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A Characterization of n-Player Strongly Monotone Scheduling Mechanisms*

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Abstract

Our work deals with the important problem of globally characterizing truthful mechanisms where players have multi-parameter, additive valuations, like *scheduling unrelated machines* or *additive combinatorial auctions*. Very few mechanisms are *known* for these settings and the question is: Can we prove that no other truthful mechanisms exist?

We characterize truthful mechanisms for n players and 2 tasks or items, as either *task-independent,* or a *player-grouping minimizer*, a new class of mechanisms we discover, which generalizes affine minimizers. We assume decisiveness, *strong monotonicity* and that the truthful payments[1] are continuous functions of players' bids.

1 Introduction

Using the power of *crowdsourcing* and *cloud computing* to compute complicated tasks that consist of multiple sub-tasks is a major challenge in multi-agent systems. When assigning the tasks to different agents/cloud providers, we have to provide the agents with the right incentives to truthfully report the times they need to complete the tasks, and execute the tasks that are assigned to them.

Combinatorial auctions constitute another important class of problems. Here multiple items are auctioned simultaneously, and we need to motivate the agents to report their true valuations for the items. Can we characterize all allocation algorithms that are truthful for these two settings?

Fifteen years ago Nisan and Ronen posed their famous – still open– question about the approximability of the optimum makespan in unrelated machine scheduling by *truthful scheduling mechanisms* [Nisan and Ronen, 2001]. In this strategic version of the unrelated machines problem, we are given n machines and m tasks, and the machines are owned by selfish agents, each of them holding the vector $t_i = (t_{ij})_{j=1}^m$ of running times (costs) on his machine i as private information. A scheduling mechanism consists of an *allocation algorithm*, and a *payment scheme* $(p_1, \ldots, p_n)$. Having received the *bid* vectors t'_i for the costs from the respective agents, the matrix t' is used as input of the allocation algorithm, and the payments to each agent are calculated according to the payment scheme. The *utility* of player i is then $p_i - cost_i$, where $cost_i$ is defined as the total running time of the received jobs on machine i, i.e., the finish time of the machine. Note that for each player, the costs incurred from the different tasks are additive. A similar problem to unrelated scheduling is that of *combinatorial auctions (CAs)* with *additive* bidder valuations: we get a model equivalent to additive CAs by assuming *negative* values of t_{ij}, and leaving the scheduling model unchanged otherwise.

We are interested in *truthful* mechanisms, where bidding $t'_i = t_i$ is a dominant strategy for every agent. It is well known that *weak monotonicity (WMON)* of the allocation function is necessary and sufficient for truthful implementability (see [Saks and Yu, 2005; Archer and Kleinberg, 2008]). Thus, the problem boils down to searching for monotone allocation algorithms for unrelated scheduling.

Nisan and Ronen conjectured that no mechanism can yield an approximation of the makespan by a factor smaller than $\min(n, m)$, the approximation ratio of the VCG mechanism, that gives each task to the player bidding the smallest cost for that task. The attempts to prove lower bounds based merely on monotonicity, have only given small constants as lower bounds [Christodoulou *et al.*, 2007; Koutsoupias and Vidali, 2007; Mu'alem and Schapira, 2007].[2] Narrowing the gap between the constant factor lower bound and the upper bound n became a paradigmatic problem of *algorithmic mechanism design*, and directed much attention to mechanisms with additive bidder valuations in general (see, eg., [Lavi, 2007]).

While monotonicity characterized truthful mechanisms well in the (single-parameter) *related machines* case (see [Epstein *et al.*, 2013] and references therein), it is much more difficult to exploit it in multi-parameter settings like unrelated scheduling. A different approach is to improve our understanding by investigating the *global structure* of WMON allocations. To this end we strive for *global, closed form* char-

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[1]The (normalized) payments are uniquely determined by the allocation function of the mechanism; thus the assumptions concern properties of the allocation.

[2]A lower bound n has been established adding the *anonymity* assumption [Ashlagi *et al.*, 2012].

acterizations. Even though it seems extremely hard to provide a complete characterization for the original problem, attempts to characterize restricted, in some way purified classes of WMON mechanisms prove to be very useful: they develop insight, while new types of allocations might be discovered along the way.

In this paper we assume *strong monotonicity (SMON)*, a condition that parallels Arrow's Independence of Irrelevant Alternatives (IIA) condition, and which is the strict version of the WMON property (see Definition 3).[3] We completely characterize SMON mechanisms for two tasks, and at the same time identify a new class of monotone allocations.

Related work. Characterizations by *weak monotonicity* [Myerson, 1981; Saks and Yu, 2005; Gui *et al.*, 2005; Archer and Kleinberg, 2008; Frongillo and Kash, 2012], and *cycle monotonicity* [Rochet, 1987] describe truthfully implementable allocations in a *local* fashion. *Complete* characterizations, of implementable allocations describe them in a *global* fashion. The most important result of this type is due to Roberts who showed that for *unrestricted* domains the only implementable social choice rules are affine maximizers [Roberts, 1979], a generalization of VCG mechanisms. However, the requirement of unrestricted valuations does not apply to most of the realistic setups with richer structure.

Characterizations for domains with high economic importance like combinatorial auctions or the scheduling domain seem very hard to obtain, even with additional restrictions. The characterizations we know for the case of n players, either concern domains where SMON can be assumed without loss of generality [Roberts, 1979; Papadimitriou *et al.*, 2008], or put additional restrictions [Lavi *et al.*, 2003; 2009; Dobzinski and Nisan, 2011; Christodoulou and Kovács, 2011]. The only *complete* characterization results for additive bidders in the multi-parameter setting are for *two players* [Dobzinski and Sundararajan, 2008; Christodoulou *et al.*, 2008]. These characterizations involve *affine minimizers*, or *threshold* mechanisms, or a combination of these.[4] We define these allocation rules next.

Definition 1 (affine minimizer). *An allocation function A is an* affine minimizer *if there exist positive multiplicative constants λ_i for each player i, and additive constants c_a one for each allocation a, such that for every input matrix $t = \{t_{ij}\}_{n\times m}$ the allocation $A(t) = \{a_{ij}\}_{n\times m}$ minimizes $\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i \cdot a_{ij} \cdot t_{ij} + c_a$ (where a_{ij} is 1 if player i gets task j and 0 otherwise).*

Threshold allocations are exactly those that admit additive payment functions over the received tasks/items [Vidali, 2009]. Restricted to SMON mechanisms, they coincide (assuming proper tie-breaking) with *task-independent* mechanisms, that allocate each task by an arbitrary monotone single item allocation. These single item mechanisms were characterized as *virtual utility maximizers* in [Mishra and Quadir, 2012].

[3] It is known [Mu'alem and Schapira, 2007] that SMON mechanims can only approximate the makespan by factor $\min(m, n)$.

[4] *Affine maximizers* become *affine minimizers* in scheduling.

Our contribution. 1. We identify a so far unknown monotone allocation rule generalizing affine minimizers, that we call *player-grouping minimizer*. A player-grouping minimizer partitions the set of players, and always allocates all tasks within one subset of the partition (called *group*) by some affine minimizer. Between any two groups of players, preferences are decided by minimizing arbitrary fixed increasing functions of the objective values of the groups:

Definition 2 (player-grouping minimizer). *Let $\{N_g\}_{g=1}^{r}$ be a partition of the set of players into r* groups, *with at least two players in each group. For each $1 \le g \le r$ let $\Phi_g : (-\infty, C_g) \to \mathbb{R}$ be an increasing continuous bijection*[5] *and A_g be an affine minimizer over the players of group g. Within each group the affine minimizer A_g decides, which players (would) receive the tasks in that group. For given bids t_i of the players in group g, let Opt_g denote the objective value of A_g.*[6] *Group s receives all the tasks, allocated according to A_s, if $\Phi_s(Opt_s) = \min_g \Phi_g(Opt_g)$ (assuming some consistent tie-breaking rule).*

Example 1. *The allocation for $n = 4$ and $m = 2$, that gives the tasks to the players who provide the minimum of the expressions $t_{11}+t_{12}$, $t_{11}+5t_{22}$, $5t_{21}+t_{12}$, $5t_{21}+5t_{22}$, and $(t_{31}+t_{32}+3)^3$, $(t_{31}+t_{42})^3$, $(t_{41}+t_{32}-1)^3$, $(t_{41}+t_{42})^3$ is a grouping minimizer. (If ties are broken by a fixed order of these eight possible allocations, then it is an SMON grouping minimizer.) Note that here players 1 and 2 form group N_1, in the respective affine minimizer $\lambda_2 = 5$ and the additive constants c_a are zero; the Φ_1 function for this group is the identity function $\Phi_1(x) = x$. Another group N_2 consists of players 3 and 4. The corresponding affine minimizer has multiplicative constants $\lambda = 1$, and additive constants 3, 0, -1 and 0 depending on the allocation within these two players; moreover, $\Phi_2(x) = x^3$.*

2. We characterize SMON mechanisms for two tasks or items as either task-independent mechanisms or player-grouping minimizers:

Theorem 1. *Every continuous decisive SMON mechanism for allocating two tasks or items, with additive bidder valuations and $t_{ij} \in \mathbb{R}$, is either a task-independent mechanism, or a grouping minimizer (if the grouping minimizer is onto*[7]*, then it is an affine minimizer).*

Since monotone allocations for two players are essentially SMON (with an appropriate tie-breaking rule), for two tasks our result generalizes the 2-player characterization in [Christodoulou *et al.*, 2008]. Moreover, grouping minimizers are very similar to *virtual utility maximizers* for a single item [Mishra and Quadir, 2012].

3. We derive a key lemma (Lemma 2) that turns out to be of 'universal' use for the SMON problem. For fixed bids t_{-i}, of all other players, the allocations to a single player i depending on his own 2-dimensional bid vector t_i, partition the

[5] The $C_g \in \mathbb{R} \cup \{+\infty\}$ with C_g being $+\infty$ for at least one g; this is needed for the tasks to be always allocated.

[6] $Opt_g = \min_{a^g} \sum_i \sum_j \lambda_i t_{ij} a^g_{ij} + c_{a^g}$, where allocations a^g give all tasks to group g.

[7] I.e. every allocation occurs for at least one input t.

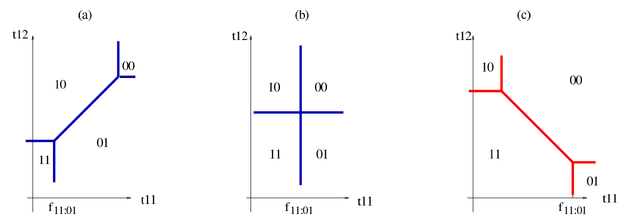

Figure 1: The allocations to a single player depending on his own 2-dimensional bid vector, partitions the bid-space according to one of these shapes.

bid-space according to one of the shapes in Figure 1 (due to the WMON property). The positions of the boundary lines in these figures correspond to the (differences of the) payments to player i for the different allocations. We investigate these boundary positions (i.e. the truthful payments), as functions of the other players' bids.[8] For SMON mechanisms, Lemma 2 implies the linearity of these payment functions in 'most' cases. We prove that linearity of all boundary functions results in the mechanism being an affine minimizer. The only exceptions where linearity needs not hold, imply either a task independent mechanism or a boundary between different player-groups in a grouping minimizer (see Figures 2 and 3).

Our technical assumptions are *decisiveness, continuity* of the payment functions, and that the costs t_{ij} can take arbitrary real values.[9] For a discussion on these, and some examples of degenerate allocation rules, see the full version [10].

Preliminaries. The following notation and observations apply to any number of tasks. Since we treat the two-tasks case in the paper, we will illustrate these notions for $m = 2$. For a more detailed treatment see, e.g., [Christodoulou *et al.*, 2008; Vidali, 2009].

An *allocation matrix* is $a = (a_1, a_2, \ldots, a_n)$, where a_k is the binary *allocation vector* of player k. We also use α, $\alpha' \ldots$ etc. to denote m-dimensional allocation vectors (for two tasks, $\alpha, a_k \in \{(00), (01), (10), (11)\}$). For two tasks, a^{ik} denotes the allocation giving the first task to player i and the second to player k (mind the difference to a_{ij}, which is a single bit). The bid matrix of all players except for player k, is denoted by t_{-k}, whereas t_{-ik} denotes the bid matrix of all players except for players i and k. For fixed t_{-k}, the *allocation regions* $R^k_\alpha = \{t_k \mid a_k(t_k, t_{-k}) = \alpha\} \subset \mathbb{R}^m$ for all possible α, partition the bid space $\mathbb{R}^m$ of player k in 2^m parts.

Definition 3. *An allocation function A satisfies SMON (WMON) if $A(t_k, t_{-k}) = a$, and $A(t'_k, t_{-k}) = a'$, where $a \neq a'$, imply $(a_k - a'_k)(t_k - t'_k) < 0$ (≤ 0).*

For WMON allocations the allocation regions of any player must have a special geometric shape: the *boundary* between any two regions R^k_α and $R^k_{\alpha'}$ (if it exists) is part of the hyperplane $(\alpha - \alpha') \cdot t_k = f^k_{\alpha:\alpha'}$, where the functions $f^k_{\alpha:\alpha'}$, that determine these boundary positions are defined as follows: In a truthful mechanism, the payment of player k depends on t_{-k} and on a_k. Let $p^k_{a_k}(t_{-k})$ denote this payment. Then

$$f^k_{\alpha:\alpha'} = p^k_\alpha(t_{-k}) - p^k_{\alpha'}(t_{-k}).$$

In general, for some t_{-k} some of the allocation areas R_α might be empty. We make the assumption that the allocation figures are complete (all the 2^m regions are always nonempty), i.e., the allocation is *decisive*:

Definition 4. *An allocation function A is* decisive*, if every player k, for every fixed bids t_{-k} of the other players and every particular allocation $\alpha \in \{0,1\}^m$ has bids t_k so that A allocates him exactly the items in α.*

Thus, for WMON allocations of two tasks, the allocation of player k as a function of (t_{k1}, t_{k2}) has a geometrical representation of one of three possible shapes (see Figure 1). For given fixed t_{-k}, we call this geometric representation the *(allocation) figure of player k.*

In Section 2.1 we investigate how the positions $f^k_{\alpha:\alpha'}$ of a player's boundaries change as a function of t_{-k}. We will show that this change is linear in t_{-k} with only a few exceptions. For fixed t_{-ik} the boundary $f^k_{\alpha:\alpha'}$ is a function of the bid t_i. We assume the continuity of $f^k_{\alpha:\alpha'}(t_i)$ for any fixed t_{-ik}. Most of the time, w.l.o.g. we consider the figure and boundaries of the first player. In this case, for $k = 1$, we omit the superscript in f^k, R^k, etc.

In the rest of the section we summarize further implications of the SMON property, the strict version of WMON. See the full version for omitted proofs. If an allocation rule $A(t)$ is SMON, then the allocation of *all* players is constant in the interior of any region R^k_α (otherwise, changing t_k would result in changing a to a', with $a_k = a'_k$, contradicting SMON). We denote by $f^k_{a:a'}(t_{-k})$ the boundary between allocations a, a' (these describe the allocation of *all* players, not just of player k) for every t_{-k} *for which such a boundary exists* in the figure of player k. Next we state a crucial elementary property of continuous SMON mechanisms:

($\star$) *For fixed bids of the other players, the boundary $f^k_{a:a'}$ in the allocation figure of player k, considered as a function of the bids t_i of any particular player $i \neq k$ depends only on $(a'_i - a_i) \cdot t_i$ ($= \sum_{j=1}^m (a'_{ij} - a_{ij}) t_{ij}$), by some strictly increasing continuous function. If $a'_i = a_i$ then the boundary position is independent of t_i.*

For example, let $a = (11, 00, 00)$, and $a' = (01, 10, 00)$, and t_3 be fixed; then $f_{a:a'}(t_2) = \varphi(t_{21})$, for a nondecreasing real function φ. Similarly, if $a = (10, 01, 00)$ and $a' = (01, 10, 00)$, then $f_{a:a'} = \psi(t_{21} - t_{22})$ for some nondecreasing function ψ. The property ($\star$) is implied by Lemma 1, and Observation 1 below:

Lemma 1. *[increasing boundaries] Let t_{-1i} be fixed, and $\mathcal{G} \subset \mathbb{R}^m$ a connected set. Assume that the boundary $f_{a:a'}$ exists for all $t_i \in \mathcal{G}$. For every $t_i, \bar{t}_i \in \mathcal{G}$ it holds that if $(a'_i - a_i) \cdot t_i < (a'_i - a_i) \cdot \bar{t}_i$, then $f_{a:a'}(t_i) \leq f_{a:a'}(\bar{t}_i)$.*

[8] Truthful normalized payments are uniquely determined by the allocation function and by t_{-i}.

[9] Note that similar assumptions are made in [Dobzinski and Nisan, 2011; Dobzinski and Sundararajan, 2008; Christodoulou *et al.*, 2008; Mishra and Quadir, 2012].

[10] Full version of the paper: https://infotomb.com//eeevx

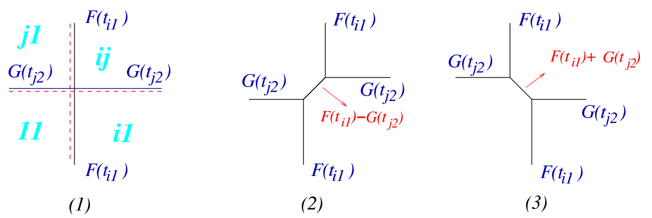

Figure 2: Types of quasi-independent allocation figures of player 1. The letters i and j show which player receives the tasks in the regions (here $j = i$ is allowed). Only in type (1) can F or G be locally nonlinear.

Corollary 1. *Let S be a subset of the tasks. If $a_{ij} = a'_{ij}$ for all $j \in S$, then $f_{a:a'}(t_i) = f_{a:a'}(\bar{t}_i)$ whenever t_i and $\bar{t}_i$ differ only on tasks in S.*

For example, let $a = (11, 00, 00)$, and $a' = (01, 10, 00)$. By the lemma (for $i = 2$), $f_{11:01}$ is increasing in t_{21}. The corollary says, that $f_{11:01}$ is a function of *only* t_{21}. S consists of task $j = 2$, because the allocation of this task to player 2 remains the same $a_{22} = a'_{22} = 0$. Consider t_2 and $\bar{t}_2$ bids such that $t_{21} = \bar{t}_{21}$. If $f_{11:01}(t_2) < f_{11:01}(t'_2)$ were the case, then by the lemma, $f_{11:01}$ would have a jump of at least $f_{11:01}(t'_2) - f_{11:01}(t'_2)$ in t_{21}. However, $f_{11:01}(t_{21})$ is continuous by assumption. We conclude the subsection with a couple of simple observations that hold in the case of continuous boundary functions.

Observation 1. *(a) [boundary points] Let $t_{-1} = (t_i, t_{-1i})$. If t_1 is a point on the boundary $f_{a:a'}(t_{-1})$, where the allocations a_i and a'_i are different, then for $t_{-i} = (t_1, t_{-1i})$ the bid t_i is a point on the boundary $f^i_{a:a'}(t_{-i})$.*

(b) [inverse boundaries] Let t_{-1i} be fixed. If for some monotone continuous univariate function $f_{a:a'}(t_i) = \varphi((a'_i - a_i) \cdot t_i)$, then $f^i_{a':a}(t_1) = \varphi^{-1}((a_1 - a'_1) \cdot t_1)$.

(c) [strictly increasing boundaries] Let t_{-1i} be fixed, and $\mathcal{G} \subset \mathbb{R}^m$ a connected set. Assume that the boundary $f_{a:a'}$ exists for all $t_i \in \mathcal{G}$. The function $f_{a:a'}(t_i)$ is strictly increasing in $(a'_i - a_i) \cdot t_i$ over $t_i \in \mathcal{G}$.

2 Characterization

In this section we sketch the proof of Theorem 1. The full version contains the detailed proofs. Figures 2 and 3 show the possible allocation figures of player 1, w.r.t. the dependence of boundaries on other players' bids. The player indices marking the four regions indicate the players who get the tasks in the respective region. As an example, consider Figure 2 (1). Here player i gets task 1 in regions R_{01} and R_{00}; and player j gets task 2 in R_{00} and R_{10}. By property $(\star)$, the position $f_{11:01}$ of the boundary between R_{11} and R_{01} is given by an increasing function of t_{i1}, that we denote by $F(t_{i1})$. Similarly, the horizontal boundary position is determined by an increasing function $G(t_{j2})$. If the figure has a shape like in Figure 2 (2), then the slanted boundary has the equation $t_{11} - t_{12} = F(t_{i1}) - G(t_{j2})$.

In general, in every allocation figure, task 1 is either given to the same player in R_{01} and in R_{00}, or to two different players, and similarly for task 2. Accordingly, the boundaries where task 1 changes owner are either only the vertical boundaries (like in types (1) to (6)), or the vertical boundaries and $f_{01:00}$ where two other players exchange the task among each other. Combining all possibilities for both tasks, we obtain the depicted cases.

Definition 5. *We call the allocations in an allocation figure* quasi-independent*, if the same player receives task 1 in R_{01} and in R_{00}, and the same (possibly other) player receives task 2 in R_{10} and in R_{00}.*

Figures 2 and 3 show all possible *quasi-independent* and *non quasi-independent* allocations, respectively, up to symmetry. Observe that for a particular boundary the $(\star)$ property only implies that it(s position) is a multivariate function, monotone in each variable. (E.g., in case (6) in Figure 3 $f_{10:00}$ is some function $f(t_{i1}, t_{j2}, t_{k2})$.) However, whenever different *other* boundaries depend on these different variables, it must be the case that the multivariate function is a so called *additively separable* function as appears in the figure. For instance, in (6), if only t_{i1} is increased (locally), then only the two vertical boundaries move, and $f()$ as a function of t_{i1} is necessarily of the form $F(t_{i1}) + C(t_{j2}, t_{k2})$ for some function F; by similar considerations, G and H functions exist s.t. $f = F(t_{i1}) - G(t_{j2}) + H(t_{k2}) + C$.

2.1 Local linearity results

The main theorems of this subsection show the linearity of different boundary functions $f_{\alpha:\alpha'}(t_i)$. Since task-independent allocations, that are in general nonlinear, have "crossing" figures (see Figure 1 (b)), it will be important to distinguish two types of allocation figures:

Definition 6. *For fixed t_{-k}, the allocation figure of k is* crossing*, if $f^k_{11:01} = f^k_{10:00}$, and $f^k_{11:10} = f^k_{01:00}$. Otherwise we call the figure* non-crossing.

Mechanisms with only crossing allocation figures are task-independent. The boundary functions $f_{\alpha:\alpha'}$ of such mechanisms need not be linear, as they indicate the critical values for getting the task in arbitrary monotone single-task allocations. In what follows, we show that if the mechanism is onto, the converse also holds: if the mechanism has an allocation figure (for some k and t_{-k}) that is non-crossing, then *all* boundary functions of the mechanism must be linear. However, if the mechanism is not onto, that is, certain allocations a^{ij} never occur, then even complete (decisive) non-crossing allocation figures might change by non-linear functions. The resulting mechanisms, which we named (player) grouping minimizers, constitute a generalization of affine minimizers. The main reason for enforced linearity of the boundaries is given in the following basic lemma. Namely, whenever these boundaries are additive separable functions of at least two variables, and so are their inverse boundaries (on another player's figure), we encounter a situation that fulfils the conditions of the lemma. For an illustration see Figure 4: α and φ are (roughly) inverse functions. The conditions of the lemma imply that if the functions β and ψ are monotone, then for *any* small enough Δ, the second curve is a parallel translation of the first one *both* in vertical and in horizontal direction. This

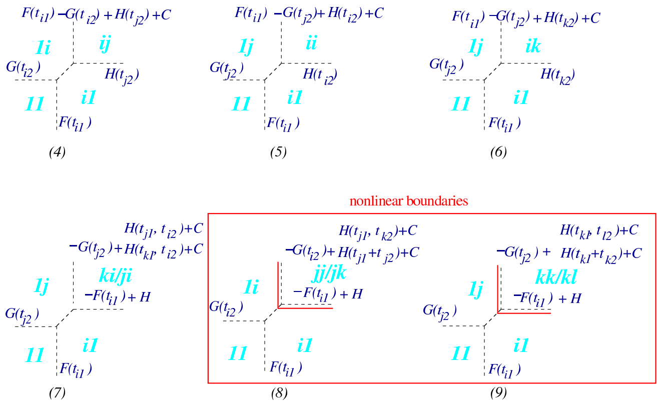

Figure 3: *Non*-quasi-independent allocation figures of player 1. Here the shapes of the figures are arbitrary. The letters indicate the players who receive the two tasks in the respective regions; $i, j, k, \ell \neq 1$ denote different players. F and G are linear, and only in (8) and (9) can H be locally nonlinear.

is possible only if the curve is a straight line (α and φ are linear).

Lemma 2. *Assume that for strictly monotone continuous real functions α, β, φ, and ψ, for every $(x, y, z) \in \mathcal{G}$ for an open set $\mathcal{G} \subseteq \mathbb{R}^3$, it holds that $(y = \alpha(x) + \beta(z)) \Leftrightarrow (x = \varphi(y) + \psi(z).)$ Moreover, we assume that an open neighborhood of (x, z) pairs exists for which $(x, \alpha(x) + \beta(z), z) \in \mathcal{G}$. Then*

- *(a) α and φ are linear functions;*
- *(b) α and φ are both increasing or both decreasing, and exactly one of β and ψ is increasing;*
- *(c) if β and ψ are also linear functions with slopes λ_β and λ_ψ, then the slopes λ_a of α and λ_φ of φ satisfy $\lambda_\alpha = -\frac{\lambda_\beta}{\lambda_\psi}$, and $\lambda_\varphi = -\frac{\lambda_\psi}{\lambda_\beta}$.*

We call a real function $\varphi : \mathbb{R} \to \mathbb{R}$ *locally linear in point* x, if a $\delta > 0$ exists such that φ is linear in the interval $(x - \delta, x + \delta)$; φ is *locally non-linear in* x, if φ is not a linear function in any open neighborhood of x. With the help of Lemma 2, in Theorems 2 and 3 we show the local linearity of the boundary functions of region R_{11} (F and G) and also of R_{00} (H) in most cases. The exceptional cases, when linearity in general does not hold, are case (1) for functions F and G, which is the allocation of a task-independent mechanism, and cases (8) and (9) for H which are typical allocations of grouping minimizers. Notice that (8), (9) are exactly the allocation types, where in region R_{00} the tasks are given only to players who do not get a job in any other region.

Theorem 2. *If an allocation figure of player 1 is constant (i.e. the allocations of* all *players in each region are constant) for some connected open set $\mathcal{G} \subset \mathbb{R}^{(n-1)\times 2}$ of t_{-1} values, then the function F (resp. G) is locally linear in every point t_{i1} (resp. t_{i2} or t_{j2}) in the projection of $\mathcal{G}$, or case (1) of Figure 2 holds over $\mathcal{G}$.*

Proof. We assume an open rectangle $\mathcal{T} \subset \mathcal{G}$ where the conditions hold. Then the result extends to an arbitrary open set $\mathcal{G}$. As an illustration we give a detailed proof of the theorem in case (2) of Figure 2 as example for how to apply Lemma 2 for the linearity of boundary functions.

In case (2), the line of the (slanted) boundary $f_{10:01}$ is given by the equation $t_{11} - t_{12} = F(t_{i1}) - G(t_{j2})$, and it is a boundary between allocations a^{1j} and a^{i1}. If we fix a boundary point t_1, then for this t_1 the figure of player i has a boundary $f^i_{10:00} = f^i_{a^{i1}:a^{1j}}$, and t_i is a point of this boundary by Observation 1 (b). Moreover, the boundary position for i is a function of the form $t_{i1} = -G^i(t_{12}) + H^i(t_{11}, t_{j2})$, for some monotone increasing functions G^i and H^i. For fixed $\bar{t}_{j2}$ holds

$$t_{12} = -F(t_{i1}) + t_{11} + G(\bar{t}_{j2})$$

$$\Updownarrow$$

$$t_{i1} = -G^i(t_{12}) + H^i(t_{11}, \bar{t}_{j2}),$$

and this is valid in some neighborhood of (t_{11}, t_{12}, t_{i1}), since it is valid on $t_{-1} \in \mathcal{T}$.

We set $y := t_{12}$, $x := t_{i1}$, $z := t_{11}$ and choose the strictly monotone, continuous functions $-F(x)$, $z + G(\bar{t}_{j2})$, $-G^i(y)$, and $H^i(z, \bar{t}_{j2})$, as $\alpha()$, $\beta()$, $\varphi()$, and $\psi(z)$, respectively. Applying the lemma yields that F is locally linear. For proving the linearity of G we must fix t_{i1} instead of t_{j2}. □

Theorem 3. *If an allocation figure of player 1 is constant for some connected open set $\mathcal{G} \subset \mathbb{R}^{(n-1)\times 2}$ of t_{-1} values, then the function $H()$ (when defined) is locally linear in each variable of H, in every point in the corresponding projection of $\mathcal{G}$, or else one of the cases (8) or (9) in Figure 3 holds.*

2.2 Global linearity results

Next, we focus on the functions F and G. We prove that local linearity extends to global linearity of these functions. More precisely, the functions $F(t_{i1})$ for all $i \neq 1$ are uniquely determined with domain $\mathbb{R}$ (unless the allocation a^{i1} does not occur at all), and the same holds for $G(t_{i2})$. Moreover, they prove to be linear, unless the mechanism is task-independent. It turns out that the requirement of Theorem 2 to have constant allocations in *all the four* regions is not necessary, concerning the functions F and G. The next observation extends Corollary 1 to the case when other players' bids are not fixed.

Observation 2. *Let $t_{-1} = (t_2, t_3, \ldots, t_n)$ and $t'_{-1} = (t'_2, t'_3, \ldots, t'_n)$, so that $t_{i1} = t'_{i1}$. If both boundaries $f_{a^{11}:a^{i1}}(t_{-1})$ and $f_{a^{11}:a^{i1}}(t'_{-1})$ exist, then they are equal.*

From now on, we can use $F^1_i(t_{i1})$ and $G^1_i(t_{i2})$ (in general $F^k_i(t_{i1})$ and $G^k_i(t_{i2})$) to denote the over their whole domain uniquely determined boundary functions of a mechanism. We will omit the superscript 1 whenever we consider the allocation of player 1. We omit the subscript i, when it is clear from the argument t_{i1} or t_{i2}. We prove that if the mechanism is not task-independent, these functions are linear and have domain $\mathbb{R}$ or $\emptyset$. The proof uses the following lemmas:

Lemma 3. *If all allocation figures of a single player are crossing, then all allocation figures of all players are crossing, and therefore the mechanism is task-independent.*

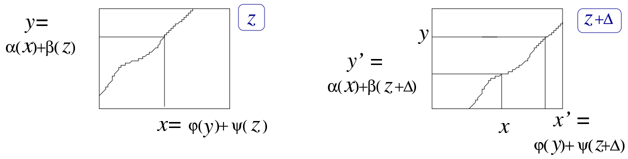

Figure 4: Illustration to Lemma 2.

Lemma 4. *Let $\bar{t}_{i1}$ be an interior point of the domain of $F(t_{i1})$. If F is locally non-linear in $\bar{t}_{i1}$, then the mechanism is task-independent.*

Theorem 4. *If the SMON mechanism is not task-independent, then in the allocation figures of player* 1,

(a) every F_i and every G_i function is linear (when defined);

(b) for any fixed t_{-1} the boundaries of the region R_{11} are $f_{11:01} = \min_{i\neq 1} F_i(t_{i1})$ and $f_{11:10} = \min_{i\neq 1} G_i(t_{i2})$;

(c) the domain of every F_i and every G_i is $\mathbb{R}$ or $\emptyset$;

(d) if a^{i1} exists then for any fixed t_{-1}, the t_{j1} values of the players $j \neq 1, i$ can be increased so that in R_{01} the allocation becomes a^{i1}; in turn, for any fixed t_{-1}, the t_{i1} can be decreased so that in R_{01} the allocation becomes a^{i1} (and similarly for t_{i2}).

The same holds for the allocation figures of every player k.

2.3 Non task-independent mechanisms

This subsection completes the characterization. We define the slopes of different boundary functions and settle the connection between them. For ease of exposition, henceforth we assume that the SMON mechanism we consider is *not* task-independent (more precisely, not threshold), and therefore has at least one non-crossing allocation figure by Lemma 4. By Theorem 4, the $F^k_i(t_{i1})$ and $G^k_i(t_{i2})$ functions are linear over the whole real domain. We introduce the notation $\lambda^k_{i,horiz}$ and $\lambda^k_{i,vert}$, respectively, for their slopes:

Notation 1. *If the allocation a^{ik} ever occurs in the mechanism, then we denote the slope of the linear function $F^k_i(t_{i1})$ by $\lambda^k_{i,horiz}$; if the allocation a^{ki} occurs, we denote the slope of the function $G^k_i(t_{i2})$ by $\lambda^k_{i,vert}$.*

Later we will prove that for non task-independent allocations $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$ must hold. Before showing this, it will be useful to first elaborate on the H functions (see Figure 3). These will turn out to be linear, unless the players are partitioned into isolated groups that never share the jobs (in particular, H is always linear in cases (4)–(7), but not necessarily in cases (8) and (9)). Note also that such H functions never occur in task-independent allocations. We treat the allocation figures of player 1, and first examine the dependence of H functions on the bids of players i, with whom player 1 sometimes shares the jobs, i.e., either of a^{i1} or a^{1i} occurs as allocation. This case also serves as the base case of the induction proof of Theorem 6.

Theorem 5. *Assume that the allocation a^{i1} occurs in the mechanism. Then whenever a boundary function $H()$ of the region R_{00} depends on t_{i1}, or t_{i2}, or on $t_{i1} + t_{i2}$, this dependence is linear with slope $\lambda^1_{i,horiz}$. Analogously, if the allocation a^{1i} exists, then for any of these arguments the function has slope $\lambda^1_{i,vert}$.*

Corollary 2. *If for some open set of t_{-k} values, the allocation to player k is of type (4)–(9) so that H depends on t_{i1} or t_{i2} or on $t_{i1}+t_{i2}$, then $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$ (if both are defined).*

Lemma 5. *For any SMON allocation $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$ (when defined), unless the allocation is task-independent.*

Next we define a partition of the players, such that restricted to any set of the partition, the mechanism is an affine minimizer.

Definition 7. *We define the* player-graph *with the set of players $[n]$ as vertices: let players i and j be connected by an edge if the allocation a^{ij} occurs in the mechanism. The players of the same connected component are called a* group.

For *neighboring* players i and k in the player-graph by Lemma 5 we can define $\lambda^k_i = \lambda^k_{i,horiz} = \lambda^k_{i,vert}$. Observe that for these λ^k_i values $\lambda^k_i = 1/\lambda^i_k$ is obvious by Observation 1 (b).

Lemma 6. *Assume that the mechanism is not task-independent and i, j, k is a triangle in the player-graph. Then $\lambda^k_i = \lambda^k_j \cdot \lambda^j_i$.*

The next theorem shows that the constant slopes λ^k_i can be defined for *any* pair of players within the same group.

Theorem 6. *Assume that the mechanism is not task-independent. For any two players i and k of the same group there exist constants $\lambda^k_i = 1/\lambda^i_k$ such that in every allocation figure of player k where the functions F^k_i, G^k_i appear or H depends on t_i, they depend linearly on t_i with slope λ^k_i.*

Observation 3. *If the mechanism is not task-independent, then for an arbitrary path $i_0, i_1, i_2, \ldots, i_t$ in the player-graph it holds that $\lambda^{i_0}_{i_t} = \lambda^{i_0}_{i_1} \cdot \lambda^{i_1}_{i_2} \cdot \ldots \cdot \lambda^{i_{t-1}}_{i_t}$.*

Corollary 3. *For any three players i, j, k of the same group, $\lambda^k_i = \lambda^k_j \cdot \lambda^j_i$.*

Due to the uniqueness and transitivity of the λ values in non task-independent allocations, we can choose $\lambda_i = \lambda^1_i$ to be the multiplicative weight of a player i in the group of player 1. (We can choose a representative player in each group to play the role of player 1.) In order to determine the affine minimizer within a connected group of players completely, we need additive constants for each allocation to these players, which we define next.

Lemma 7. *Consider w.l.o.g. the connected group of player 1. There exist constants c^{ii} and c^{ij} for arbitrary members i and j of the group, such that within this group the mechanism allocates according to an affine minimizer with multiplicative constants λ^1_i and additive constants c^{ii} and c^{ij} (given that this group of players receives the tasks).*

Clearly, for every connected group g we can choose a representative player k_g, and determine the multiplicative and additive constants $\lambda^{k_g}_i$, c^{ij} and c^{ii} accordingly. We assume that $k_1 = 1$. It remains to elaborate on the rules of the allocation between two different groups of players. Let Opt_g be the optimum value of group g, that is $Opt_g = \min_{i,j \in g}(\lambda_i^{k_g} t_{i1} + \lambda_j^{k_g} t_{j2} + c^{ij})$. The characterization result (Theorem 1) is now an immediate corollary of Lemma 8.

Lemma 8. *For every connected group g of players, there exists an increasing continuous function Φ_g with domain $(-\infty, C_g)$ and $\lim_{-\infty} \Phi_g = -\infty$ s.t. (optimal players of) group g with minimum value of $\Phi_g(Opt_g)$ receive the tasks.*

3 Discussion

The most immediate open question is, whether our result extends to SMON mechanisms with many tasks. The characterization can be generalized if we make the following strong assumption: for any two tasks u and v, if two players (ever) share these tasks in the 2-dimensional projection allocation obtained by fixing the bids for every other task to some t^{-uv} matrix, then they also share the tasks given any other fixed values t'^{-uv}. This property holds for all the mechanisms that we know, but it is not clear why it can be assumed right away. We conjecture that every continuous decisive SMON mechanism for allocating m items, with additive bidder valuations and $t_{ij} \in \mathbb{R}$, is the product of grouping minimizers and a task-independent mechanism.

Another question is, if Lemma 2 can in any way be helpful to gain insight into the nature of WMON allocations. The lemma suggests the intuition, that in order to obtain new types of WMON allocation rules, we need to find ones where the boundary functions are *not* additive separable functions of the relevant variables.[11]

[11]E.g., are not of the form $F(t_{i1}) + G(t_{j2})$, like in Figure 2 (3), but of some other $H(t_{i1}, t_{j2})$.

References

[Archer and Kleinberg, 2008] A. Archer and R. Kleinberg. Truthful germs are contagious: a local to global characterization of truthfulness. In *EC*, pages 21–30, 2008.

[Ashlagi *et al.*, 2012] Itai Ashlagi, Shahar Dobzinski, and Ron Lavi. Optimal lower bounds for anonymous scheduling mechanisms. *Mathematics of Operations Research*, 37(2), 2012.

[Christodoulou and Kovács, 2011] G. Christodoulou and A. Kovács. A global characterization of envy-free truthful scheduling of two tasks. In *WINE*, pages 84–96, 2011.

[Christodoulou *et al.*, 2007] G. Christodoulou, E. Koutsoupias, and A. Vidali. A lower bound for scheduling mechanisms. In *Proc. 18th SIAM Symp. on Disc. Algs. (SODA)*, 2007.

[Christodoulou *et al.*, 2008] G. Christodoulou, E. Koutsoupias, and A. Vidali. A characterization of 2-player mechanisms for scheduling. In *ESA*, pages 297–307, 2008.

[Dobzinski and Nisan, 2011] Shahar Dobzinski and Noam Nisan. Multi-unit auctions: beyond roberts. In *Proceedings 12th ACM Conference on Electronic Commerce (EC)*, pages 233–242, 2011.

[Dobzinski and Sundararajan, 2008] S. Dobzinski and M. Sundararajan. On characterizations of truthful mechanisms for combinatorial auctions and scheduling. In *EC*, pages 38–47, 2008.

[Epstein *et al.*, 2013] L. Epstein, A. Levin, and R. van Stee. A unified approach to truthful scheduling on related machines. In *SODA*, pages 1243–1252, 2013.

[Frongillo and Kash, 2012] Rafael M. Frongillo and Ian A. Kash. General truthfulness characterizations via convex analysis. *CoRR*, abs/1211.3043, 2012.

[Gui *et al.*, 2005] H. Gui, R. Müller, and R. V. Vohra. Dominant strategy mechanisms with multidimensional types. In *Computing and Markets*, 2005.

[Koutsoupias and Vidali, 2007] E. Koutsoupias and A. Vidali. A lower bound of 1+phi for truthful scheduling mechanisms. pages 454–464, Krumlov, Czech Republic, 26-31 August 2007.

[Lavi *et al.*, 2003] R. Lavi, A. Mu'alem, and N. Nisan. Towards a characterization of truthful combinatorial auctions. In *FOCS*, pages 574–583, 2003.

[Lavi *et al.*, 2009] Ron Lavi, Ahuva Mu'alem, and Noam Nisan. Two simplified proofs for roberts theorem. *Social Choice and Welfare*, 32(3):407–423, 2009.

[Lavi, 2007] Ron Lavi. Searching for the possibility: impossibility border of truthful mechanism design. *SIGecom Exchanges*, 7(1):25–29, 2007.

[Mishra and Quadir, 2012] Debasis Mishra and Abdul Quadir. Deterministic single object auctions with private values. Technical report, 2012. http://EconPapers.repec.org/RePEc:ind:isipdp:12-06.

[Mu'alem and Schapira, 2007] A. Mu'alem and M. Schapira. Setting lower bounds on truthfulness. In *SODA*, pages 1143–1152, 2007.

[Myerson, 1981] R. Myerson. Optimal auction design. *Mathematics of Oper. Res.*, 6:58–73, 1981.

[Nisan and Ronen, 2001] Noam Nisan and Amir Ronen. Algorithmic mechanism design. *Games and Economic Behaviour*, 35:166–196, 2001.

[Papadimitriou *et al.*, 2008] Christos H. Papadimitriou, Michael Schapira, and Yaron Singer. On the hardness of being truthful. In *FOCS*, pages 250–259, 2008.

[Roberts, 1979] K. Roberts. The characterization of implementable choice rules. In J.J. Laffont, editor, *Aggregation and revelation of preferences*. Studies in Public Economics, Amsterdam, North Holland, 1979.

[Rochet, 1987] J. Rochet. A necessary and sufficient condition for rationalizability in a quasilinear context. *Journal of Mathematical Economics*, 16:191–200, 1987.

[Saks and Yu, 2005] M. E. Saks and L. Yu. Weak monotonicity suffices for truthfulness on convex domains. In *EC*, pages 286–293, 2005.

[Vidali, 2009] A. Vidali. The geometry of truthfulness. In *WINE*, pages 340–350, 2009.