Robust Survey Aggregation with Student-t Distribution and Sparse Representation

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Abstract

Most existing survey aggregation methods assume that the sample data follow Gaussian distribution. However, these methods are sensitive to outliers, due to the thin-tailed property of Gaussian distribution. To address this issue, we propose a robust survey aggregation method based on Student-t distribution and sparse representation. Specifically, we assume that the samples follow Student-t distribution, instead of the common Gaussian distribution. Due to the Student-t distribution, our method is robust to outliers, which can be explained from both Bayesian point of view and non-Bayesian point of view. In addition, inspired by James-Stain estimator (JS) and Compressive Averaging (CAvg), we propose to sparsely represent the global mean vector by an adaptive basis comprising both data-specific basis and combined generic basis. Theoretically, we prove that JS and CAvg are special cases of our method. Extensive experiments demonstrate that our proposed method achieves significant improvement over the state-of-the-art methods on both synthetic and real datasets.

1 Introduction

Surveys are a common way to investigate the characteristics, behaviors, or opinions of target population. Nowadays, surveys have been widely used in business [Arnot *et al.*, 2006], data mining [Benton *et al.*, 2016], *etc*. Survey-based companies (*e.g.*, Nielsen and Kantar) earn billions of dollars per year in over 100 countries. However, conducting a survey is very expensive. For example, each short phone-call interview of a survey is charged on the order of \$20-\$30 in the United States, and thus even small-scale surveys can easily cost up to tens of thousands of dollars [Chen and Huang, 2006]. In addition, survey data often contain noise or outliers [Bamnett and Lewis, 1994], which have considerable effects on survey aggregation. Such high cost of conducting surveys and the issue of outliers in the survey data motivate the study of advanced survey aggregation methods that could utilize the available samples well while being robust to the outliers.

The fundamental problem in aggregating survey data is to estimate the true mean response of each group of interest. A natural idea is to estimate the true means with sample means, that is, simply averaging all the samples collected within each group, which is usually referred to as sample Averaging (Avg). In fact, if the number of groups is not larger than two, Avg is uniformly better than any other method in terms of mean square error (MSE). However, if the number of groups is larger than two, it has been proved that Avg is not necessarily the best according to the Stein's paradox [Stein, 1956]. James and Stein [1961] constructed James-Stein estimator (JS) that recovers each single mean using information from all groups and outperforms Avg. Efron and Morris [1972] cast JS into an empirical Bayesian framework and showed that it is a result of maximum posterior with a prior regularization. Recently, Shi *et al.* [2016] proposed Compressive Averaging (CAvg) to improve JS with the compressive sensing technique [Candès and Wakin, 2008]. Feldman *et al.* [2012] proposed Multi-Task Averaging (MTAvg) by adding a penalty term to improve Avg, from the perspective of multi-task learning.

For the convenience of analysis, most existing survey aggregation methods assume that the samples are from Gaussian distribution. However, these methods do not work well on the data with outliers or heavy tails, due to the thin-tailed property of Gaussian distribution. Unfortunately, real data often exhibit heavy-tailed phenomena [Nair *et al.*, 2013], and are common with outliers [Bamnett and Lewis, 1994]. Therefore, a robust survey aggregation method is highly demanded.

To handle the data with outliers or heavy tails, we propose a robust survey aggregation method based on Student-t distribution and sparse representation. Unlike the most existing methods based on the Gaussian distribution, we assume that the samples are from Student-t distribution, which has been proved to be a robust generalization of Gaussian distribution. Besides, inspired by JS and CAvg, we shrink the sample means to a global mean vector, and apply a sparse recovery technique to recover the global mean vector. To obtain a flexible sparse representation of the global mean vector, we design an adaptive basis which is a union of combined generic basis and data-specific basis. Theoretically, we show the robustness of our method from both Bayesian point of view and non-Bayesian point of view, and prove that both JS and CAvg are special cases of our method. Experimental

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results on synthetic and real data demonstrate the superiority (especially on the data with outliers) of the proposed method over the state-of-the-art methods.

2 Preliminaries

In a typical survey problem, it is common to study the characteristics of different groups, such as working time, age, or income in different regions. Specifically, assume that there are T groups in total and n_i samples in group i. The quantity of interest is the true mean of certain characteristic (e.g., the mean of working hours) within each group, denoted as μ_i for group i $(i = 1, 2, \ldots, T)$. Samples from group i are denoted as x_{ij} $(j = 1, 2, \ldots, n_i)$. The task is to estimate $\boldsymbol{\mu} = \{\mu_i | i = 1, \ldots, T\}$ from the dataset $\mathcal{D} = \{x_{ij} | i = 1, \ldots, T; j = 1, \ldots, n_i\}$ as accurately as possible.

2.1 Sample Averaging

A straightforward method is to estimate the true mean μ_i with the sample mean $\bar{x}_i$, namely, $\hat{\mu}_i^{Avg} = \bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$. It can be proved that sample Averaging (Avg) can be interpreted as maximum likelihood estimation, given that the samples of each group are from independent and identically distributed (*i.i.d.*) Gaussian distribution, *i.e.*,

$$x_{ij} \overset{ind}{\sim} \mathcal{N}(\mu_i, \sigma_i^2), \quad \begin{matrix} i = 1, 2, \ldots, T \\ j = 1, 2, ..., n_i. \end{matrix} \tag{1}$$

Formally, $\hat{\mu}_i^{Avg}$ is the result of maximum likelihood estimation with the assumption (1), *i.e.*,

$$\min_{\mu_i} -\ln \prod_{j=1}^{n_i} p(x_{ij}|\mu_i, \sigma_i) \Leftrightarrow \min_{\mu_i} \sum_{j=1}^{n_i} \frac{(x_{ij} - \mu_i)^2}{\sigma_i^2}. \tag{2}$$

According to the analysis in [Stein, 1956], when the number of groups is one or two (*i.e.*, $T \leq 2$), Avg is uniformly better than any other method in terms of MSE. Otherwise, Avg cannot be guaranteed to be the best.

2.2 James-Stein Estimator

To improve Avg when the number of groups is larger than two, James-Stein estimator[1] (JS) was proposed in [James and Stein, 1961], which estimates the true mean of each group as

$$\hat{\mu}_i^{JS} = f + (1 - \gamma)_+ \cdot (\bar{x}_i - f), \tag{3}$$

where $f = \frac{1}{T}\sum_{i=1}^{T} \bar{x}_i$ is the global mean, $\gamma = (T - 3)\left(\sum_{i=1}^{T} \frac{n_i}{\sigma_i^2}(\bar{x}_i - f)^2\right)^{-1}$, and $(\cdot)_+ = \max(0, \cdot)$; σ_i^2 is the variance of group i (usually unknown in real data) and is replaced by a standard unbiased estimate. Note that unlike Avg, JS shrinks the sample means $\bar{x}_i$'s towards the global mean f. Efron and Morris [1972] proved that JS is the maximum posterior estimation with the Gaussian sample assumption (1) and the following assumption,

$$\mu_i \overset{ind}{\sim} \mathcal{N}(f, A), \quad i = 1, 2, \ldots, T, \tag{4}$$

where A is the variance of the prior. Compared with Avg, JS adds the prior assumption (4) with respect to μ_i.

[1]The original James-Stein estimator has many variants. Following [Feldman *et al.*, 2012; Shi *et al.*, 2016], in this paper, we use the positive part James-Stein estimator with independent unequal variances [Bock, 1975; Casella, 1985].

2.3 Compressive Averaging

It has been shown in [Shi *et al.*, 2016] that JS performs poorly when the distance between $\bar{x}_i$ and f becomes large (see the example in [Shi *et al.*, 2016], Section 3.1). To overcome this drawback, Shi *et al.* [2016] proposed an improvement of JS based on compressive sensing, named Compressive Averaging (CAvg). Specifically, CAvg assumes

$$\mu_i \overset{ind}{\sim} \mathcal{N}(f_i, A), \quad i = 1, 2, \ldots, T, \tag{5}$$

where $\{f_1, f_2, \ldots, f_T\}$, denoted as $\mathbf{f}$, is the global mean vector, which is assumed to be sparsely represented by a given basis. Besides, CAvg assumes that the sample means $\bar{x}_i$'s follow Gaussian distribution, *i.e.*,

$$\bar{x}_i \overset{ind}{\sim} \mathcal{N}\left(\mu_i, \sigma_i^2/n_i\right), \quad i = 1, 2, \ldots, T. \tag{6}$$

At the first glance, the assumptions (6) and (1) seem different ((1) can lead to (6) directly, but not vice versa). However, these two assumptions actually make no difference for maximum posterior (to estimate μ_i) and maximum marginal likelihood (to estimate parameters $\mathbf{f}$ and A), which will be explained next separately. For the maximum posterior, its result in CAvg is (see (11) in [Shi *et al.*, 2016])

$$\hat{\mu}_i^{CAvg} = f_i + \left(1 - \frac{\sigma_i^2}{An_i + \sigma_i^2}\right)(\bar{x}_i - f_i), \tag{7}$$

which is also the result of maximum posterior based on the assumptions (1) and (5), *i.e.*,

$$\begin{aligned} &\min_{\mu_i} -\ln\left(\prod_{j=1}^{n_i} p(x_{ij}|\mu_i, \sigma_i)\, p(\mu_i|f_i, A)\right) \\ &\Leftrightarrow \min_{\mu_i} \sum_{j=1}^{n_i} \frac{(x_{ij} - \mu_i)^2}{2\sigma_i^2} + \frac{(\mu_i - f_i)^2}{2A}. \end{aligned} \tag{8}$$

For the maximum marginal likelihood, we will show that the assumptions (6) and (1) have no difference in Section 4. Since the assumptions (6) and (1) have no difference for both maximum posterior and maximum marginal likelihood, we claim that CAvg is consistent with the Gaussian sample assumption (1).

2.4 Multi-Task Averaging

The idea of using information from all groups to improve the estimate of some quantities of a single group falls into the realm of multi-task learning in artificial intelligence and machine learning community. From this perspective, Feldman *et al.* [2012] proposed Multi-Task Averaging (MTAvg), which estimates μ_i $(i = 1, 2, \ldots, T)$ by minimizing

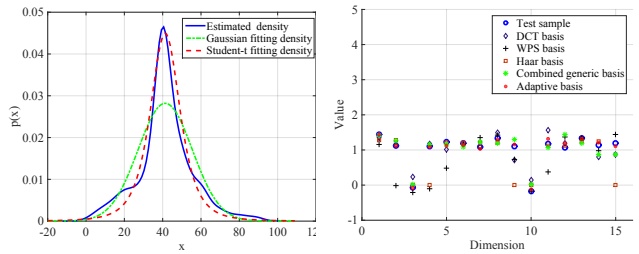
$$\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{n_i} \frac{(x_{ij} - \mu_i)^2}{\sigma_i^2} + \frac{\gamma}{T^2}\sum_{r=1}^{T}\sum_{s=1}^{T} S_{rs}(\mu_r - \mu_s)^2, \tag{9}$$

where $\mathbf{S}$ is a pairwise task similarity matrix. MTAvg can be interpreted as estimating the means of T Gaussians with an intrinsic Gaussian Markov random field prior [Feldman *et al.*, 2012]. Note that the square loss in MTAvg corresponds to the Gaussian sample assumption (1).

3 Robust Survey Aggregation

3.1 Motivation

As discussed in Section 2, all the above methods, including Avg, JS, CAvg, and MTAvg, share the common Gaussian sample assumption (1) explicitly or implicitly. Based on such assumption, the inference is relatively easy. However, this assumption results in severe problems in practice. I) Real data often exhibit heavy-tailed phenomena, which are beyond the Gaussian distribution. Fig. 1(a) shows the heavy-tailed phenomenon in real dataset Hrs1 from the yearly General Social Survey [Smith *et al.*, 2015]. The density of Hrs1 is estimated by kernel density estimation [Silverman, 1986]. Student-t distribution and Gaussian distribution fit the data respectively with the method of moments. It can be observed that the Student-t fitting density is much closer to the estimated density of Hrs1 than the Gaussian fitting density, because the real distribution has heavy tails; II) real data often contain noise or outliers, which deteriorate the performance of Gaussian based methods. The effect of outliers can be seen in Table 1 (the experimental details will be introduced in Section 5), from which we can observe that MSEs of Avg, JS, CAvg, and MTAvg increase a lot (all over 60%) when 2% of the data are contaminated with outliers.



(a) Density fitting for Hrs1 (b) Signal sparse representation

Figure 1: (a) Heavy-tailed phenomenon in the real data Hrs1. (b) Sparse recovery results by OMP with different bases.

The issues caused by the Gaussian sample assumption (1) motivate us to design a robust survey aggregation method.

3.2 Derivation of Robust Survey Aggregation

To handle the survey data with outliers or heavy tails, we propose Robust Averaging (RAvg), in which the samples are assumed to follow Student-t distribution, *i.e.*,

$$x_{ij} \overset{ind}{\sim} St\left(\mu_i, \nu, \sigma_i^2\right), \quad \begin{matrix} i = 1, 2, \ldots, T \\ j = 1, 2, ..., n_i, \end{matrix} \tag{10}$$

where ν is degrees of freedom; $\boldsymbol{\sigma}^2 = \{\sigma_i^2 | i = 1, \ldots, T\}$ is the set of scale parameters, which can be replaced by a standard unbiased estimate, similarly as in JS and CAvg.

The reasons for using the Student-t sample assumption (10) instead of the Gaussian sample assumption (1) are as follows: I) Student-t sample assumption (10) is a natural generalization of Gaussian sample assumption (1). When degrees of freedom ν approaches $+\infty$, Student-t sample assumption (10) reduces to Gaussian sample assumption (1). II) Student-t distribution, which is a heavy-tailed distribution, has been proved to be a robust alternative to Gaussian distribution in [O'Hagan, 1979; Shah *et al.*, 2014; Tang *et al.*, 2016], from Bayesian point of view. Besides, the robustness from Student-t sample assumption (10) can be also shown from non-Bayesian view, which will be discussed in detail in Section 4.

Therefore, we expect our method based on the Student-t sample assumption (10) is more robust to outliers and can be applied to more types of data.

Similar to CAvg, we also assume the true means

$$\mu_i \overset{ind}{\sim} \mathcal{N}(f_i, A) \quad i = 1, 2, ..., T, \tag{11}$$

where f_i encodes a flexible prior about μ_i, and A controls how strong the prior is.

Based on the assumptions (10) and (11), we minimize the negative log marginal likelihood of RAvg to learn the parameters $A, \nu, \mathbf{f} = \{f_i | i = 1, \ldots, T\}$, followed by estimating the target μ_i via minimum negative log posterior.

The marginal likelihood can be derived as follows based on the assumptions (10) and (11),

$$\begin{aligned} p(\mathcal{D}|\mathbf{f}, A, \nu) &= \int p(\mathcal{D}|\boldsymbol{\mu}, \boldsymbol{\sigma}, \nu) p(\boldsymbol{\mu}|\boldsymbol{f}, A) \mathrm{d}\boldsymbol{\mu} \\ &= \int \exp\left(\Psi(\boldsymbol{\mu})\right) \mathrm{d}\boldsymbol{\mu}, \end{aligned} \tag{12}$$

where $\Psi(\boldsymbol{\mu}) = \sum_{i=1}^{T} \left(\sum_{j=1}^{n_i} \ln p(x_{ij}|\mu_i, \sigma_i, \nu) + \ln p(\mu_i|f_i, A) \right)$.

Since (12) is analytically intractable, we rely on Laplace approximation [Jylänki *et al.*, 2011] for approximate inference. Particularly, we use a second order Taylor expansion of $\Psi(\boldsymbol{\mu})$ locally around $\hat{\boldsymbol{\mu}} = \arg_{\boldsymbol{\mu}} \max \Psi(\boldsymbol{\mu})$, and obtain $\Psi(\boldsymbol{\mu}) \simeq \Psi(\hat{\boldsymbol{\mu}}) - \frac{1}{2}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})$. Thus, an approximation, $q(\mathcal{D}|\mathbf{f}, A, \nu)$, to the marginal likelihood is

$$\begin{aligned} &p(\mathcal{D}|\mathbf{f}, A, \nu) \simeq q(\mathcal{D}|\mathbf{f}, A, \nu) \\ &= \exp\left(\Psi(\hat{\boldsymbol{\mu}})\right) \int \exp\left(-\frac{1}{2}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \right) \mathrm{d}\boldsymbol{\mu}, \end{aligned}$$

where $\boldsymbol{\Sigma}^{-1}$ is the negative Hessian matrix of $\Psi(\boldsymbol{\mu})$ at $\hat{\boldsymbol{\mu}}$. Particularly, $\boldsymbol{\Sigma}^{-1}$ is a diagonal matrix with

$$\boldsymbol{\Sigma}_{ii}^{-1} = \sum_{j=1}^{n_i} (\nu + 1) \frac{\nu\sigma_i^2 - (\hat{\mu}_i - x_{ij})^2}{\left(\nu\sigma_i^2 + (\hat{\mu}_i - x_{ij})^2\right)^2} + \frac{1}{A} \ (i = 1, \ldots, T).$$

Based on the following fact,

$$\int \frac{1}{(2\pi)^{T/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \right) \mathrm{d}\boldsymbol{\mu} = 1,$$

an approximation to the negative log marginal likelihood is

$$-\ln q(\mathcal{D}|\mathbf{f}, A, \nu) = -\Psi(\hat{\boldsymbol{\mu}}) - \ln\left((2\pi)^{T/2} |\boldsymbol{\Sigma}|^{1/2} \right) \tag{13}$$

$$\begin{aligned} &= \sum_{i=1}^{T} \left[\frac{\nu + 1}{2} \sum_{j=1}^{n_i} \ln\left(1 + \frac{1}{\nu\sigma_i^2} (x_{ij} - \hat{\mu}_i)^2 \right) + \frac{(\hat{\mu}_i - f_i)^2}{2A} \right. \\ &\quad \left. - n_i \ln\left(\frac{\Gamma((\nu + 1)/2)}{\sqrt{\nu\pi}\sigma_i \Gamma(\nu/2)} \right) + \ln\sqrt{A} + \frac{1}{2} \ln \boldsymbol{\Sigma}_{ii}^{-1} \right]. \end{aligned}$$

The details of learning the parameters A, ν, and $\mathbf{f}$ by minimizing (13) will be introduced in Section 3.3. Once the parameters A, ν, and $\mathbf{f}$ are learned, the estimation of μ_i can be obtained by minimum negative log posterior of RAvg, *i.e.*, minimizing the following objective:

$$
\begin{aligned}
&-\ln(\prod_{j=1}^{n_i} p\left(x_{ij}|\mu_i,\sigma_i,\nu\right) p\left(\mu_i|f_i,A\right)) \\
&\propto \frac{\nu+1}{2}\sum_{j=1}^{n_i}\ln\left(1+\frac{1}{\nu\sigma_i^2}\left(x_{ij}-\mu_i\right)^2\right)+\frac{\left(\mu_i-f_i\right)^2}{2A}. \quad (14)
\end{aligned}
$$

The exponent of (14) is a product of positive convex functions, and thus minimizing the exponent of (14) (equivalent to minimizing (14)) belongs to the convex multiplicative programming [Konno and Kuno, 1995], which can be efficiently obtained by the method in [Kuno *et al.*, 1993] (also refer to the convex multiplicative programming in robust logistic regression [Ding and Vishwanathan, 2010]).

3.3 Parameter Learning

When minimizing the negative log marginal likelihood in (13) to learn the parameters $A, \nu, \mathbf{f}$, we need to restrict $\mathbf{f}$ by adding some assumptions. Inspired by [Shi *et al.*, 2016], we assume that $\mathbf{f}$ can be sparsely represented by a given basis $\Phi = \{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_K\}$ with $\boldsymbol{\phi}_i \in \mathbb{R}^{T\times 1}$, *i.e.*, $\mathbf{f} = \Phi\boldsymbol{\alpha} = \sum_{i=1}^{K}\alpha_i\boldsymbol{\phi}_i$, where $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_K\}$ is a sparse coefficient vector with $||\boldsymbol{\alpha}||_0 \leq k \ll K$, in which $||\boldsymbol{\alpha}||_0$ is ℓ_0 norm that counts the non-zero elements in $\boldsymbol{\alpha}$.

Then, we solve the minimum negative log marginal likelihood in (13) by updating two sets of parameters $\{A, \nu\}$ and $\boldsymbol{\alpha}$ alternatively until the objective of (13) converges.

- **Fix $\boldsymbol{\alpha}$, solve for** A, ν. (13) is differentiable with respect to A, ν, and thus we can use gradient based optimization methods. An alternative approach is to estimate a rough range of A, ν and search the optimum with linear search algorithm, considering that A, ν are scalars.
- **Fix** A, ν, **solve for** $\boldsymbol{\alpha}$. The optimization problem is

$$
\min_{\boldsymbol{\alpha}}\sum_{i=1}^{T}\frac{\left(\hat{\mu}_i-f_i\right)^2}{2A}, \qquad s.t.: \quad ||\boldsymbol{\alpha}||_0 \leq k. \quad (15)
$$

This problem resembles the sparse recovery formulation in compressive sensing [Candès and Wakin, 2008], which can be solved by classical algorithms, such as Orthogonal Matching Pursuit (OMP) [Tropp and Gilbert, 2007]. For simplicity and efficiency, we adopt the linear search algorithm to solve for A, ν, and use OMP to solve for $\boldsymbol{\alpha}$.

3.4 Adaptive Basis

As discussed above, the global mean vector $\mathbf{f}$ can be sparsely represented by a given basis Φ. Nonetheless, it is non-trivial to choose the basis, since different types of signals typically have different characteristics [Bruckstein *et al.*, 2009; Dai *et al.*, 2015]. In CAvg, Shi *et al.* [2016] empirically used the single Daubechies least-asymmetric wavelet packet (WPS), which cannot always lead to sparse solutions. Taking Fig. 1(b) for example, the WPS basis performs poorly on the electricity consumption data ("elec35_nor", a built-in dataset in Matlab).

The poor performance of WPS motivates us to design a more generalizable basis that can be used for a wide range of signals of interest. A straightforward idea is to unify different types of generic bases (*e.g.*, discrete cosine transform (DCT) basis, WPS basis, Haar basis) in a combined representation [Lesage *et al.*, 2005], denoted as combined generic basis $\tilde{\Phi}_c$,

$$
\tilde{\Phi}_c = [\Phi_1, \Phi_2, \ldots, \Phi_M],
$$

where $\Phi_j \in \mathbb{R}^{T\times K_j}, j = 1, 2, \ldots, M$ are different types of generic bases. Compared with the single basis, the major advantage of the combined generic basis $\tilde{\Phi}_c$ is the relative simplicity of the pursuit algorithm and better generalization ability to different signals. However, combined generic basis $\tilde{\Phi}_c$ generally cannot work well on an arbitrary unseen family of signals of interest, where an adaptive data-specific basis usually performs better [Bruckstein *et al.*, 2009].

To obtain a data-specific basis, K-SVD [Aharon *et al.*, 2006] algorithm has been widely used in sparse representation. When applying K-SVD algorithm to the survey data, the problem is that the sample size is usually small for K-SVD. To address this issue, denoting the data-specific basis learned by K-SVD algorithm as $\tilde{\Phi}_l$, we cascade combined generic basis $\tilde{\Phi}_c$ (a prior) and data-specific basis $\tilde{\Phi}_l$ as

$$
\Phi = [\tilde{\Phi}_c, \tilde{\Phi}_l],
$$

which is our final adaptive basis.

To demonstrate the effectiveness of our adaptive basis, we randomly select 50 samples for training and 1 sample for testing, from the electricity consumption data. The test sample and sparse recovery results produced by OMP with various bases, namely DCT basis, WPS basis, Haar basis, combined generic basis (union of DCT, WPS and Haar basis), and our adaptive basis are shown in Fig. 1(b), from which we can observe that our adaptive basis produces the best results while the generic bases fail to achieve satisfactory results.

4 Theoretical Analysis

In this section, we prove that CAvg and JS are special cases of our RAvg method, and show that our RAvg method is more robust to outliers than Avg, JS, CAvg, and MTAvg from the view of loss and penalty.

Theorem 1 *CAvg is a special case of RAvg with $\nu \to +\infty$ and WPS basis.*

Proof. When $\nu \to +\infty$, the minimum negative log posterior of RAvg (*i.e.*, (14)) converges to that of CAvg (*i.e.*, (8)). Then, as long as the parameters A, $\mathbf{f}$ in the posterior are also identical, CAvg and RAvg will generate the same estimation for μ_i. As the parameters $A, \mathbf{f}$ are learned by minimum negative log marginal likelihood, we only need to prove that their minimum negative marginal likelihoods are equivalent when using the same WPS basis.

With $\nu \to +\infty$, it is not hard to prove that $\hat{\boldsymbol{\mu}}$ and $\boldsymbol{\Sigma}^{-1}$ in the approximate procedure have the forms

$$
\hat{\mu}_i = \frac{\sigma_i^2 f_i + n_i A\bar{\mu}_i}{\sigma_i^2 + n_i A}, \qquad \boldsymbol{\Sigma}_{ii}^{-1} = \frac{n_i}{\sigma_i^2} + \frac{1}{A}. \quad (16)
$$

By substituting $\hat{\mu}_i$ and $\mathbf{\Sigma}_{ii}$ in (16) into (13) and with simplifications, we can obtain

$$\min_{\mathbf{f},A} -\ln q(\mathcal{D}|\mathbf{f}, A, \nu)$$

$$\Leftrightarrow \min_{\mathbf{f},A} \sum_{i=1}^{T} \left[\frac{\sum_{j=1}^{n_i} (x_{ij} - f_i)^2}{An_i + \sigma_i^2} + \frac{An_i \left(\sum_{j=1}^{n_i} x_{ij}^2 - n_i \bar{x}_i^2 \right)}{\sigma_i^2 \left(An_i + \sigma_i^2 \right)} + \log \left(A + \frac{\sigma_i^2}{n_i} \right) \right]. \quad (17)$$

Recall that σ_i^2 is replaced by a standard unbiased estimate, with $\nu \to +\infty$, we can obtain the following two equations:

$$\sigma_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = \frac{\left(\sum_{j=1}^{n_i} x_{ij}^2 - n_i \bar{x}_i^2 \right)}{n_i - 1}, \quad (18)$$

$$\begin{aligned} \sum_{j=1}^{n_i} (x_{ij} - f_i)^2 &= \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i + \bar{x}_i - f_i)^2 \\ &= (n_i - 1)\sigma_i^2 + n_i (\bar{x}_i - f_i)^2 . \quad (19) \end{aligned}$$

By substituting (18) and (19) into (17) and omitting the constant, we obtain

$$\min_{\mathbf{f},A} \sum_{i=1}^{T} \left[\frac{(\bar{x}_i - f_i)^2}{A + \sigma_i^2 / n_i} + \log \left(A + \frac{\sigma_i^2}{n_i} \right) \right], \quad (20)$$

which is exactly the same as the minimum negative log marginal likelihood of CAvg (see [Shi *et al.*, 2016], (10)), *i.e.*, minimum negative marginal likelihood of RAvg is equivalent to that of CAvg. So we complete the proof here. □

With (18) and (19), it is easy to prove that CAvg is consistent with the Gaussian sample assumption implicitly, as mentioned in Section 2.3. Specifically, for CAvg, when replacing the assumption (6) with the assumption (1), the minimum negative log marginal likelihood of CAvg is equivalent to (20). Besides, as discussed in Section 2.3, the assumptions (6) and (1) also make no difference for maximum posterior. Thus, CAvg is consistent with the Gaussian sample assumption (1) implicitly.

Theorem 2 *JS is a special case of RAvg with $\nu \to +\infty$, the equal elements in $\mathbf{f}$, and identical parameter estimation.*

Proof. When $\nu \to +\infty$, the Student-t sample assumption (10) reduces to the Gaussian sample assumption (1). When the elements in $\mathbf{f}$ of RAvg, *i.e.*, $f_1, \ldots, f_T$, are equal, JS and RAvg have the same assumptions. Furthermore, when the parameters A and $\mathbf{f}$ of RAvg are estimated using the method in JS (see [Efron and Morris, 1972; Casella, 1985]), RAvg will produce the same estimation as JS. □

Proposition 1 *From the view of loss and penalty, RAvg is more robust than Avg, JS, CAvg, and MTAvg.*

From Bayesian view, we know that RAvg is more robust than those Gaussian based methods (*i.e.*, Avg, JS, CAvg, and MTAvg) due to the Student-t sample assumption (10), as discussed in Section 3.2. In fact, from non-Bayesian view, the robustness of RAvg can also be explained as follows. The optimization objectives of all the methods for μ_i, Avg (2), JS ((8) with all f_i's equal), CAvg (8), MTAvg (9), and RAvg (14) can be interpreted as a loss term plus a penalty (for Avg, the penalty is 0). We can see that for Avg, JS, CAvg and MTAvg, the loss function is square loss. In contrast, the loss function of RAvg is a *log-square* loss, which is much more robust.

5 Experiments

In this section, we conduct comprehensive experiments on both synthetic and real datasets. Our baselines include Avg, JS, MTAvg, and CAvg. For fair comparison, OMP is also used to calculate $\boldsymbol{\alpha}$ in CAvg and k is set as 3 for both RAvg and CAvg, following [Shi *et al.*, 2016]. The evaluation metric is the mean square error (MSE) between the true mean and the estimated mean of each group. Note that the true means are known for synthetic data, but unknown for real data. Following [Shi *et al.*, 2016], the true means are replaced by the sample means when 100% of the data are used.

5.1 Datasets

The datasets for the experiments include

- **Synthetic Data A.** This dataset is created according to the generative procedure in [Shi *et al.*, 2016]. The sample assumption is Gaussian, in which there are 10 groups with 100 samples in each group.
- **GSS.** General Social Survey (GSS) is a well-known public survey dataset [Smith *et al.*, 2015]. We take out the variables **Age, Hrs1, Paedu, and Spedu** from GSS. **Age** means the age of the respondent, with 30 groups, each of which contains $1372 \sim 4492$ samples. **Hrs1** refers to the number of working hours the respondent had last week, with 29 groups, each of which contains $741 \sim 2739$ samples. **Paedu** stands for the highest school year completed by the respondent's father, with 30 groups, each of which contains $1051 \sim 2258$ samples. **Spedu** denotes the highest school year completed by the respondent's spouse, with 30 groups, each of which contains $721 \sim 1458$ samples.

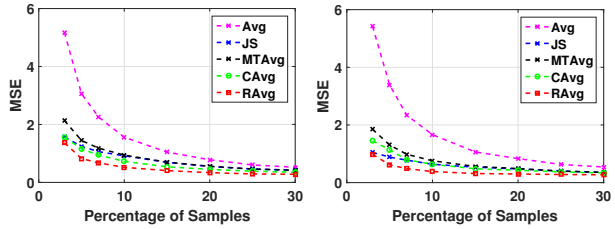
5.2 Performance on Synthetic and Real Datasets

Table 1: MSEs of different methods on *Synthetic Data A*.

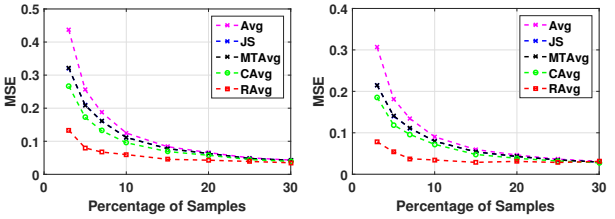
	Avg	JS	MTAvg	CAvg	RAvg
Original data	2.575	2.201	2.131	2.106	2.013
With 2% outliers	4.192	3.656	3.598	3.603	2.372
Increased MSE	1.617	1.455	1.467	1.503	0.359

The experimental results on the Synthetic Data A are reported in Table 1, from which we observe that RAvg has comparable result with CAvg even though the samples are from Gaussian distribution. The underlying reason is that the Student-t sample assumption includes the Gaussian sample assumption as a special case and our basis is adaptive. Moreover, when 2% of the samples are contaminated with outliers, RAvg is statistically better than other methods with 95% confidence.

For the real datasets, we randomly select some samples from each dataset and use these samples to estimate the true means (the means when 100% of the data are used). We vary

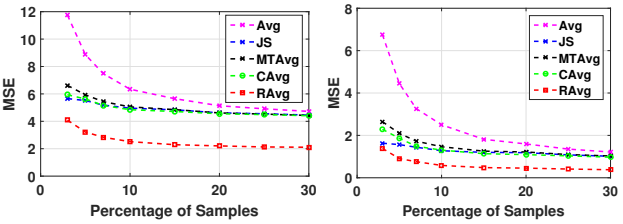


(a) Age (b) Hrs1

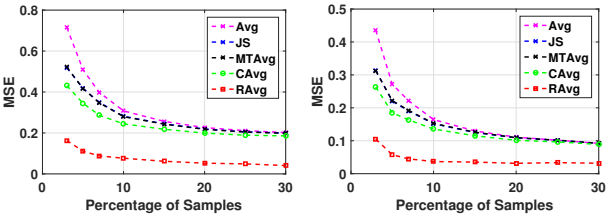


(c) Paedu (d) Spedu

Figure 2: Results of different methods on real datasets.



(a) Age with 2% outliers (b) Hrs1 with 2% outliers



(c) Paedu with 2% outliers (d) Spedu with 2% outliers

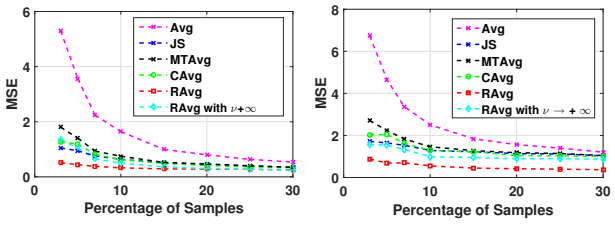
Figure 3: Results of different methods on the real datasets with 2% outliers.

the sampling percentage from 3% to 30%, run each method for 30 runs, and compute the average MSE of 30 runs. The results of different methods on the real datasets are plotted in Fig. 2, from which we can observe that our RAvg method outperforms all the baselines for all percentages of samples.

In the experiments, RAvg takes more time than baselines (Specifically, $3 \sim 5$ times more than CAvg). However, the sample size is usually small, since each sample point is expensive (see Introduction). Thus, time complexity is not the key issue for survey aggregation.

5.3 Robustness to Outliers

The robustness of RAvg has been shown on the synthetic data, as demonstrated in Table 1. In order to evaluate the robustness on real datasets, 2% of the samples are contaminated with outliers and the results of all methods are reported in Fig. 3. Comparing Fig. 2 and Fig. 3, we can see that once the data are



(a) Hrs1 (b) Hrs1 with 2% outliers

Figure 4: (a) Results of different methods on Hrs1. (b) Results of different methods on Hrs1 with 2% outliers.

contaminated with outliers, MSEs of Avg, JS, MTAvg, and CAvg increase significantly while RAvg remains steady and achieves much better performance than other methods, which demonstrates the robustness of RAvg on the real datasets.

5.4 Component Analysis

To investigate the individual contribution of the Student-t sample assumption and the adaptive basis in our proposed RAvg method, we introduce RAvg with $\nu \to +\infty$ (in practice, ν is fixed to 100), which reduces to CAvg except using adaptive basis, based on Theorem 1. Therefore, the gap between RAvg with $\nu \to +\infty$ and RAvg is caused by the Student-t sample assumption, and the gap between RAvg with $\nu \to +\infty$ and CAvg is the contribution of the adaptive basis. Taking the Hrs1 dataset as an example, Fig. 4 shows the results, where we can observe that the adaptive basis reduces the MSE, and the Student-t sample assumption further reduces the MSE, since the density of Hrs1 has heavy tails, as illustrated in Fig. 1(a). It is worth noting that the contribution of the Student-t assumption is more significant when the data are contaminated with outliers, as shown in Fig. 4(b).

6 Conclusion

In this paper, we propose a robust survey aggregation method, RAvg, based on Student-t sample assumption and sparse representation. In particular, due to the Student-t sample assumption, our method is robust to outliers, which is analyzed in detail from Bayesian point of view and loss-penalty point of view. For a better sparse representation of the global mean vector, we design a simple yet effective adaptive basis, which further improves the performance. Theoretically, we have proved that JS and CAvg are special cases of our method. Extensive experiments have demonstrated that our method significantly outperforms the state-of-the-art methods, especially in the presence of outliers.

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