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Refutation in Dummett Logic Using a Sign to Express the Truth at the Next Possible World

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Abstract

In this paper we use the Kripke semantics characterization of Dummett logic to introduce a new way of handling non-forced formulas in tableau proof systems. We pursue the aim of reducing the search space by strictly increasing the number of forced propositional variables after the application of non-invertible rules. The focus of the paper is on a new tableau system for Dummett logic, for which we have an implementation.

1 Introduction

In this paper we present a tableau calculus and theorem prover for propositional Dummett logic. By exploiting the linearly ordered Kripke semantics of Dummett logic, we devise a tableau calculus working on two semantical levels: the present and the next possible world. In this way we can guarantee that as the construction of the tableau proceeds, moving from a world to another, the known (forced) information strictly increases. Moreover, the calculus can be equipped with specialized rules to reduce the branching. As a result, the decision procedure is speeded up.

Dummett logic has been extensively investigated both by people working in computer science and in logic. The history of this logic starts with Gödel, who studied the family of logics semantically characterizable by a sequence of n-valued ($n > 2$) matrices [Gödel, 1986]. In paper [Dummett, 1959] Dummett studied the logic semantically characterized by an infinite valued matrix which is included in the family of logics studied by Gödel and proved that such a logic is axiomatizable by adding to any Hilbert system for propositional intuitionistic logic the axiom scheme $(p \to q) \vee (q \to p)$. Moreover, it is well-known that such a logic is semantically characterizable by linearly ordered Kripke models. Dummett logic has been extensively studied also in recent years for its relationships with computer science [Avron, 1991] and fuzzy logics [Hajek, 1998]. For a survey in proof theory in Gödel-Dummett logics we quote [Baaz *et al.*, 2003].

To perform automated deduction both tableau and sequent calculi have been proposed. Paper [Avellone *et al.*, 1999] provides tableau calculi whose distinguishing feature is a multiple premise rule for implicative formulas that are not-forced. The syntactical way to express this semantical meaning is by the sign **F**. We recall that the sign **F** comes from Smullyan [Smullyan, 1968; Fitting, 1969] and labels those formulas that in a sequent calculus occur in the right-hand side of $\Rightarrow$. A tableau calculus derived from those of [Avellone *et al.*, 1999] is provided in paper [Fiorino, 2001]. Its main feature is that the depth of every deduction is linearly bounded in the length of the formula to be proved. Hypersequent systems for the implicative fragment of propositional Dummett logic are provided in [Metcalfe *et al.*, 2003].

The approach of [Avellone *et al.*, 1999], based on characterizing Dummett logic by means of the multiple premise rule, has been criticized because, from the perspective of worst case analysis, there are simple examples of sets of formulas giving rise to a factorial number of branches in the number of formulas in the set. Paper [Avron and Konikowska, 2001] shows how to get rid of the multiple premise rule. New rules are provided whose correctness is strictly related to the semantics of Dummett logic. These ideas have been further developed and in paper [Larchey-Wendling, 2007] a graph-theoretic decision procedure is described and implemented. The approach introduced in [Avron and Konikowska, 2001] has also disadvantages with respect to the multiple premise rule proposed in [Avellone *et al.*, 1999] and these disadvantages have been considered in [Fiorino, 2010], where also a new version of the multiple premise rule is proposed. This version, from a practical point of view, can reduce the branching when compared with the original one. Paper [Fiorino, 2010] also provides an implementation that outperforms the one of [Larchey-Wendling, 2007], thus proving that the approach based on the multiple premise rule of [Avellone *et al.*, 1999] deserves attention also from the practical point of view. As a matter of fact, on the one hand the rules of [Avron and Konikowska, 2001] give rise to two branches at most, on the other hand there are cases of formulas that multiple premise calculi decide with a number of steps lower than the calculi based on [Avron and Konikowska, 2001].

In this paper we continue our investigation around multiple premise calculi for Dummett logic. The tool we use to prove our results is the characterization of Dummett logic via linearly ordered Kripke models, whose elements are considered *worlds* ordered w.r.t. $\leq$ and $\alpha \leq \beta$ means that, roughly speaking, β is in a subsequent point of the time w.r.t. α.

With the aim to reduce the size of the proofs, we provide a calculus whose main feature is the way the non-forced formulas are handled. In tableau calculi proofs start by supposing that at the present world A is not forced. A proof of our calculus starts by adding a further semantical constraint, namely, by supposing that the present world is the last where A is not forced. This further constraint implies that in the present we have information about a different semantical status of A in the future: if we conclude that there exists the next world, then we deduce that in such a future world A is forced (this implies that starting from such a next world, A is equivalent to $\top$). Such information can be handled to draw deductions about the semantical status of the formulas both in the present and in the next world (hereafter, when is clear from the context, we omit "world" after "present" and and "future"). As a consequence, our calculus handles two kinds of signed formulas: the first kind describes the semantical status of a formula at the present, the second kind describes the semantical status of a formula at the next world. The introduction of the signs related to the future allows to introduce specialized rules that exploit the knowledge about the future to draw deductions about the present. In particular, we refer to the introduction of specialized and single conclusion rules to handle the formulas of the kind $\mathbf{F}(A \to B)$ (to be read "at the present world the fact $A \to B$ is not known"), thus reducing the branching generated by the multiple premise rule.

The semantical constraint used in calculus has also the advantage to change the effect of the application of the non-invertible rules. Roughly speaking, non-invertible rules draw conclusions about the next world having as premise facts about the present. The application of these rules has an effect in the proof-search, usually requiring the introduction of backtracking to guarantee the completeness of the proof-search procedure. The semantical effect is that in the construction of the counter model a new state of knowledge is added, corresponding to the facts derived by the non-invertible rule. In the known sequent/tableau calculi for intermediate logics there is no guarantee that in the conclusion of the non-invertible rules at least one fact that in the premise was explicitly unknown becomes explicitly known in the conclusion. In the semantical construction this means that there is no guarantee that between two subsequent worlds, knowledge increases. The two semantical levels used in our calculus allow us to provide non-invertible rules such that there is at least one fact that in the premise is explicitly signed as not known and in the conclusion is explicitly signed as known. This implies that moving from the present to the next state of knowledge there is at least one explicitly unknown fact of the present that becomes explicitly known in the future. This has also a correspondence in the construction of the Kripke counter model, where there are no two elements exactly forcing the same propositional variables. The knowledge that a formula A is equivalent to $\top$ can be exploited to replace all the occurrences of A with $\top$. Such a replacement reduces the size of the set to be decided. Further reductions are possible by applying replacements based on the truth table of the connectives. Rules based on the replacement have been proved effective to dramatically reduce the search space both in classical and intuitionistic logic [Avellone *et al.*, 2008; Massacci, 1998]. The idea of reasoning on two states of knowledge can be used also in other intermediate logics. As an example it can be used to develop a loop-free analytic calculus for propositional intuitionistic logic.

2 Basic definitions, the calculus and general considerations

We consider the propositional language based on a denumerable set of propositional variables $\mathcal{PV}$, the boolean constants $\top$ and $\bot$ and the logical connectives $\neg, \wedge, \vee, \to$. We call *atoms* the elements of $\mathcal{PV} \cup \{\top, \bot\}$. In the following, formulas (respectively set of formulas and propositional variables) are denoted by letters $A, B, C\ldots$ (respectively $S, T, U,\ldots$ and $p, q, r,\ldots$) possibly with subscripts or superscripts.

From the introduction we recall that *Dummett Logic* ($\mathbf{Dum}$) can be axiomatized by adding to any axiom system for propositional intuitionistic logic the axiom scheme $(p \to q) \vee (q \to p)$ and a well-known semantical characterization of $\mathbf{Dum}$ is by *linearly ordered Kripke models*. In the paper *model* means a linearly ordered Kripke model, namely a structure $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$, where $\langle P, \leq, \rho\rangle$ is a linearly ordered set with ρ minimum with respect to $\leq$ and $\Vdash$ is the *forcing relation*, a binary relation on $P \times (\mathcal{PV} \cup \{\top, \bot\})$ such that: (i) if $\alpha \Vdash p$ and $\alpha \leq \beta$, then $\beta \Vdash p$; (ii) for every $\alpha \in P$, $\alpha \Vdash \top$ holds and $\alpha \Vdash \bot$ does not hold. Hereafter we denote the members of P by means of lowercase letters of the Greek alphabet.

The forcing relation is extended in a standard way to arbitrary formulas of our language as follows:

1. $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$;
2. $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$;
3. $\alpha \Vdash A \to B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash A$ implies $\beta \Vdash B$;
4. $\alpha \Vdash \neg A$ iff for every $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash A$ does not hold.

We write $\alpha \nVdash A$ when $\alpha \Vdash A$ does not hold. It is easy to prove that for every formula A the *persistence* property holds: If $\alpha \Vdash A$ and $\alpha \leq \beta$, then $\beta \Vdash A$. We say that β is *immediate successor* of α iff $\alpha < \beta$ and there is no $\gamma \in P$ such that $\alpha < \gamma < \beta$. A formula A *is valid in a model* $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$ if and only if $\rho \Vdash A$. It is well-known that $\mathbf{Dum}$ coincides with the set of formulas valid in all models.

The rules of our calculus $\mathbb{D}$ are in Figure 1 and 2. The calculus $\mathbb{D}$ works on signed formulas, that is well-formed formulas prefixed with one of the *signs* $\mathbf{T}$ (with $\mathbf{T}A$ to be read "the fact A is known at the present world"), $\mathbf{F}$ (with $\mathbf{F}A$ to be read "the fact A is not known at the present"), $\mathbf{F_l}$ (with $\mathbf{F_l}A$ to be read "this is the last world where A is not known") and $\mathbf{F_n}$ (with $\mathbf{F_n}A$ to be read "A is not known at the next world"), and on sets of signed formulas (hereafter we omit the word "signed" in front of "formula" in all the contexts where no confusion arises). Formally, the meaning of the signs is provided by the relation *realizability* ($\rhd$) defined as follows: Let $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$ be a model, let $\alpha \in P$, let H be a signed formula and let S be a set of signed formulas. We say that α *realizes* H (respectively α *realizes* S and $\underline{K}$ *realizes* S), and we write $\alpha \rhd H$ (respectively $\alpha \rhd S$ and $\underline{K} \rhd S$), if the following conditions hold:

1. $\alpha \rhd \mathbf{T}A$ iff $\alpha \Vdash A$;
2. $\alpha \rhd \mathbf{F}A$ iff $\alpha \nVdash A$;
3. $\alpha \rhd \mathbf{F_n}A$ iff there exists $\beta > \alpha$ such that $\beta \rhd \mathbf{F}A$;
4. $\alpha \rhd \mathbf{F_l}A$ iff $\alpha \rhd \mathbf{F}A$ and for every $\beta > \alpha$, $\beta \rhd \mathbf{T}A$;
5. $\alpha \rhd S$ iff α realizes every formula in S;

Before to enter into technical details, let us justify the introduction of the signs $\mathbf{F_l}$ and $\mathbf{F_n}$ in the object language by means of a motivating case. Let us suppose that a world α of a model $\underline{K}$ realizes $S_{\mathbf{F}\rightarrow} = \{\mathbf{F}(A_1 \rightarrow B_1), \ldots, \mathbf{F}(A_n \rightarrow B_n)\}$. Then there exists the last element $\beta \geq \alpha$ such that $\beta \rhd \mathbf{T}A_j, \mathbf{F}B_j, S_{\mathbf{F}\rightarrow} \setminus \{\mathbf{F}(A_j \rightarrow B_j)\}$. Without loss of generality we let $j = 1$. Analogously, there exists an element $\gamma \geq \beta$ such that $\gamma \rhd \mathbf{T}A_i, \mathbf{F}B_i, \mathbf{T}A_1, S_{\mathbf{F}\rightarrow} \setminus \{\mathbf{F}(A_1 \rightarrow B_1), \mathbf{F}(A_i \rightarrow B_i)\}$. Without loss of generality we let $i = 2$. Now, $\beta < \gamma$ or $\beta = \gamma$. If $\beta < \gamma$, then, since β is last element where $\mathbf{F}B_1$ holds, it follows that $\gamma \rhd \mathbf{T}B_1$ holds. If $\beta = \gamma$, then β is the last element where both B_1 and B_2 are not forced. The example shows that we can give a rule ($\mathbf{F} \rightarrow$) taking into account that if $\alpha \rhd \mathbf{F}(A_i \rightarrow B_i)$, for $i = 1, \ldots, n$, then the set $\{\mathbf{T}A_i, \mathbf{F}B_i\}$ is realized in $\theta \in \{\alpha, \beta\}$ and for every world $\gamma > \theta$, $\gamma \rhd \mathbf{T}B_i$ holds. The sign $\mathbf{F_l}$ aims to codify such a semantical property of B_i.

Generalizing the case given above, let $S_{\mathbf{F}\rightarrow} = \{\mathbf{F}(A_1 \rightarrow B_1), \ldots, \mathbf{F}(A_n \rightarrow B_n)\}$ and let S be a set of formulas. Let us suppose that $\alpha \rhd S, S_{\mathbf{F}\rightarrow}$. Then, for $i = 1, \ldots, n$, let us consider the element β_i such that $\beta_i \rhd \mathbf{T}A_i, \mathbf{F}B_i$ and for every $\gamma > \beta_i, \gamma \rhd \mathbf{T}A_i, \mathbf{T}B_i$. Thus β_i is the maximum element such that $\beta_i \rhd \mathbf{T}A_i, \mathbf{F}B_i$. Since $\alpha \rhd \mathbf{F}(A_i \rightarrow B_i)$, such an element β_i exists. Let $\beta_j = \min\{\beta_1, \ldots, \beta_n\}$, with $j \in \{1, \ldots, n\}$. There are two cases: (i) $\beta_j = \alpha$, then we conclude that β realizes the set $S, \mathbf{T}A_j, \mathbf{F_l}B_j, S_{\mathbf{F}\rightarrow} \setminus \{\mathbf{F}(A_j \rightarrow B_j)\}$; (ii) $\beta_j > \alpha$, then β_j realizes the set $S_c, \mathbf{T}A_j, \mathbf{F_l}B_j, S_{\mathbf{F}\rightarrow} \setminus \{\mathbf{F}(A_j \rightarrow B_j)\}$, where, by the meaning of the signs, S_c consists of: (i) the formulas of S signed with $\mathbf{T}$; (ii) the formulas $\mathbf{T}A$ such that $\mathbf{F_l}A \in S$.

With the aim of reducing the size of the proofs, we add further considerations justifying the introduction of the sign $\mathbf{F_n}$. Let us suppose that beside $\alpha \rhd S, S_{\mathbf{F}\rightarrow}$ we also know that for every model $\beta_1, \ldots, \beta_{j-1} > \beta_j$ holds. This rules out that $\beta_j \rhd \mathbf{T}A_i, \mathbf{F_l}B_i$, for $i = 1, \ldots, j-1$, and implies that there exists an element γ such that $\beta_j < \gamma$ and $\gamma \rhd \mathbf{F}(A_i \rightarrow B_i)$. The sign $\mathbf{F_n}$ aims to codify that the formulas $\mathbf{F}(A_i \rightarrow B_i)$ have the following semantical property: there exists an element γ such that $\beta_j < \gamma$ and $\gamma \rhd \mathbf{F}(A_i \rightarrow B_i)$, thus the element β_j is not the maximum element realizing $\mathbf{F}(A_i \rightarrow B_i)$. By the meaning of $\mathbf{F_n}$ it follows that $\beta_j \rhd S_c, \mathbf{F_n}(A_1 \rightarrow B_1), \ldots, \mathbf{F_n}(A_{j-1} \rightarrow B_{j-1}), \mathbf{T}A_j, \mathbf{F}B_j, S_{\mathbf{F}\rightarrow} \setminus \{\mathbf{F}(A_1 \rightarrow B_1), \ldots, \mathbf{F}(A_{j-1} \rightarrow B_{j-1})\}$. These are the intuitions justifying the rules $\mathbf{F_n}$ and $\mathbf{F} \rightarrow$. In particular the aim of $\mathbf{F} \rightarrow$ is to distinguish if the last element realizing $\mathbf{F}(A \rightarrow B)$ is the same element α realizing the premise or an element β such that $\alpha < \beta$. The rule $\mathbf{F_n}$ handles the formulas of the kind $\mathbf{F_n}A$. For every formula there exists a world β such that $\alpha < \beta$ and β is the last element such that $\beta \nVdash A$. The minimum of such β does not force any formula in evidence in the premise.

The intuition behind the rule $\mathbf{F_n}$ can be explained as follows. Let us suppose that $\alpha \rhd S, \mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_u$. Thus there exists β_i such that $\alpha < \beta_i$ and $\beta_i \rhd \mathbf{F_l}A_i$, for $i = 1, \ldots, u$. We notice that β_i realizes all the $\mathbf{T}$ formulas in S and $\beta_i \rhd \mathbf{T}C$ if $\mathbf{F_l}C \in S$. Moreover, if $\beta_1 = \min\{\beta_1, \ldots, \beta_u\}$, then $\beta_1 \rhd \mathbf{F_l}A_1, \mathbf{F}A_2, \ldots, \mathbf{F}A_u$. If β_2 is the minimum and we know that there is no model realizing $\alpha \rhd S, \mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_u$ having β_1 as the minimum, then, since $\beta_2 < \beta_1$ holds, we conclude that $\beta_2 \rhd \mathbf{F_n}A_1, \mathbf{F_l}A_2, \mathbf{F}A_3, \ldots, \mathbf{F}A_u$. Analogously, if β_i is the minimum and we know that there is no model realizing $\alpha \rhd S, \mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_u$ having $\beta_1, \ldots$ or β_{i-1} as the minimum, then, since $\beta_i < \beta_1, \ldots, \beta_{i-1}$ holds, we conclude that $\beta_i \rhd \mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_{i-1}, \mathbf{F_l}A_i, \mathbf{F}A_{i+1}, \ldots, \mathbf{F}A_u$.

From the meaning of the signs we get the conditions that make a set of formulas inconsistent. A set S is *inconsistent* if one of the following conditions holds:
-$\{\mathbf{T}A, \mathbf{F}A\} \subseteq S$; -$\{\mathbf{T}A, \mathbf{F_l}A\} \subseteq S$; -$\{\mathbf{T}A, \mathbf{F_n}A\} \subseteq S$;
-$\{\mathbf{F_l}A, \mathbf{F_n}A\} \subseteq S$; -$\{\mathbf{F_n}A, \mathbf{F_l}\neg B\} \subseteq S$;
-$\{\mathbf{F_n}A, \mathbf{F_l}\bot\} \subseteq S$.
We emphasize that inconsistency conditions involving $\mathbf{F_n}$ are related to the existence of a future world α and in such a α the other formula of the pair is not realizable. Moreover the case $\{\mathbf{F_n}A, \mathbf{F_l}A\} \subseteq S$ is redundant since it is subsumed by $\{\mathbf{T}A, \mathbf{F}A\} \subseteq S$. It is easy to prove the following

Proposition 1. *If a set of formulas S is inconsistent, then for every Kripke model $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ and for every $\alpha \in P$, $\alpha \not\rhd S$.*

Proof. We only consider the case $\{\mathbf{F_n}A, \mathbf{F_l}\neg B\} \subseteq S$. By absurd, let us suppose that $\alpha \rhd S$, then $\alpha \rhd \mathbf{F_l}\neg B$ and $\alpha \rhd \mathbf{F_n}A$. Since $\alpha \rhd \mathbf{F_n}A$, there exists $\beta \in P$ such that $\alpha < \beta$. By definition of $\mathbf{F_l}$, it follows that $\alpha \nVdash \neg B$ and for every $\beta \in P$, if $\alpha < \beta$, then $\beta \Vdash \neg B$. Note that such a β exists. Since $\underline{K}$ is a linearly ordered Kripke model, by definition of negation it follows that $\alpha \Vdash \neg B$. Thus we have that $\alpha \nVdash \neg B$ and $\alpha \Vdash \neg B$, absurd. The other cases are easy to prove. $\square$

A proof table (or proof tree) for S is a tree, rooted in S and obtained by the subsequent instantiation of the rules of the calculus. The premise of the rules are instantiated in a duplication-free style: in the application of the rules we always consider that the formulas in evidence in the premise are not in S. We say that a rule $\mathcal{R}$ applies to a set U when it is possible to instantiate the premise of $\mathcal{R}$ with the set U and we say that a rule $\mathcal{R}$ applies to a formula $H \in U$ (respectively the set $\{H_1, \ldots, H_n\} \subseteq U$) to mean that it is possible to instantiate the premise of $\mathcal{R}$ taking S as $U \setminus \{H\}$ (respectively $U \setminus \{H_1, \ldots, H_n\}$).

A *closed proof table* is a proof table whose leaves are all inconsistent sets. A closed proof table is a proof of the calculus and a formula A is provable iff there exists a closed proof table for $\{\mathbf{F_l}A\}$. Note that deductions in our proof system start with a formula signed with $\mathbf{F_l}$. This corresponds to suppose that there exists the **last** element of a Kripke model where A is not forced. In the tableau systems the deduction starts with $\mathbf{F}A$, that corresponds to suppose that there exists an element, **not necessarily the last**, of a Kripke model such that A is not forced. We refer to [Hähnle, 2001] for full details on tableaux systems.

$$\frac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge \qquad \frac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A|S, \mathbf{F}B}\mathbf{F}\wedge \qquad \frac{S, \mathbf{T}\neg(A \wedge B)}{S, \mathbf{T}\neg A|S, \mathbf{T}\neg B}\mathbf{T}\neg\wedge \qquad \frac{S, \mathbf{T}\neg\neg(A \wedge B)}{S, \mathbf{T}\neg\neg A, \mathbf{T}\neg\neg B}\mathbf{T}\neg\neg\wedge$$

$$\frac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A|S, \mathbf{T}B}\mathbf{T}\vee \qquad \frac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee \qquad \frac{S, \mathbf{T}\neg(A \vee B)}{S, \mathbf{T}\neg A, \mathbf{T}\neg B}\mathbf{T}\neg\vee \qquad \frac{S, \mathbf{T}\neg\neg(A \vee B)}{S, \mathbf{T}\neg\neg A|S, \mathbf{T}\neg\neg B}\mathbf{T}\neg\neg\vee$$

$$\frac{S, \mathbf{F}\neg A}{S, \mathbf{T}\neg\neg A}\mathbf{F}\neg \qquad \frac{S, \mathbf{T}\neg\neg\neg A}{S, \mathbf{T}\neg A}\mathbf{T}\neg\neg\neg \qquad \frac{S, \mathbf{T}\neg(A \to B)}{S, \mathbf{T}\neg\neg A, \mathbf{T}\neg B}\mathbf{T}\neg\to \qquad \frac{S, \mathbf{T}\neg\neg(A \to B)}{S, \mathbf{T}\neg A|S, \mathbf{T}\neg\neg B}\mathbf{T}\neg\neg\to$$

$$\frac{S, \mathbf{T}A, \mathbf{T}(A \to B)}{S, \mathbf{T}A, \mathbf{T}B}\text{MP} \qquad \frac{S, \mathbf{T}((A \wedge B) \to C)}{S, \mathbf{T}(A \to (B \to C))}\mathbf{T}\to\wedge \qquad \frac{S, \mathbf{T}(\neg A \to B)}{S, \mathbf{T}\neg\neg A|S, \mathbf{T}B}\mathbf{T}\to\neg$$

$$\frac{S, \mathbf{T}((A \vee B) \to C)}{S, \mathbf{T}(A \to p), \mathbf{T}(B \to p), \mathbf{T}(p \to C)}\mathbf{T}\to\vee, \text{ with } p \text{ a new atom}$$

$$\frac{S, \mathbf{T}((A \to B) \to C)}{S, \mathbf{F}(A \to p), \mathbf{T}(p \to C), \mathbf{T}(B \to p)|S, \mathbf{T}C}\mathbf{T}\to\to, \text{ with } p \text{ a new atom}$$

$$\frac{S, \mathbf{F_1}(A \vee B)}{S, \mathbf{F}A, \mathbf{F_1}B|S, \mathbf{F}B, \mathbf{F_1}A}\mathbf{F_1}\vee \qquad \frac{S, \mathbf{F}(A \to B)}{S, \mathbf{T}A, \mathbf{F_1}B|S, \mathbf{F_n}(A \to B)}\mathbf{F}\to$$

$$\frac{S, \mathbf{F_1}(A \to B)}{S, \mathbf{T}A, \mathbf{F_1}B}\mathbf{F_1}\to \qquad \frac{S, \mathbf{F_1}(\neg A)}{S_{cl}, \mathbf{T}A}\mathbf{F_1}\neg \qquad \frac{S, \mathbf{F_1}(A \wedge B)}{S, \mathbf{F_1}A, \mathbf{F_1}B|S, \mathbf{F_1}A, \mathbf{T}B|S, \mathbf{T}A, \mathbf{F_1}B}\mathbf{F_1}\wedge$$

$$\frac{S, \mathbf{T}(A \to B)}{S_{cl}, \mathbf{T}\neg A|S, \mathbf{T}B}\mathbf{T}\to\text{-cl} \quad \frac{S, \mathbf{T}\neg\neg A}{S, \mathbf{T}A}\mathbf{T}\neg\neg\text{-cl}$$

provided S only contains $\mathbf{T}$-formulas

$$where \quad S_{cl} \quad = \quad \{\mathbf{T}A|\mathbf{T}A \in S\} \cup \{\mathbf{T}\neg A|\mathbf{F}A \in S\} \cup \{\mathbf{T}\neg A|\mathbf{F_1}A \in S\}$$

Figure 1: The invertible rules of $\mathbb{D}$.

$$\frac{S, \mathbf{T}\neg\neg A}{S_{cl}, \mathbf{T}A|S_\phi, \mathbf{T}A}\mathbf{T}\neg\neg\text{-Atom}, \text{ provided } S \text{ does not contain } \mathbf{F_n}\text{-formulas.}$$

where

$$S_\phi \quad = \quad \{\mathbf{T}A|\mathbf{T}A \in S\} \cup \{\mathbf{T}A|\mathbf{F_1}A \in S\} \text{ and}$$
$$S_{cl} \quad = \quad \{\mathbf{T}A|\mathbf{T}A \in S\} \cup \{\mathbf{T}\neg A|\mathbf{F}A \in S\} \cup \{\mathbf{T}\neg A|\mathbf{F_1}A \in S\};$$

$$\frac{S, \mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_u}{V_1|\ldots|V_j|\ldots|V_u}\mathbf{F_n}$$

where:
for $j = 1, \ldots, u,\ V_j = S_c \cup \{\mathbf{F_n}A_1, \ldots, \mathbf{F_n}A_{j-1}, \mathbf{F_1}A_j, \mathbf{F}A_{j+1}, \ldots, \mathbf{F}A_u\}$;
$S_c = \{\mathbf{T}A|\mathbf{T}A \in S\} \cup \{\mathbf{T}A|\mathbf{F_1}A \in S\}$;

Figure 2: The non-invertible rules of $\mathbb{D}$.

$$\frac{S, \mathbf{F}(A \to B), \mathbf{F_1}B}{S, \mathbf{T}A, \mathbf{F_1}B}\mathbf{F}\to_1 \qquad \frac{S, \mathbf{T}A}{S[A/\top], \mathbf{T}A}\text{Replace }\mathbf{T}$$

$$\frac{S, \mathbf{F_1}\bot}{S_{cl}}\mathbf{F_1}\bot, \qquad \frac{S, \mathbf{F_n}A, \mathbf{F_1}B}{S, \mathbf{F_n}A[B/\top], \mathbf{F_1}B}\text{Replace }\mathbf{F_1}$$

Figure 3: Optmization rules

3 Correctness, completeness and termination

To obtain the correctness of $\mathbb{D}$ with respect to Dummett logic, we proceed by showing that the existence of a proof table for $\{\mathbf{F_1}A\}$, implies the validity of A in Dummett logic. The main step consists in establishing that the rules of the calculus preserve realizability:

Proposition 2. *For every rule of $\mathbb{D}$, if a model realizes the premise, then there exists a model realizing at least one of the* *conclusions.*

The correctness of rule $\mathbf{F_n}$ proceeds following the considerations given in previous section to justify the introduction of $\mathbf{F_n}$ and is similar to the proof given in [Avellone *et al.*, 1999]. Some other cases are given in [Fiorino,]. We emphasize that rule $\mathbf{T}\neg\neg$-Atom is correct also when A is not an atom.

Theorem 1. *If there exists a closed proof table for A, then A is valid in* $\mathbf{Dum}$.

Proof. By hypothesis there exists a proof table starting from $\{\mathbf{F_l}A\}$ whose leaves are all inconsistent sets. An inconsistent set is not realizable and the rules of $\mathbb{D}$ preserve the realizability, thus $\mathbf{F_l}A$ is not realizable. By absurd, let us suppose that there exist a model $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$ and a world $\alpha \in P$ such that $\alpha \nVdash A$. Then, there exists $\beta \in P$ s.t. $\alpha \leq \beta$, $\beta \nVdash A$ and for every $\gamma \in P$ s.t $\beta < \gamma$, $\gamma \Vdash A$ holds. Thus $\beta \rhd \mathbf{F_l}A$ and, since the rules of $\mathbb{D}$ preserve the realizability, the leaves of the proof table are realized, which is impossible. We conclude that that for every model $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$ and every $\alpha \in P$, $\alpha \Vdash A$, that is A is valid in $\mathbf{Dum}$. □

As regard the completeness, the calculus contains two non-invertible rules. It is possible to devise a complete strategy that relies on respecting a particular sequence in the application of the rules: $\mathbf{T}\neg\neg$-Atom is applied if no other rule is applicable and $\mathbf{F_n}$ is applied if no other rule but $\mathbf{T}\neg\neg$-Atom is applicable. We remark that $\mathbf{F_n}$-formulas are introduced by rule $\mathbf{F}\rightarrow$, thus their main connective is the implication. Rules $\mathbf{F_n}$ and $\mathbf{T}\neg\neg$-Atom apart, the other rules of $\mathbb{D}$ are invertible, thus they are applicable at any point of the deduction without affecting the completeness. We conclude that it is possible to devise a decision procedure based on $\mathbb{D}$ where no backtracking step is necessary (see [Fiorino,] for the details).

By inspecting the rules of the calculus it follows that the procedure terminates. In particular the rightmost conclusion of the rule $\mathbf{F}\rightarrow$ only change the sign of the formula in evidence in the premise, but we remark that the rule $\mathbf{F_n}$ handles these kind of formulas, thus the combined action of $\mathbf{F}\rightarrow$, $\mathbf{F_n}$ and $\mathbf{F_l}\rightarrow$ allows us to prove that there is no an infinite loop due to the handling $\mathbf{F}\rightarrow$-formulas. Moreover, by the structure of the rules $\mathbf{T}\neg\neg\wedge$, $\mathbf{T}\neg\neg\vee$, $\mathbf{T}\neg\neg\rightarrow$ and $\mathbf{T}\neg\neg\neg$, it follows that rules $\mathbf{F}\neg$ and $\mathbf{T}\neg\rightarrow$ do not cause an infinite loop. Finally, if we split the rules $\mathbf{F}\neg$ and $\mathbf{T}\neg\rightarrow$ in different cases according to A, by following [Fiorino, 2001] we can prove that the depth of every proof table of $\mathbb{D}$ is linear in the size of the formula to be proved.

4 Optimization rules and implementation

Additional rules can be introduced to $\mathbb{D}$. Although unnecessary for completeness, in practice, they can be useful to reduce the search space. We discuss them in the following. The aim of the sign $\mathbf{F_l}$ is to label those formulas that after an application of $\mathbf{F_n}$ become equivalent to $\top$. As a matter of fact, after an application of the rule $\mathbf{F_n}$, the formulas signed with $\mathbf{F_l}$ become signed with $\mathbf{T}$, thus they are equivalent to $\top$. Hence the rule Replace $\mathbf{T}$ of Fig. 3 is applicable. The occurrences of $\top$ are removed by using the well-known boolean simplification rules [Fiorino, ; Massacci, 1998]. The rule Replace $\mathbf{T}$ and the boolean simplification rules are examples of optimization rules allowing in many circumstances the reduction of search space. Another example is the rule Replace $\mathbf{F_l}$, that exploits the meaning of $\mathbf{F_l}$ and $\mathbf{F_n}$ to perform a replacement: if $\alpha \rhd \mathbf{F_n}A, \mathbf{F_l}B$ holds, then we conclude $\beta \rhd \mathbf{F}A, \mathbf{T}B$, with β immediate successor of α. Thus $\beta \rhd \mathbf{F}A[B/\top]$ and $\alpha \rhd \mathbf{F}A[B/\top]$ hold. Compared with the calculus in paper [Fiorino, 2010], introducing the sign $\mathbf{F_l}$ has a price: proof tables can be wider, because of the combined action of $\mathbf{F}\rightarrow$ and $\mathbf{F_n}$. To reduce this problem there is the rule $\mathbf{F}\rightarrow_1$, which is not necessary to the completeness and represents another example of optimization rule that exploits the information conveyed by the $\mathbf{F_l}$-formulas. To better exploit $\mathbf{F_l}$-formulas, $\mathbb{D}$ can be extended with the sign $\mathbf{T_n}$: Let $\underline{K} = \langle P,\leq,\rho,\Vdash\rangle$ be a model and let $\alpha \in P$, $\alpha \rhd \mathbf{T_n}A$ holds iff for every $\beta > \alpha$, $\beta \rhd \mathbf{T}A$. Given a set S, a further inconsistent condition is $\{\mathbf{F_n}A, \mathbf{T_n}\bot\} \subseteq S$. The sign $\mathbf{T_n}$ makes explicit the semantical status of $\mathbf{F_l}$-formulas in the next worlds. It is a derived sign and to get the completeness it is not necessary to provide rules to handle $\mathbf{T_n}$-formulas. The sign $\mathbf{T_n}$ allows to draw conclusions about the next world, as an example:

$$\frac{S, \mathbf{T}\neg A, \mathbf{T_n}B}{S, \mathbf{T}\neg A[B/\top], \mathbf{T_n}B}\text{Replace }\mathbf{T_n}-special$$

provided a $\mathbf{F_n}$-formula is in S

$$\frac{S, \mathbf{T}A, \mathbf{F_l}B}{S, \mathbf{T}A, \mathbf{T_n}A[B/\top], \mathbf{F_l}B}\text{Replace }\mathbf{F_l}-dup$$

$$\frac{S, \mathbf{T}A, \mathbf{T_n}B}{S, \mathbf{T}A, \mathbf{T_n}A[B/\top], \mathbf{T_n}B}\text{Replace }\mathbf{T_n}-dup$$

Note that Replace $\mathbf{T_n}$-dup and Replace $\mathbf{F_l}$-dup copy the premise $\mathbf{T}A$, thus loop-free mechanisms need to be implemented.

We have developed a prolog prototype based on the extension of $\mathbb{D}$ as described above, where to avoid infinite loops in the application of Replace $\mathbf{F_l}$-dup and Replace $\mathbf{T_n}$-dup a special labelling on the copied formulas is implemented. In Fig. 4 the implementations LC-cmodels [Larchey-Wendling, 2007], EPDL [Fiorino, 2010] and our prototype, called Dum-prover, are compared on the formulas of ILTP library [Raths *et al.*, 2007]. We see that on many formulas Dum-prover outperforms EPDL. As regard family formulas SYJ202, the pigeon principle, the two provers produce the same proof.

5 Conclusions and future work

We have presented a calulus based on the idea that to prove a given formula A, the refutation starts by supposing that *there exists the last world where A is not forced*. The consequence of this semantical assumption is a calculus having rules that draw deductions about the (immediate) future and rules exploiting the information about facts known in the future to deduce information about the present. We have also developed the prolog prototype Dum-prover whose timings outperforms EPDL and LC-cmodels and show that the calculus is suitable to perform practical automated deduction in propositional Dummett logic. As regard to possible extensions, we plan to modify $\mathbb{D}$ to be able to decide n-valued Gödel logics by bounding the number of applications of non-invertible

Formula	LC-cmodels	EPDL	Dum-prover
SYJ201+1.002	43.16	0.21	0.02
SYJ201+1.003	362	2.73	0.06
SYJ201+1.004	2183	33.63	0.11
SYJ201+1.005	12186	358.07	0.19
SYJ202+1.004	4.16	0.12	0.16
SYJ202+1.005	43.79	1.02	1.34
SYJ202+1.006	N.A.	9.32	12.60
SYJ202+1.007	N.A.	100	131.15
SYJ205+1.009	N.A.	12.04	0.57
SYJ205+1.010	N.A.	30.44	0.75
SYJ205+1.011	N.A.	75.57	1.04
SYJ205+1.012	N.A.	187.17	1.28
SYJ207+1.003	46	0.27	0.03
SYJ207+1.004	200	3.33	0.07
SYJ207+1.005	805	37.78	0.11
SYJ207+1.006	3280	419.80	0.29

Formula	LC-cmodels	EPDL	Dum-prover
SYJ208+1.009	N.A.	6.55	1.40
SYJ208+1.010	N.A.	14.75	2.51
SYJ208+1.011	N.A.	30.06	4.09
SYJ208+1.012	N.A.	71.28	6.84
SYJ211+1.017	381	0.17	0.45
SYJ211+1.018	458	0.20	0.51
SYJ211+1.019	522	0.22	0.59
SYJ211+1.020	589	0.26	0.68
SYJ212+1.011	N.A.	0.25	0.05
SYJ212+1.012	N.A.	0.49	0.05
SYJ212+1.013	N.A.	1.18	0.05
SYJ212+1.014	N.A.	2.22	0.06

Figure 4: LC-cmodels, EPDL and Dum-prover on ILTP formulas.

rules inside proof-search branches. Finally, by exploiting the semantics of the sign $\mathbf{F}_1$, we are working to develop a loop-free tableau calculus for propositional Dummett logic having the subformula property.

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