15) implies $\tilde{u}_k(B_k) \geq \frac{p_k(1-p_k\alpha_k)}{(p_k+1)(n-1)-1} = W_n(\alpha_k)$ which can easily be checked because $\alpha_k \in I^2_{(n,p_k)}$.

(c): $\alpha_k \in I^3_{(n,p_k)}$. Then by construction, $W_n(\alpha_k) = \frac{p_k(\alpha_k+p_k-1)}{np_k^2-p_k-n+2}$. On the one hand, by (12),

$$\begin{aligned}\tilde{u}_k(B_k) \geq &W_{n-1}(\tilde{\alpha}_k)\frac{((n-1)p_k + n - 2)}{np_k^2 - p_k - n + 2} \times \\ &(\alpha_k + p_k - 1)\end{aligned} \tag{16}$$

On the other hand, by (13),

$$\begin{aligned}\tilde{\alpha}_k &\leq \frac{(np_k^2 - p_k - n + 2)\alpha_k}{((n-1)p_k + n - 2)(\alpha_k + p_k - 1)} \\ &\leq \frac{p_k + 1}{p_k((p_k + 1)(n-1) - 1)} \text{ for } \alpha_k \in I^3_{(n,p_k)}\end{aligned}$$

Using 2. of Property 1 if $n \geq 3$ with this last inequality of $\tilde{\alpha}_k$ or $W_1(x) = 1$ if $n = 2$, we deduce that $W_{n-1}(\tilde{\alpha}_k) \geq \frac{p_k}{(p_k+1)(n-1)-1}$. Inequality (16) implies $\tilde{u}_k(B_k) \geq \frac{p_k(\alpha_k+p_k-1)}{np_k^2-p_k-n+2} = W_n(\alpha_k)$ which can easily be checked because $\alpha_k \in I^3_{(n,p_k)}$.

The induction is proved and the result follows. □

6 Discussion

Recall that THRESHOLD provides a solution which is a base B of a matroid with a relative utility $\tilde{u}_i(B) \geq W_n(\alpha_i)$ for each agent i which is tight in $I_{(n,1)} \cup I^1_{(n,p)}$ for any integers $n \geq 1$ and $p \geq 2$.

Unlike V_n, W_n is not monotonic, so it is not obvious that we may guarantee a relative utility of $W_n(\alpha)$ for each agent. However, for $n = 2$ agents, a slight modification of the selected agent in each call of THRESHOLD, allows us to provide a guarantee of at least $\max\{W_2(\alpha_i); W_2(\alpha)\}$ for each agent $i \in N = \{1, 2\}$. Moreover, this latter bound can be shown tight when $\alpha \in I_{(2,1)} \cup \left(\cup_{p\geq 2} I^1_{(2,p)} \cup I^2_{(2,p)}\right)$. An interesting challenge is to know if a bound of $\max\{W_n(\alpha_i); W_n(\alpha)\}$ for each agent $i \in N$ can be reached when $n \geq 3$.

Another perspective is to study the same approach for a more general structure like matroid intersection.

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