Robust Constraint Satisfaction and Local Hidden Variables in Quantum Mechanics

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Abstract

Motivated by considerations in quantum mechanics, we introduce the class of robust constraint satisfaction problems in which the question is whether every partial assignment of a certain length can be extended to a solution, provided the partial assignment does not violate any of the constraints of the given instance. We explore the complexity of specific robust colorability and robust satisfiability problems, and show that they are NP-complete. We then use these results to establish the computational intractability of detecting local hidden-variable models in quantum mechanics.

1 Introduction and Summary of Results

Since the 1930's, the study of hidden variables and locality has occupied a prominent place in the foundations of quantum mechanics. The hidden-variable program in quantum mechanics was developed in an attempt to explain the celebrated Einstein-Podolsky-Rosen paradox [Einstein *et al.*, 1935]. The main idea behind this program is that the original formulation of quantum mechanics is "incomplete", but it can be "completed" via the introduction of hidden variables, that is, quantities that cannot be seen or measured, but which control the observable behaviour. A hidden-variable model is supposed to be consistent with the predictions of quantum mechanics, while at the same time possessing certain desirable properties of classical systems, such as locality and the observer-independence of properties of physical systems. Later on, the hidden-variable program was dealt a blow by the *no-go* theorems, which are results establishing that certain quantum mechanics predictions cannot be explained via hidden-variable models. Bell's theorem [Bell, 1964], the Kochen-Specker theorem [Kochen and Specker, 1967], and Hardy's paradox [Hardy, 1993] are among the best-known no-go theorems.

More recently, researchers have embarked on the development of unifying mathematical frameworks in which the key properties (such as locality, determinism, and independence) of hidden-variable models can be expressed, the relations between these properties can be studied, and existence and non-existence (no-go) theorems can be stated in precise terms and proved rigorously. In particular, [Brandenburger and Yanofsky, 2008] developed a probabilistic framework for hidden-variable models, while [Abramsky, 2013] developed a purely relational framework. In the latter framework, the focus is on n-dimensional relational models, $n \geq 1$, of the form (M, O, e), where M is the cartesian product of n sets $M_1, \ldots, M_n$ of *measurements*, O is the cartesian product of n sets $O_1, \ldots, O_n$ of *outcomes*, and $e \subseteq M \times O$. [Abramsky, 2013] explored the connections between the key properties of relational hidden-variable models and, in particular, characterized when such a model belongs to the class $\mathrm{HV}(n)$ of local hidden-variable models. A consequence of this characterization is that the membership problem for $\mathrm{HV}(n)$ is in NP. However, the exact complexity of this decision problem, which is also denoted by $\mathrm{HV}(n)$, was left open. This was the original motivation for the work reported here.

In this paper, we introduce the class of robust constraint satisfaction problems[1] in which the question is whether every partial assignment of a certain length can be extended to a solution, provided the partial assignment does not violate any of the constraints of the given instance. Special cases of robust constraint satisfaction problems were studied earlier in totally different contexts. In particular, [Beacham, 2000] studied a robust version of 3-HYPERGRAPH 2-COLORABILITY in an investigation of problems with no frozen variables (no backbone), while [Gottlob, 2012] studied a robust version of SATISFIABILITY in the context of minimal constraint networks. However, the general concept of a robust constraint satisfaction problem had not been formulated earlier.

We focus on two robust constraint satisfaction problems that turn out to be just the right tool needed to settle the computational complexity of local hidden-variable models. Specifically, we show that ROBUST 3-COLORABILITY and ROBUST 3SAT are NP-complete problems. The former problem asks: given a graph G, is it 3-colorable and also is it true that every partial assignment of one of three colors to two independent nodes u and v can be extended to a 3-coloring of G? The latter problem asks: given a 3CNF formula φ, is it true that every partial assignment to three variables can be extended to a satisfying truth assignment of φ, provided it does not directly contradict any clause of φ?

Armed with these two new NP-completeness results, we show that for every $n \geq 2$, testing for membership in $\mathrm{HV}(n)$

[1]This notion is unrelated to the notion of robust constraint satisfaction algorithm recently introduced by [Barto and Kozik, 2012]

is a NP-complete problem, thus resolving the problem left open in [Abramsky, 2013]. In fact, we obtain a complete picture for the computational complexity of the parameterized subclasses of $\mathrm{HV}(n)$ that also take into account the size of the domains O_i, $1 \leq i \leq n$, of possible outcomes. More precisely, for every $n \geq 2$ and every $k \geq 2$, let $\mathrm{HV}(n)/k$ be the subclass of $\mathrm{HV}(n)$ consisting of all hidden-variable models (M, O, e) in which each domain O_i of outcomes has at most k elements. We show that $\mathrm{HV}(2)/2$ is in PTIME, while $\mathrm{HV}(n)/k)$ is NP-complete, for all other values of n and k. In particular, both $\mathrm{HV}(2)/3$ and $\mathrm{HV}(3)/2$ are NP-complete.

Roadmap. In Section 2, we define robust constraint satisfaction problem and present several examples. The complexity of robust colorability and robust satisfiability problems is studied in Section 3. In Section 4, we give some background material about hidden-variable models and then show that detecting hidden variables is NP complete. Section 5 concludes the paper with a brief summary and outlook on future work.

2 Robust Constraint Satisfaction

We first review some basic notions from database theory. A *relation schema* is a finite set of variables (a.k.a. *attributes*), where each variable x has an associated domain $dom(x)$ of values. A *relation* r over a schema S (also denoted by $schema(r)$) is a set of tuples, where each tuple t is a mapping that assigns a value $t(x) \in dom(x)$ to each variable $x \in S$. If $|S| = s$, then we say that r is s-ary. If $S' = \{x_1, \ldots, x_n\} \subseteq S$ and t is a tuple over S, then $t[S']$ (and also $t[x_1, \ldots, x_n]$) denotes the restriction of the function t to the variables of S. If $S' \subseteq S$, then the *projection* $\pi_{S'}(r)$ is the relation $\{t[S'] | t \in r\}$ over the schema S'.

Next we recall the standard definition of a constraint satisfaction problem (see also [Tsang, 1993; Dechter, 2003; Rossi *et al.*, 2006]).

A CONSTRAINT SATISFACTION PROBLEM (CSP) is a decision problem P that has the following characteristics.

• An instance I of P consists of a finite set $var(I) = \{x_1, \ldots, x_v\}$ of variables with associated finite domains $dom(x_i)$, $1 \leq i \leq v$, and a finite set of *constraints* $constr(I) = \{c_1, \ldots, c_m\}$. The *domain* $dom(I)$ of I is the union $\bigcup_{x \in var(I)} dom(x)$. Each constraint $c \in constr(I)$ is a relation whose schema $schema(c)$ is a subset of $var(I)$ and whose variable-domains are inherited from I. The set $schema(c)$ is also called the *scope* of c, denoted by $scope(c)$.

• Given an instance I of P, the question is to decide whether I is *satisfiable* (or, *solvable*), that is to say, whether there is a mapping $h : var(I) \longrightarrow dom(I)$, such that $h[schema(c)]$ belongs to c, for each constraint c. Each such mapping is called a *solution* of I.

Let P be a CSP. A *partial assignment* for an instance I of P is an assignment of values to a subset V of the variables of I, that is, a mapping $t : V \longrightarrow dom(I)$. We say that I is *satisfiable under* t if t can be extended to a solution of I.

Let k be a positive integer. A kCSP problem is a CSP problem P such that for every instance I of P and every constraint c of I, the scope of c has at most k variables.

Clearly, every CSP is in NP. Many important NP-complete problems can be viewed as CSPs whose constraints are of a certain form. For example, 3SAT can be viewed as a 3CSP whose constraints represent the satisfying assignments of the different types of clauses with 3 literals. Similarly, GRAPH 3-COLORABILITY is a 2CSP, where a given graph $G = (V, E)$ is viewed as a CSP-instance I having $var(I) = V$, $dom(I) = \{1, 2, 3\}$ (the three colors), and the same constraint $c = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), 3, 2)\}$, for each edge $(u, v) \in E$.

Here, we will also be interested in 3-HYPERGRAPH 2-COLORABILITY, another well known NP-complete problem. An instance of this problem is a 3-hypergraph, i.e., a pair $H = (V, R)$, where R is a ternary relation on V. The question is whether there is an assignment of one of two colors to each node of V such that for every triple (3−hyperedge) in R, two of its three nodes are assigned different color. It is easy to see that 3-HYPERGRAPH 2-COLORABILITY is a 3CSP.

In what follows, we introduce the notion of *robust constraint satisfaction* and present a number of examples.

Let P be a CSP and r a non-negative integer. Intuitively, the *r-robust version of* P, denoted by rROBUST P, is the decision problem that, given an instance I of P, asks whether I is satisfiable under every partial assignment to r variables that does not directly violate any constraint of I. The following definition makes this precise.

Definition 2.1 Let P be a CSP and let $r \geq 0$ an integer.

• Let I be an instance of P, V a set of variables from I, and $t : V \longrightarrow dom(I)$ a partial assignment for I. We say that t is *compatible* [2] *with* I if for every constraint c of I such that $V \cap scope(c) \neq \emptyset$, we have that $t[V \cap scope(c)]$ belongs to the projection $\pi_{V \cap scope(c)}(c)$ of c on $V \cap scope(c)$.

• The *r-robust version of* P, denoted by rROBUST P, is the following decision problem. The instances of rROBUST P are exactly those of P. Given such an instance I, the question is whether I is satisfiable under *every* partial assignment $t : V \rightarrow dom(I)$ such that $|V| = r$ and t is compatible with I.

• If P is a kCSP, for some $k \geq 1$, then we will write ROBUST P, instead of kROBUST P.

According to the preceding definition, if P is a CSP, then 0ROBUSTP = P. Several examples are in order now. We begin with robust satisfiability.

Let φ be a 3CNF formula. A partial assignment t on three variables is compatible with φ precisely when no clause of φ is falsified by t. Therefore, ROBUST 3SAT asks: Given a 3CNF formula φ, is it true that if t is a partial assignment on three variables of φ that falsifies no clause of φ, then t can be extended to a satisfying assignment of φ?

If φ is a kCNF formula with $k > 3$, then every partial assignment on three variables is compatible with φ. Therefore, 3ROBUST kSAT, where $k > 3$, asks: Given a kCNF formula φ, is it true that if t is a partial assignment on 3 variables, then t can be extended to a satisfying assignment of φ?

[2] The notion of a compatible assignment is related to that of a *locally consistent instantiation*, as defined, e.g., in [Bessiere, 2006]. Specifically, a partial assignment is compatible with I if and only if it is a locally consistent instantiation with respect to the instance I^* obtained from I by adding the projections of all constraints of I. The relationship of our notion of compatibility with other notions of local consistency (cf. [Freuder, 1978; Dechter, 1992]) will be discussed in the full version of this paper.

Next, we consider 2-robust colorability problems.

Let $G = (V, E)$ be a graph and let u and v be two vertices of V. If t is a partial assignment assigning colors $t(u)$ and $t(v)$ to u and v, then t *is compatible with* G if either u and v are connected via an edge and $t(u) \neq t(v)$ or u and v are independent. Therefore, ROBUST GRAPH 3-COLORABILITY amounts to the following decision problem: Given a graph $G = (V, E)$ is it true that G is 3-colorable and every partial assignment of one of three colors to two independent nodes u and v can be extended to a 3-coloring of G?

If $H = (V, R)$ is a 3-hypergraph, then every partial assignment of one of two colors to two nodes of V is compatible with H. Therefore 2ROBUST 3-HYPERGRAPH 2-COLORABILITY is the following decision problem: Given a 3-hypergraph $H = (V, R)$, is it true that every partial assignment of one of two colors to two nodes u and v can be extended to a 2-coloring of G?

In the full paper we will also discuss the relationship between robust CSPs and *minimal constraint networks* as studied by [Montanari, 1974; Gaur, 1995; Dechter and Pearl, 1992; Cros, 2003; Dechter, 2003], and [Gottlob, 2012], and the relationship between robust CSPs and *join consistency*, as introduced and studied by [Honeyman *et al.*, 1980].

3 On the Complexity of Robust CSP

It is easy to see that if P is a constraint satisfaction problem and r is a non-negative integer, then rROBUSTP is in NP; this follows easily from the fact that the intersection of polynomially-many NP problems is in NP. As regards lower bounds, only two results about robust constraint satisfaction problems had been obtained earlier. First, [Beacham, 2000] studied 2ROBUST 3-HYPERGRAPH 2-COLORABILITY under the name UNFROZEN 3-HYPERGRAPH 2-COLORABILITY.

Theorem 3.1 ([Beacham, 2000]) 2ROBUST 3-HYPERGRAPH 2-COLORABILITY *is* NP*-complete.*

In an entirely different context, [Gottlob, 2012] studied rROBUST SAT under the name r*-supersymmetric* SAT.

Theorem 3.2 ([Gottlob, 2012]) rROBUST $3(r+1)$SAT *is* NP*-complete, where r is an arbitrary non-negative integer. In particular,* 3ROBUST 12SAT *is* NP*-complete.*

Unlike the case of 2ROBUST 3-HYPERGRAPH 2-COLORABILITY, the complexity of ROBUST GRAPH 3-COLORABILITY has not been studied so far. In fact, we could find no reference for this problem in the literature, even though it is a natural computational problem. Furthermore, the complexity of ROBUST 3SAT has not been identified either. We now show that these two problems are NP-complete.

Theorem 3.3 ROBUST GRAPH 3-COLORABILITY *is* NP*-complete.*

Proof Sketch. As pointed out earlier, this problem is in NP. We show that it is NP-hard by first reducing 2ROBUST 3-HYPERGRAPH 2-COLORABILITY to this problem in logarithmic space, and then using Theorem 3.1. Consider a 3-hypergraph H whose edges are $C_1, \ldots, C_r$. We transform H into a graph $G = (V, E)$ as follows: The graph G contains a node p for each node p of H. Moreover, G contains a special node X that is connected to each such node p. Finally, for each 3-hyperedge C_i of H, the graph G contains a triangle T_i with fresh nodes a_i, b_i, c_i, such that each node of C_i is connected to exactly one node of this triangle. This completes the construction of G. This is illustrated in Figure 1 for a hypergraph H having hyperedges $\{p, q, r\}$, $\{p, r, s\}$, and $\{q, r, s\}$.

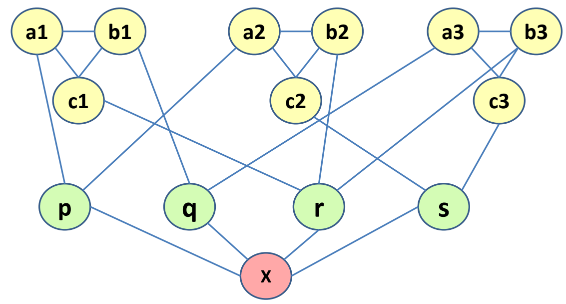

Figure 1: *Graph G for a 3-hypergraph H having hyperedges $\{p, q, r\}$, $\{p, r, s\}$, and $\{q, r, s\}$.*

Claim: H is a "yes"-instance of 2ROBUST 3-HYPERGRAPH 2-COLORABILITY if and only if G is a "yes"-instance of ROBUST GRAPH 3-COLORABILITY.

To prove the "if-direction" of the claim, assume G is a "yes"-instance of ROBUST GRAPH 3-COLORABILITY. Let p, q be two vertices of H, for which we fix a coloring f, that is, $f(p)$ and $f(q)$ are some given fixed colors from $\{0, 1\}$. Because of the well-known symmetry of 3-HYPERGRAPH 2-COLORABILITY with respect to dual colorings, it is actually sufficient to specify whether $f(p) = f(q)$ or $f(p) \neq f(q)$. We need to show that f can be extended to a correct 2-coloring f^* to all vertices of H. Given that G is a "yes"-instance of ROBUST GRAPH 3-COLORABILITY, there must be a 3-coloring $g : V(G) \longrightarrow \{0, 1, 2\}$ of G that is consistent with f , i.e., that assigns the same colors to any nodes p and q if and only if $f(p) = f(q)$. Thus, assume g is such a 3-coloring. Observe that, because of node X, all vertices of G that are also vertices of H must be colored by two colors only, say, w.l.o.g., colors from $\{0, 1\}$. Let f^* be the corresponding 2-coloring of the vertices of H. Obviously, f^* is consistent with f. It remains to show that f^* is effectively a correct hypergraph 2-coloring for H. Assume this is not the case. Then, for some set C_i with vertices p, q, r, $f(p) = f(q) = f(r)$. For the corresponding nodes p, q, r, we then have $g(p) = g(q) = g(r)$, and therefore the triangle $\langle a_i, b_i, c_i \rangle$ would need to be colored with two colors only, which is impossible. Therefore, f^* is a correct 2-coloring of H that, moreover, is consistent with f, and thus H is a "yes"-instance of 2ROBUST 3-HYPERGRAPH 2-COLORABILITY. This proves the "if-direction" of the claim.

The "only-if-direction" of the claim is shown in a similar spirit, but requires a rather involved case-analysis. Due to space considerations, we have to defer it to the full paper. ∎

Theorem 3.4 ROBUST 3SAT *is* NP*-complete.*

Proof Sketch. As pointed out earlier, this problem is in NP. We show that it is NP-hard by first reducing 3ROBUST 12SAT to this problem in logarithmic space, and then using Theorem 3.2. Let φ be a 12CNF formula $C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each C_i is a clause of the form $\ell_i^1 \vee \cdots \vee \ell_i^{12}$ and each ℓ_i^j is a literal, for $1 \leq i \leq m$ and $1 \leq j \leq 12$. For each clause C_i, let $p_i^2, \ldots p_i^{10}$ be nine fresh propositional variables. Let φ^* be the 3CNF formula obtained from φ by replacing each clause C_i by the following conjunction Γ_i of 3-clauses:

$$(\ell_i^1 \vee \ell_i^2 \vee p_i^2) \wedge (\neg p_i^2 \vee \ell_i^3 \vee p_i^3) \wedge \cdots \wedge (\neg p_i^{j-1} \vee \ell_i^j \vee p_i^j)$$

$$\wedge (\neg p_i^j \vee \ell_i^{j+1} \vee p_i^{j+1}) \wedge \cdots \wedge (\neg p_i^{10} \vee \ell_i^{11} \vee \ell_i^{12}).$$

Note that this transformation is just a special instantiation of the well-known logarithmic-space reduction of SAT to 3SAT. Therefore it is easy to see that φ is satisfiable if and only if φ^* is satisfiable. Here, however, we need to show that the following holds:
Claim: φ is a "yes" instance of 3ROBUST 12SAT if and only if φ^* is a "yes" instance of ROBUST 3SAT.

The easier part is the "if-direction". Assume φ^* is a "yes" instance of ROBUST 3SAT. To show that φ is then a "yes" instance of 3ROBUST 12SAT, take any partial truth value assignment (tva) f to three variables of φ. This can be extended to a total satisfying tva f^+ to φ^*. The restriction of f to the ℓ-variables is also a satisfying tva to φ. This is easy to check. The "only-if-direction" of the claim, proven in detail in the full paper, is rather non-trivial. Here, we only provide a rough intuition about the technique. Assume that φ is a "yes" instance of 3ROBUST 12SAT. We must show that φ^* is a "yes" instance of ROBUST 3SAT. The key idea is that, given a partial tva t^* on three variables of φ^*, one can transform t^* to a carefully chosen partial tva t on three variables of φ, and then use the 3-robustness of φ to extend t to a satisfying assignment of φ, which, in turn, can be used to extend t^* to a satisfying assignment of φ^*.

To obtain t from t^*, we carefully replace each truth value assignment of t^* to a to p-variable (i.e., a propositional variable of φ^* that does not occur in φ) by an appropriate tva to an ℓ-variable (i.e., a to variable that occur in both φ and φ^*). For convenience, rather than dealing with tva's to propositional variables, we consider tvas to *literals*. For example, rather than saying p_i is assigned *true*, we say that literal $\neg p_i$ is assigned *false* (which is obviously equivalent). We obtain t from t^* by replacing each assignment of t^* that assigns *false* to a p-literal λ by an assignment that assigns *true* to an ℓ-literal $\lambda' = proxy(\lambda)$, the so called *the proxy of* λ. Intuitively, setting $proxy(\lambda)$ to *true* will ensure that the clause in which λ occurs is satisfied by some literal different from λ; therefore, λ can be safely set to *false* without endangering the overall satisfiability of ϕ^*. In the simplest case, $proxy(\lambda)$ is just the unique ℓ-literal in the same clause as λ. If this ℓ-literal were forced to be *true*, then t would make the clause true, even if we assigned *false* to λ. But choosing this simple proxy for λ is not always possible, because this ℓ-literal itself may be required to be *false* by t^*. However, we are able to show that it is actually always possible to find an appropriate proxy for each ℓ-literal λ. To this aim, we define three functions π_1, π_2, and π_3 from the p-literals of each conjunction Γ_i to the ℓ-literals of the same Γ_i, and we prove that one of $\pi_1(\lambda)$, $\pi_2(\lambda)$, $\pi_3(\lambda)$ can always be used as proxy for λ. The functions π_1, π_2, and π_3 are illustrated in Figure 2.

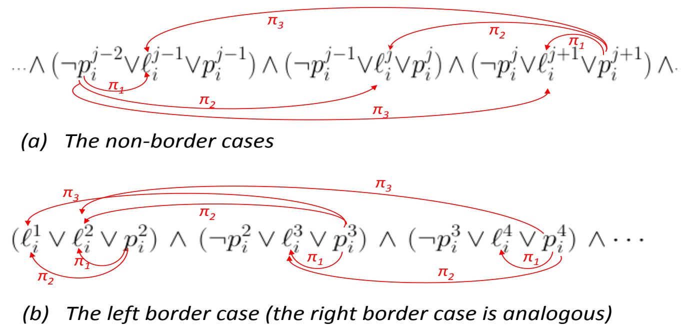

Figure 2: *Candidate proxies π_1, π_2, and π_3.*

By replacing the (at most three) tvas to p-variables of the partial tva t^* by their respective proxy-assignments, we obtain a (partial) tva t to φ. Since φ is a "yes"-instance of 3ROBUST 12SAT, we have that t can be extended to a full satisfying tva t^+ of φ. We then prove that this assignment t^+ can, in turn, be extended to a full satisfying tva $t^\ddagger$ of φ^* that is consistent with t. This shows that φ^* is a "yes"-instance of ROBUST 3SAT, which settles the "only-if-direction". ∎

4 Quantum Mechanics and Hidden Variables

Background on Local Hidden Variables Consider the following scenario. There is some system on which measurements of various quantities can be performed. Two experimenters, Alice and Bob, can each select one of several different measurements to perform, and observe one of several different outcomes. We assume that Alice and Bob are spatially separated, and there is no communication with each other while the measurements are being performed. When measurements are selected by Alice and Bob, some corresponding outcomes will be observed. These individual occurrences or "runs" of the system are the basic events. Repeated runs allow relative frequencies to be tabulated, which can be summarized by a probability distribution. We can abstract from the probabilities themselves, and focus only on the *support* of the probability distributions — distinguishing those events which are *possible* (have non-zero probability) from those which can never happen.

Here is an example of such a support table:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
(a',b)	0	1	1	1
(a,b')	0	1	1	1
(a',b')	1	1	1	0

Table 1

In this example, Alice can select measurement settings a or a', and Bob can select b or b'. Each measurement can result in an outcome of 0 or 1. We consider the joint outcomes when Alice and Bob both select measurements, and tabulate which are possible. For example, if Alice chooses measurement setting a and Bob chooses b', then according to the above table, the joint outcome $(1,0)$, *i.e.*, outcome 1 for a and 0 for b', is possible, while $(0,0)$ is not possible.

How can such *correlated outcomes* be explained, given our assumption about Alice and Bob? One mechanism is to assume that there are preset values for each of the measurements which Alice and Bob can make, independently of the measurement made by the other. Such preset values are specified by functions $h : \{a, a', b, b'\} \to \{0, 1\}$. We allow for the fact that there may be several possible such preset values, which are determined by circumstances beyond our control, by saying that the behaviour of the system is "covered" by a *set* of such functions $\{h_1, \ldots, h_p\}$. Such a set generates a support table by the rule that an entry for measurements (x, y) and outcomes (u, v) is set to 1 if and only if for some i, $h_i(x) = u$ and $h_i(y) = v$.

We can now ask the question: can the above support table be realized by a set of functions in such a fashion? The answer is that it *cannot*. This can be shown by using only the following information from the table: the joint outcome $(0, 0)$ is possible for measurements (a, b), while the joint outcome $(0, 0)$ is impossible for the pairs $(a'b)$ and (a, b'), and the joint outcome $(1, 1)$ is impossible for the pair (a', b').

What makes this fact remarkable is that the above support table can be *realized physically*. That is, we can generate a two-qubit quantum state, and local spin measurements for Alice and Bob corresponding to the measurements a, a', b, b', such that quantum mechanics predicts, and experiment confirms, this support table. Moreover, these predictions are verified even under conditions of spatial separation of the two subsystems corresponding to Alice and Bob. The particular construction we have described is known as *Hardy's paradox* [Hardy, 1993], a variant of *Bell's theorem* [Bell, 1964].

Thus Nature realizes *non-local correlations*, which do not admit any explanation in terms of values possessed by the physical quantities under consideration independently of the actual combination of measurements which are performed. Such an explanation is called a *local hidden-variable model*, so Hardy's paradox is showing that no such local hidden variable model exists for the empirically observed table of possible behaviours we gave above. This leads to the following natural computational problem: given a possibility table, determine whether it has a local hidden-variable model. We focus on this problem next.

Complexity of Local Hidden-Variable Models [Abramsky, 2013] introduced a relational framework for studying the notions of hidden variables and locality in quantum mechanics. Assume we have n different experimenters, for some $n \geq 1$. The ith experimenter can select measurements from a set M_i of available measurements, whose possible outcomes form a set O_i, $1 \leq i \leq n$. Let $M = \Pi_{i=1}^n M_i$ be the set of all combinations of available measurements, and let $O = \Pi_{i=1}^n O_i$ be the set of all combinations of possible outcomes. A *relational model of dimension* n is a relational structure of the form (M, O, e), where $e \subseteq M \times O$.

For example, Table 1 gives rise to the relational model (M, O, e) of dimension 2, where $M = M_1 \times M_2$ with $M_1 = \{a, a'\}$ and $M_2 = \{b, b'\}$, $O = O_1 \times O_2$ with $O_1 = O_2 = \{0, 1\}$, and e is the relation consisting of all tuples (m_1, m_2, o_1, o_2) such that the entry in the cell determined by row (m_1, m_2) and column (o_1, o_2) is equal to 1.

In [Abramsky, 2013], the notion of a *local hidden-variable model* was rigorously defined and the properties of the classes $\mathrm{HV}(n)$ of all local hidden-variable models of dimension n were investigated. Each class $\mathrm{HV}(n)$ can be identified with the following decision problem, which, for simplicity, is also denoted by $\mathrm{HV}(n)$: Given a relational model (M, O, e) of dimension n, is it in $\mathrm{HV}(n)$?

As shown in [Abramsky, 2013], for every $n \geq 1$, the class $\mathrm{HV}(n)$ consists of all relational models (M, O, e) that satisfy the following formula $\Psi(n)$ of second-order logic:

$$(\forall \mathbf{x})(\exists \mathbf{y})R(\mathbf{x}, \mathbf{y}) \wedge [(\forall \mathbf{x})(\forall \mathbf{y})(R(\mathbf{x}, \mathbf{y}) \rightarrow \exists f_1 \cdots \exists f_n (\bigwedge_{i=1}^{n}((f_i(x_i) = y_i) \wedge (\forall \mathbf{u})R(\mathbf{u}, f_1(u_1), \ldots, f_n(u_n)))],$$

where the variables in $\mathbf{x}$ and $\mathbf{u}$ range over M, the variables in $\mathbf{y}$ range over O, and the relational symbol R is interpreted by the relation e of the model.

Going back to our example, the relational model (M, O, e) arising from Table 1 does *not* satisfy the formula $\Psi(2)$. To see this and towards a contradiction, assume that it does. Since the tuple $(a, b, 0, 0)$ belongs to e, there are functions f_1 and f_2 such that (i) $f_1(a) = 0$ and $f_2(b) = 0$; (ii) for all (m_1, m_2) in M, we have that $(m_1, m_2, f_1(m_1), f_2(m_2))$ belongs to e. Since $(a', b, 0, 0) \notin e$, it follows that $f_1(a') = 1$. Similarly, since $(a, b', 0, 0) \notin e$, it follows that $f_2(b') = 1$. Hence, $(a', b', 1, 1) \in e$, which is a contradiction, since the entry determined by row (a', b') and column $(1, 1)$ in Table 1 is 0.

An important consequence of the aforementioned characterization of $HV(n)$ in terms of second-order logic is that $\mathrm{HV}(n)$ is in NP, for every $n \geq 1$. The exact computational complexity of $\mathrm{HV}(n)$, however, was left as an open problem. Note that if $n = 1$, then $\mathrm{HV}(1)$ is actually first-order definable, hence it is in LOGSPACE (in fact, it is in the lower complexity class AC_0). This is so because a relational model (M, O, e) of dimension 1 is in $\mathrm{HV}(1)$ if and only if e is *total*, that is, it satisfies the formula $\forall x \exists y R(x, y)$, where x ranges over M and y ranges over O.

The main result of this section is that $\mathrm{HV}(n)$ is NP-complete, for every $n \geq 2$. In fact, we obtain a complete picture for the complexity of the parameterized subclasses of $\mathrm{HV}(n)$ that also take into account the size of the domains O_i, $1 \leq i \leq n$, of possible outcomes. More precisely, for every $n \geq 2$ and every $k \geq 2$, we write $\mathrm{HV}(n)/k$ to denote the subclass of $\mathrm{HV}(n)$ consisting of relational models (M, O, e) in which, for every $i \leq n$, we have that $|O_i| \leq k$.

Theorem 4.1 *The following statements are true.*

1. $\mathrm{HV}(2)/2$ *is in* NLOGSPACE.
2. $\mathrm{HV}(2)/k$ *is* NP-*complete, for every* $k \geq 3$. *In particular,* $\mathrm{HV}(2)/3$ *is* NP-*complete.*
3. $\mathrm{HV}(n)/k$ *is* NP-*complete, for every* $n \geq 3$ *and every* $k \geq 2$. *In particular,* $\mathrm{HV}(3)/2$ *is* NP-*complete.*

Proof Sketch. We show that $\mathrm{HV}(2)/2$ is in NLOGSPACE by using Theorem 5 in [Blass and Gurevich, 1986], which, in effect, asserts that the model-checking problem for narrow Henkin quantifiers is in NLOGSPACE. By definition, a *narrow Henkin quantifier* is a Henkin quantifier of the form $\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix}$, where y_i, for $i = 1, 2$, depends only on x_i

and takes values from the set $\{0,1\}$. Over relational models $(M_1 \times M_2, O_1 \times O_2, e)$ of dimension 2 in which the sets O_1 and O_2 of possible outcomes are of size 2, the existential second-order formula

$$\exists f_1 \exists f_2((f_1(x_1) = y_1) \wedge (f_2(x_2) = y_2) \wedge$$
$$(\forall u_1, u_2) R(u_1, u_2, f_1(u_1), f_2(u_2)))$$

can be expressed using narrow Henkin quantifiers, hence its model-checking problem is in NLOGSPACE. Since NLGO-SPACE is closed under first-order quantification and Boolean operations, it follows that $\mathrm{HV}(2)/2$ is in NLOGSPACE.

We show that $\mathrm{HV}(2)/3$ is NP-complete by giving a logarithmic-space reduction of ROBUST GRAPH 3-COLORABILITY to $\mathrm{HV}(2)/3$, and then using Theorem 3.3. Specifically, given a graph $G = (V, E)$, we construct in logarithmic space a relation $e(G) \subseteq V \times V \times C \times C$, where $C = \{1, 2, 3\}$, consisting of the following quadruples: a quadruple (u, u, i, i), for every $u \in V$ and every $i \in C$; a quadruple (u, v, i, j), for every $u, v \in V$ such that $(u, v) \in E$, and for every $i, j \in C$ such that $i \neq j$; and a quadruple (u, v, i, j), for every $u, v \in V$ such that $u \neq v$, $(u, v) \notin E$, and for every $i, j \in C$.

We now claim G is robustly 3-colorable if and only if $(V \times V, C \times C, e(G))$ is in $\mathrm{HV}(2)/3$.

To prove the preceding claim, assume first that G is robustly 3-colorable; in particular, G is 3-colorable. We have to show that the relation $e(G)$ satisfies the formula $\Psi(2)$ that defines $HV(2)$. First, it is obvious from the construction of $e(G)$ that for all u and v in V, there are i and j in O such that $(u, v, i, j) \in e(G)$. Next, assume that $(u, v, i, j) \in e(G)$. We distinguish two cases. If $u = v$ or if $(u, v) \in E$, then, since G is 3-colorable, there is a 3-coloring $c : V \mapsto O$ of G such that $c(u) = i$ and $c(v) = j$ (to find such a 3-coloring of G, we may need to permute the colors in a given 3-coloring of G). We can then satisfy the formula $\Psi(2)$ by putting $f_1 = f_2 = c$. If u and v are two independent nodes, then, using the robust 3-colorability of G, we can find a 3-coloring $c : V \mapsto O$ of G such that $c(u) = i$ and $c(v) = j$. As before, we can satisfy the formula $\Psi(2)$ by putting $f_1 = f_2 = c$.

For the other direction, assume that the relation $e(G)$ satisfies the formula $\Psi(2)$. We have to show that the graph $G = (V, E)$ is robustly 3-colorable. Let x and z be two independent nodes in G and let i and j be two colors in $O = \{1, 2, 3\}$. We have to show that there is a 3-coloring c of G such that $c(x) = u$ and $c(z) = j$. By the construction of $e(G)$ and since $(x, z) \notin E$, we have that $(x, z, i, j) \in e(G)$. Since $e(G)$ satisfies $\Psi(2)$, we have that there are functions f_1 and f_2 such that $f_1(x) = i$, $f_2(z) = j$, and, for all u, v in V, it is the case that $(u, v, f_1(u), f_2(v)) \in e(G)$. By the construction of $e(G)$, we have that $f_1(w) = f_2(w)$, for every $w \in V$; this is so because if the first two coordinates in a quadruple in $e(G)$ are equal to each other, then the last two coordinates of that quadruple are equal to each other as well. Moreover, again by the construction of $e(G)$, we have that the function f_1 must be a 3-coloring of G; this is so because if the first two coordinates in a quadruple in $e(G)$ form an edge, then the last two coordinates must be different from each other, which means that $f_1(u) \neq f_1(v)$. Using an analogous construction, we show that $\mathrm{HV}(3)/2$ is NP-complete by giving a logarithmic-space reduction of ROBUST 3SAT to $\mathrm{HV}(2)/3$, and then using Theorem 3.4.

The remaining cases follow easily from the preceding two NP-hardness results. Details are given in the full paper. ∎

5 Conclusion and Future Work

We have discovered a surprising link between constraint satisfaction and a natural problem in quantum mechanics: deciding whether a given set of measurements can be explained through local hidden variables, and may thus be explained through classical physics. We settled an open question by showing that this problem is NP-complete, and we were able to delimit the precise boundaries of NP-completeness for it. This is, to the best of our knowledge, one of the very few results that pinpoint the (classical) computational complexity of a decision problem in quantum mechanics. The only other significant result of which we are aware is due to Pitowsky [Pitowsky, 1991], and concerns a totally different problem, namely, testing a vector for membership in a *correlation polytope*. The connection to quantum mechanics is that certain correlation polytopes are defined by Bell inequalities [Clauser *et al.*, 1969; Clauser and Horne, 1974].

Our results for local hidden-variable models were shown in the context of the possibilistic (non-probabilistic) relational quantum mechanical framework of [Abramsky, 2013]. It is worth commenting at this point on the significance of results that are couched in terms of the support tables of the probability distributions, rather than the probabilities themselves. As shown in [Abramsky, 2013], such results are *stronger* than those involving probabilities, thus no-go theorems at the possibilistic level imply those at the probabilistic level. The quantum foundations literature has many well-known examples of inequality-free or probability-free proofs of variants of Bell's theorem [Greenberger *et al.*, 1990; Mermin, 1990; Cabello, 2001; Hardy, 1993; Zimba and Penrose, 1993], which are essentially performed at the level of the support tables. At the same time, the relational level of description brings to the fore striking and surprising connections between ideas in quantum foundations, and a number of notions which have been well-studied in Computer Science; see e.g. [Abramsky, 2012].

As a main and novel tool in obtaining our results, we used the new general concept of robust CSP, and we studied two particular robust CSPs, namely ROBUST GRAPH 3-COLORABILITY and ROBUST 3SAT, that are directly relevant to the HV(n) problem of hidden-variable detection. We believe, however, that the concept of robust CSP is of interest in its own right; actually, we believe that by having studied two very specific robust CSPs here, we have only scratched the tip of an iceberg. In the near future, we plan to embark on a systematic study of robust CSPs aiming to identify general principles that govern robustness and its complexity.

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