A Characterization Theorem for a Modal Description Logic

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Abstract

Modal description logics feature modalities that capture dependence of knowledge on parameters such as time, place, or the information state of agents. E.g., the logic $S5_{\mathcal{ALC}}$ combines the standard description logic $\mathcal{ALC}$ with an $S5$-modality that can be understood as an epistemic operator or as representing (undirected) change. This logic embeds into a corresponding modal first-order logic $S5_{FOL}$. We prove a modal characterization theorem for this embedding, in analogy to results by van Benthem and Rosen relating $\mathcal{ALC}$ to standard first-order logic: We show that $S5_{\mathcal{ALC}}$ with only local roles is, both over finite and over unrestricted models, precisely the bisimulation-invariant fragment of $S5_{FOL}$, thus giving an exact description of the expressive power of $S5_{\mathcal{ALC}}$ with only local roles.

1 Introduction

Modal description logics extend the static knowledge model of standard description logics by adding modalities capturing, e.g., the temporal evolution of the state of the world or the dependence of knowledge on the information available to individual agents. Their semantics is typically *two-dimensional* [Gabbay *et al.*, 2003], i.e. it is defined over interpretations involving two sets of *individuals* and *worlds*, respectively, and concepts are interpreted as subsets of the Cartesian product of these two sets. For instance, temporal description logics (surveyed, e.g., by Lutz et al. [2008]) have a frame structure on the set of worlds, in the same way as in the semantics of standard temporal logics such as CTL; they support statements such as 'every person that is currently a child will eventually become an adult in the future'.

A simpler variant of the same idea is to give up directedness of temporal evolution and instead introduce a modality that reads 'at some other point in time', so that, continuing the previous example, one can express only that that every person that is currently a child is an adult at some other time. This coarser granularity buys a simplification of the semantics in which the set of worlds is just a set (equivalently, a frame whose transition relation is an equivalence), i.e. a model of the modal logic $S5$. Modal description logics with an $S5$ modality have been used prominently as *description logics of change*, and are able to encode a restricted form of temporal entity-relationship models if the description logic is strong enough (specifically, contains $\mathcal{ALCQI}$) [Artale *et al.*, 2007].

One of the simplest description logics of change in this sense is $S5_{\mathcal{ALC}}$, i.e. the extension of the standard description logic $\mathcal{ALC}$ [Baader *et al.*, 2003] with an $S5$ change modality. In fact, there are many other readings for the $S5$ modality. In particular, $S5$ modalities standardly feature in epistemic logics, and indeed $S5_{\mathcal{ALC}}$ was originally introduced as an epistemic description logic [Wolter and Zakharyaschev, 1999b]. As a variant of this view, $S5_{\mathcal{ALC}}$ and its $\mathcal{EL}$ fragment have been considered as a corner case of probabilistic description logics for subjective uncertainty, with probabilities mentioned in concepts restricted to 0 or 1 [Gutiérrez-Basulto *et al.*, 2017]. In the current work, we focus on $S5_{\mathcal{ALC}}$ as one of the most basic modal description logics, and use it as a starting point for the *correspondence theory* of modal description logics.

Specifically, $S5_{\mathcal{ALC}}$ embeds as a fragment into the modal first-order logic $S5_{FOL}$, which extends standard first-order logic with an $S5$ modality and lives over the same type of semantic structures as $S5_{\mathcal{ALC}}$. This situation is analogous to the one with $\mathcal{ALC}$ itself, which embeds as a fragment into ordinary first-order logic (FOL). For $\mathcal{ALC}$, it is straightforward to check that its concepts are *bisimulation-invariant*, i.e. bisimilar individuals satisfy the same $\mathcal{ALC}$-concepts. This constitutes in effect an upper bound on the expressivity of $\mathcal{ALC}$: any property that fails to be bisimulation-invariant (such as 'individual x is related to itself under role r') is not expressible in $\mathcal{ALC}$. Remarkably, it can be shown that this is also a lower bound: *every* bisimulation-invariant *first-order* property can be expressed in $\mathcal{ALC}$, a fact first proved by van Benthem [1976] and later shown to hold true also over finite structures by Rosen [1997]. In other words, $\mathcal{ALC}$ is precisely the bisimulation-invariant fragment of FOL; we refer to theorems of this type as *modal characterization theorems*. In this terminology, the object of this paper is to establish a modal characterization theorem for $S5^{loc}_{\mathcal{ALC}}$, the fragment of $S5_{\mathcal{ALC}}$ determined by admitting only local (i.e. non-modalized) roles: We show that both over unrestricted and over finite interpretations, $S5^{loc}_{\mathcal{ALC}}$ *is precisely the bisimulation-invariant fragment of* $S5_{FOL}$, where both bisimulation invariance and equivalence to a modal formula are understood over two-dimensional interpretations. Technically,

*Work forms part of DFG project *ProbDL2* (SCHR 1118/6-2)

we follow a generic recipe suggested by Otto [2004], which relies on *locality* w.r.t. Gaifman distance. In fact, the main challenge in our proof is to identify a suitable notion of Gaifman distance for $S5_{FOL}$, and relate it to numbers of rounds played in bisimulation games (Remark 4.3). Summing up, we pin down the exact expressivity of $S5^{loc}_{\mathcal{ALC}}$ as a fragment of $S5_{FOL}$; to our best knowledge, this is the first time a modal characterization theorem is obtained for a many-dimensional modal logic or a modal description logic.

Proofs are sometimes omitted or only sketched; see [Wild and Schröder, 2017] for a full version.

Related Work In the one-dimensional case, the original van Benthem / Rosen characterization theorem has been extended in various directions, e.g. for logics with frame conditions [Dawar and Otto, 2005], coalgebraic modal logics [Schröder *et al.*, 2015], fragments of XPath [ten Cate *et al.*, 2010; Figueira *et al.*, 2015; Abriola *et al.*, 2017], neighbourhood logic [Hansen *et al.*, 2009], modal and first order logic with team semantics [Kontinen *et al.*, 2015], modal μ-calculi (within monadic second order logics) [Janin and Walukiewicz, 1995; Enqvist *et al.*, 2015], and PDL (within weak chain logic) [Carreiro, 2015].

In the many-dimensional setting, all existing characterization results that we are aware of look in the other direction, from the perspective of modal first-order logics: they characterize modal first-order logics as fragments of even more expressive two-sorted first order logics that make the worlds explicit. Specifically, van Benthem [2001] proves this for unrestricted frames in the modal dimension, i.e., in the nomenclature scheme we use here, for K_{FOL}, while Sturm and Wolter [2001] characterize $S5_{FOL}$ as as a fragment of two-sorted FOL; in both cases, the relevant notion of equivalence is essentially bisimilarity in the modal dimension, and Ehrenfeucht-Fraïssé equivalence in the first-order dimension. Both results are proved only over unrestricted models, and their proofs rely on compactness. The characterization theorem for $S5_{FOL}$ can in fact be combined with our characterization of $S5_{\mathcal{ALC}}$ as a fragment of $S5_{FOL}$ to obtain, over unrestricted models, a stronger characterization of $S5_{\mathcal{ALC}}$ as the bisimulation-invariant fragment of the two-sorted first-order correspondence language (Corollary 4.13 below).

2 $S5$-modalized $\mathcal{ALC}$ and FOL

We recall the syntax of modalized $\mathcal{ALC}$ as introduced by Wolter and Zakharyaschev [1999b], restricting to a single modality: Concepts C, D of $S5^{loc}_{\mathcal{ALC}}$ (*$S5$-modalized $\mathcal{ALC}$ with only local roles*) are given by the grammar

$$C, D ::= A \mid \neg C \mid C \sqcap D \mid \exists r.\, C \mid \Box C$$

where as usual A ranges over a set $\mathsf{N_C}$ of *(atomic) concept names* and r over a set $\mathsf{N_R}$ of *role names*. The remaining Boolean connectives $\top, \bot, \sqcup$, as well as universal restrictions $\forall r.\, C$, are encoded as usual. The *rank* of an $S5^{loc}_{\mathcal{ALC}}$-concept C is the maximal nesting depth of modalities $\Box$ and existential restrictions $\exists r$ in C (e.g. $\exists r.\, \Box A$ has rank 2).

An $S5$-*interpretation*

$$\mathcal{I} = (W^{\mathcal{I}}, \Delta^{\mathcal{I}}, ((-)^{\mathcal{I},w})_{w \in W^{\mathcal{I}}})$$

consists of nonempty sets $W^{\mathcal{I}}$, $\Delta^{\mathcal{I}}$ of of *worlds* and *individuals*, respectively, and for each world $w \in W^{\mathcal{I}}$ a standard $\mathcal{ALC}$ interpretation $(-)^{\mathcal{I},w}$ over $\Delta^{\mathcal{I}}$, i.e. for each concept name $A \in \mathsf{N_C}$ a subset $A^{\mathcal{I},w} \subseteq \Delta^{\mathcal{I}}$, and for each role name $r \in \mathsf{N_R}$ a binary relation $r^{\mathcal{I},w} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. We refer to $\Delta^{\mathcal{I}}$ as the *domain* of $\mathcal{I}$. The interpretation $C^{\mathcal{I},w} \subseteq \Delta^{\mathcal{I}}$ of a composite concept C at a world w is then defined recursively by the usual clauses for the $\mathcal{ALC}$ constructs ($(\neg C)^{\mathcal{I},w} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I},w}$, $(C \sqcap D)^{\mathcal{I},w} = C^{\mathcal{I},w} \cap D^{\mathcal{I},w}$, $(\exists r.\, C)^{\mathcal{I},w} = \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I},w}.\, (d,e) \in r^{\mathcal{I},w}\}$), and by

$$(\Box C)^{\mathcal{I},w} = \{d \in \Delta^{\mathcal{I}} \mid \forall v \in W^{\mathcal{I}}.\, d \in C^{\mathcal{I},v}\}.$$

In words, $\Box C$ denotes the set of individuals that belong to C in all worlds. As usual, we write $\Diamond$ for the dual of $\Box$, i.e. $\Diamond C$ abbreviates $\neg\Box\neg C$ and denotes the set of all individuals that belong to C in some world. We write $\mathcal{I}, w, d \models C$ if $d \in C^{\mathcal{I},w}$.

$S5$-interpretations are *two-dimensional* in the sense that concepts are effectively interpreted as subsets of Cartesian products $W^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and the modalities $\Box$ and $\exists r$ are interpreted by relations that move only in one dimension of the product: $\Box$ moves only in the world dimension and keeps the individual fixed, and vice versa for $\exists r$. Thus, $S5^{loc}_{\mathcal{ALC}}$ and $S5_{\mathcal{ALC}}$ (Remark 2.2) are examples of *many-dimensional modal logics* [Marx and Venema, 1996; Gabbay *et al.*, 2003].

As indicated in the introduction, there are various readings that can be attached to the modality $\Box$. E.g. if we see $\Box$ as a *change modality* [Artale *et al.*, 2007], and, for variety, consider spatial rather than temporal change, then the concept

$$\exists\, \mathsf{isMarriedTo}.\, (\neg C \sqcap \Diamond C)$$
$$\text{where } C = \exists\, \mathsf{wantedBy}.\, \mathsf{LawEnforcement}$$

describes persons married to fugitives from the law, i.e. to persons that are wanted by the police in some place but not here. As an example where we read $\Box$ as an epistemic modality 'I know that' [Wolter and Zakharyaschev, 1999b; Gabbay *et al.*, 2003], the concept

$$\Box\exists\, \mathsf{has}.\, (\mathsf{Gun} \sqcap \mathsf{Concealed} \sqcap \Diamond \mathsf{Loaded})$$

applies to people who I know are armed with a concealed gun that as far as I know might be loaded.

A more expressive modal language is *$S5$-modalized first-order logic with constant domains*, which we briefly refer to as $S5_{FOL}$. Formulas ϕ, ψ of $S5_{FOL}$ are given by the grammar

$$\phi, \psi ::= R(x_1, \dots, x_n) \mid x = y \mid \neg\phi \mid \phi \sqcap \psi \mid \exists x.\, \phi \mid \Box\phi$$

where x, y and the x_i are variables from a fixed countably infinite reservoir and R is an n-ary predicate from an underlying *language* of predicate symbols with given arities. The quantifier $\exists x$ binds the variable x, and we have the usual notions of free and bound variables in formulas. The *rank* of a formula ϕ is the maximal nesting depth of modalities $\Box$ and quantifiers $\exists x$ in ϕ; e.g. $\exists x.\, \Box(\forall y.\, r(x,y))$ has rank 3. This is exactly the $S5$ modal first-order logic called $\mathcal{QML}$ by Sturm and Wolter [2001]. From now on we fix the language to be the *correspondence language* of $S5_{\mathcal{ALC}}$, which has a unary predicate symbol A for each concept name A and a binary predicate symbol r for each role name r. The semantics of

$S5_{FOL}$ is then defined over $S5$-interpretations, like $S5^{loc}_{\mathcal{ALC}}$. It is given in terms of a satisfaction relation $\models$ that relates an interpretation $\mathcal{I}$, a world $w \in W^{\mathcal{I}}$, and a *valuation* η assigning a value $\eta(x) \in \Delta^{\mathcal{I}}$ to every variable x on the one hand to a formula ϕ on the other hand. The relation $\models$ is defined by the expected clauses for Boolean connectives, and

$$\mathcal{I}, w, \eta \models R(x_1, \dots, x_n) \Leftrightarrow (\eta(x_1), \dots, \eta(x_n)) \in R^{\mathcal{I},w}$$
$$\mathcal{I}, w, \eta \models x = y \Leftrightarrow \eta(x) = \eta(y)$$
$$\mathcal{I}, w, \eta \models \exists x.\, \phi \Leftrightarrow \mathcal{I}, w, \eta[x \mapsto d] \models \phi \text{ for some } d \in \Delta^{\mathcal{I}}$$
$$\mathcal{I}, w, \eta \models \Box\phi \Leftrightarrow \mathcal{I}, v, \eta \models \phi \text{ for all } v \in W^{\mathcal{I}}$$

(where $\eta[x \mapsto d]$ denotes the valuation that maps x to d and otherwise behaves like η). That is, the semantics of the first-order constructs is as usual, and that of $\Box$ is as in $S5_{\mathcal{ALC}}$. We often write valuations as vectors $\bar{d} = (d_1, \dots, d_n) \in (\Delta^{\mathcal{I}})^n$, which list the values assigned to variables $x_1, \dots, x_n$ if the free variables of ϕ are contained in $\{x_1, \dots, x_n\}$.

To formalize the obvious fact that $S5^{loc}_{\mathcal{ALC}}$ *is a fragment of* $S5_{FOL}$, we extend the usual standard translation to $S5_{\mathcal{ALC}}$: a translation ST_x that maps $S5_{\mathcal{ALC}}$-concepts C to $S5_{FOL}$-formulas $\mathsf{ST}_x(C)$ with a single free variable x is given by

$$\mathsf{ST}_x(A) = A(x)$$
$$\mathsf{ST}_x(\exists r.\, C) = \exists y.\, (r(x, y) \sqcap \mathsf{ST}_y(C)) \quad (y \text{ fresh})$$

and commutation with all other constructs. Then ST_x preserves the semantics, i.e.

Lemma 2.1. *For every $S5^{loc}_{\mathcal{ALC}}$-concept C, interpretation $\mathcal{I}$, $w \in W^{\mathcal{I}}$, and $d \in \Delta^{\mathcal{I}}$, we have*

$$\mathcal{I}, w, d \models C \quad \textit{iff} \quad \mathcal{I}, w, d \models \mathsf{ST}_x(C).$$

Remark 2.2. Modalized $\mathcal{ALC}$ has been extended with *modalized roles* [Wolter and Zakharyaschev, 1999a], i.e. roles of the form $\Box r$ or $\Diamond r$, interpreted as

$$(\Box r)^{\mathcal{I},w} = \{(d, e) \mid \forall w \in W^{\mathcal{I}}.\, (d, e) \in r^{\mathcal{I},w}\}$$
$$(\Diamond r)^{\mathcal{I},w} = \{(d, e) \mid \exists w \in W^{\mathcal{I}}.\, (d, e) \in r^{\mathcal{I},w}\}.$$

The $S5$-modalized description logic in this extended sense has been termed $S5_{\mathcal{ALC}}$ by Gabbay et al. [2003]; so in our notation $S5^{loc}_{\mathcal{ALC}}$ is the fragment of $S5_{\mathcal{ALC}}$ without modalized roles. Since modalized roles $\Box r$, $\Diamond r$ have an interpretation that is independent of the world while that of basic roles r varies between worlds, the latter are called *local* roles, explaining the slightly verbose terminology used above. We will see that $S5_{\mathcal{ALC}}$ fails to be bisimulation-invariant, and is therefore strictly more expressive than $S5^{loc}_{\mathcal{ALC}}$.

3 Bisimulation and Invariance

We proceed to introduce the relevant notion of bisimulation for $S5$-interpretations. This is just the usual notion of bisimilarity, specialized to the two-dimensional shape of $S5$-interpretations and the $S5$ structure of the world dimension; explicitly:

Definition 3.1 (Bisimulation)**.** A *bisimulation* between interpretations $\mathcal{I}$, $\mathcal{J}$ is a relation

$$R \subseteq (W^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \times (W^{\mathcal{J}} \times \Delta^{\mathcal{J}})$$

such that whenever $(w, d)R(v, e)$, then

1. $d \in A^{\mathcal{I},w}$ iff $e \in A^{\mathcal{J},v}$ for all $A \in \mathsf{N_C}$;
2. for every $w' \in W^{\mathcal{I}}$ there is $v' \in W^{\mathcal{J}}$ such that $(w', d)R(v', e)$;
3. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.
4. for every $(d, d') \in r^{\mathcal{I},w}$ ($r \in \mathsf{N_R}$) there is e' such that $(e, e') \in r^{\mathcal{J},v}$ and $(w, d')R(v, e')$
5. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.

We say that $\mathcal{I}, w, d$ and $\mathcal{J}, v, e$ are *bisimilar*, and write

$$\mathcal{I}, w, d \approx \mathcal{J}, v, e$$

if there exists a bisimulation R such that $(w, d)R(v, e)$.

We record explicitly that $S5^{loc}_{\mathcal{ALC}}$ is *bisimulation-invariant*, a fact that is immediate from bisimulation invariance of basic multi-modal logic (over all interpretations, including $S5$-interpretations). As a general manner of speaking, whenever P is any property that applies to triples $\mathcal{I}, w, d$ consisting of an $S5$-interpretation $\mathcal{I}$, $w \in W^{\mathcal{I}}$, and $d \in \Delta^{\mathcal{I}}$ (e.g. P could be an $S5^{loc}_{\mathcal{ALC}}$-concept or an $S5_{FOL}$-formula with one free variable), then we say that P is *bisimulation-invariant*, or just $\approx$*-invariant*, if whenever $\mathcal{I}, w, d \approx \mathcal{J}, v, e$ then $\mathcal{I}, w, d$ has property P iff $\mathcal{J}, v, e$ has property P. We will extend this terminology without further comment to other notions of equivalence that we introduce later, such as bisimilarity up to finite depth and Ehrenfeucht-Fraïssé equivalence. Moreover, we will consider restrictions of these notions to finite $S5$-interpretations; e.g. *bisimulation-invariance over finite $S5$-interpretations* of a property P is defined like bisimulation-invariance of P above but with $\mathcal{I}$ and $\mathcal{J}$ assumed to be finite. In these terms, we have

Lemma 3.2 (Bisimulation invariance)**.** *Every $S5^{loc}_{\mathcal{ALC}}$-concept is $\approx$-invariant.*

Example 3.3. As indicated in the introduction, bisimulation invariance is an upper bound on the expressivity of $S5_{\mathcal{ALC}}$. As an extremely simple example, the formula $r(x, x)$ of $S5_{FOL}$ fails to be $\approx$-invariant and is therefore, by Lemma 3.2, not equivalent to (the standard translation of) any $S5^{loc}_{\mathcal{ALC}}$-concept. Bisimulation invariance also separates $S5^{loc}_{\mathcal{ALC}}$ from $S5_{\mathcal{ALC}}$ (Remark 2.2): the $S5_{\mathcal{ALC}}$-concept $\exists\Diamond r.\, A$ fails to be $\approx$-invariant and is therefore not expressible in $S5^{loc}_{\mathcal{ALC}}$.

Bisimulation games As usual, bisimilarity can equivalently be captured in terms of games. Explicitly:

Definition 3.4 (Bisimulation game)**.** Let $\mathcal{I}$, $\mathcal{J}$ be $S5$-interpretations, and let $(w_0, d_0) \in W^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $(v_0, e_0) \in W^{\mathcal{J}} \times \Delta^{\mathcal{J}}$. The *bisimulation game for $\mathcal{I}, w_0, d_0$ and $\mathcal{J}, v_0, e_0$* is played by players S (*Spoiler*) and D (*Duplicator*), where D means to establish bisimilarity and S aims to disprove it. A *configuration* of the game is a quadruple $((w, d), (v, e)) \in (W^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \times (W^{\mathcal{J}} \times \Delta^{\mathcal{J}})$, with $((w_0, d_0), (v_0, e_0))$ being the initial configuration. A *round* consists of one move by S and a subsequent move by D, with the following alternatives in the current configuration $((w, d), (v, e))$:

1. S may pick a world $w' \in W^{\mathcal{I}}$, and D then needs to pick a world $v' \in W^{\mathcal{J}}$; the new configuration then is $((w', d), (v', e))$.

2. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.
3. S may pick a role $r \in \mathsf{N_R}$ and an individual $d' \in \Delta^{\mathcal{I}}$ such that $(d, d') \in r^{\mathcal{I},w}$. Then D needs to pick an individual $e' \in \Delta^{\mathcal{J}}$ such that $(e, e') \in r^{\mathcal{J},v}$; the new configuration reached is $((w, d'), (v, e'))$.
4. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.

We will call the first two kinds of moves W-*moves* and the other two kinds Δ-*moves*. If one of the players cannot move, then the other one wins. A configuration $((w, d), (v, e))$ is *winning for* S if $d \in A^{\mathcal{I},w}$ and $e \notin A^{\mathcal{J},v}$ for some concept name $A \in \mathsf{N_C}$, or vice versa; and S wins if a winning configuration for S is reached. Infinite plays that do not visit a winning configuration for S are won by D.

The following is then standard:

Lemma 3.5. *We have $\mathcal{I}, w, d \approx \mathcal{J}, v, e$ iff D wins the bisimulation game for $\mathcal{I}, w, d$ and $\mathcal{J}, v, e$.*

The bisimulation game can be restricted to a finite number of rounds, capturing bisimulation up to finite depth:

Definition 3.6 (Finite-depth bisimulation)**.** The *n-round bisimulation game* for $n \geq 0$ is played in the same way as the bisimulation game but only for at most n rounds. The winning conditions are the same as in the bisimulation game except that D now wins if no winning configuration for S has been reached after n rounds. We say that $\mathcal{I}, w, d$ and $\mathcal{J}, v, e$ are *depth-n bisimilar*, and write

$$\mathcal{I}, w, d \approx_n \mathcal{J}, v, e,$$

if D wins the n-round bisimulation game for $\mathcal{I}, w, d$ and $\mathcal{J}, v, e$.

Again, the following is standard:

Lemma 3.7 (Invariance under finite-depth bisimulation)**.** *Every $S5^{loc}_{\mathcal{ALC}}$-concept C of rank at most n is $\approx_n$-invariant.*

For technical purposes, we shall need a normalization of the bisimulation game based on the observation that due to the $S5$ structure of the worlds, S can never gain an advantage from playing more than one consecutive W-move. Formally:

Definition 3.8 (Alternating bisimulation game)**.** The *alternating bisimulation game* is played like the bisimulation game but with a restriction on the sequence of moves: Each *round* in the alternating bisimulation game consists of two *phases*,

1. S may decide to make a W-move, and in this case D also makes a W-move, according to Item 1 or Item 2 of Definition 3.4, and then
2. S and D each play exactly one Δ-move according to Item 3 or Item 4 of Definition 3.4

(where D needs to avoid winning configurations for S at all times). The alternating bisimulation game also comes in two variants, the unbounded and the n-round game, with the proviso that at the end of an n-round game, there may be one extra phase of type 1 above. We write

$$\mathcal{I}, w, d \approx^{\mathsf{alt}} \mathcal{J}, v, e \quad \text{and} \quad \mathcal{I}, w, d \approx^{\mathsf{alt}}_n \mathcal{J}, v, e$$

if D has a winning strategy in the alternating and in the n-round alternating bisimulation game for $\mathcal{I}, w, d$ and $\mathcal{J}, v, e$, respectively.

The unrestricted game is equivalent to the alternating game in the following sense:

Lemma 3.9. *For interpretations $\mathcal{I}, \mathcal{J}$ and $(w, d) \in W^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $(v, e) \in W^{\mathcal{J}} \times \Delta^{\mathcal{J}}$:*

1. *If $\mathcal{I}, w, d \approx_{2n+1} \mathcal{J}, v, e$, then $\mathcal{I}, w, d \approx^{\mathsf{alt}}_n \mathcal{J}, v, e$.*
2. *If $\mathcal{I}, w, d \approx^{\mathsf{alt}}_n \mathcal{J}, v, e$, then $\mathcal{I}, w, d \approx_n \mathcal{J}, v, e$.*
3. *$\mathcal{I}, w, d \approx \mathcal{J}, v, e$ iff $\mathcal{I}, w, d \approx^{\mathsf{alt}} \mathcal{J}, v, e$.*

4 The Modal Characterization Theorem

We proceed to state our main result and sketch its proof: *$S5^{loc}_{\mathcal{ALC}}$ is the bisimulation-invariant fragment of $S5_{FOL}$*, both over finite and over unrestricted $S5$-interpretations. Formally,

Theorem 4.1 (Modal characterization)**.** *Let $\phi = \phi(x)$ be an $S5_{FOL}$-formula with one free variable x. If ϕ is $\approx$-invariant (over finite $S5$-interpretations), then there exists an $S5^{loc}_{\mathcal{ALC}}$-concept C such that ϕ is logically equivalent to $\mathsf{ST}_x(C)$ (over finite $S5$-interpretations). Moreover, the rank of C is exponentially bounded in the rank of ϕ.*

While modal characterization theorems over unrestricted structures can often be proved using model-theoretic tools such as compactness [van Benthem, 1976], proofs that apply also to finite structures typically need to work with some form of *locality* [Otto, 2004]. In the basic, one-dimensional case, this is Gaifman locality [Gaifman, 1982], which is based on the notion of *Gaifman distance* in a first-order model: The *Gaifman graph* of the model connects two of its points if they occur together in some tuple that is in the interpretation of some relation in the model, and the Gaifman distance is then just the graph distance in the Gaifman graph. We adapt these notions for our purposes as follows.

Definition 4.2. The *Gaifman graph* of an $S5$-interpretation $\mathcal{I}$ is the undirected graph with vertex set $\Delta^{\mathcal{I}}$ that has an edge between d and e iff $d \neq e$ and either $(d, e) \in r^{\mathcal{I},w}$ or $(e, d) \in r^{\mathcal{I},w}$ for some role name r and some $w \in W^{\mathcal{I}}$. The *Gaifman distance* $D : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathbb{N} \cup \{\infty\}$ is just graph distance (length of the shortest connecting path) in the Gaifman graph, and for any tuple $\bar{d} = (d_1, \ldots, d_k) \in (\Delta^{\mathcal{I}})^k$, the *neighbourhood* $U^{\ell}(\bar{d})$ of $\bar{d}$ with radius ℓ is given by

$$U^{\ell}(\bar{d}) = \{e \in \Delta^{\mathcal{I}} \mid \min_{i=1}^{k} D(d_i, e) \leq \ell\}.$$

Remark 4.3. It may be slightly surprising that Gaifman graphs for $S5$-interpretations live only in the individual dimension, so that implicit steps between worlds are effectively discounted (a point where the $S5$ structure on worlds becomes important). The technical reason for this is that it does not seem easily possible to include the worlds in the Gaifman graph without creating unduly short paths. The fact that world steps count 0 in the Gaifman distance creates a certain amount of tension with the fact that bisimulation games do feature explicit W-moves (Definition 3.4). Our alternating bisimulation games (Definition 3.8) serve mainly to address this point.

Definition 4.4 (Locality)**.** The *restriction* $\mathcal{I}|_U$ of an $S5$-interpretation $\mathcal{I}$ to a subset $U \subseteq \Delta^{\mathcal{I}}$ is given by $W^{\mathcal{I}|_U} = W^{\mathcal{I}}$, $\Delta^{\mathcal{I}|_U} = U$, $A^{\mathcal{I}|_U,w} = A^{\mathcal{I},w} \cap U$ for $A \in \mathsf{N_C}$, and $r^{\mathcal{I}|_U,w} = r^{\mathcal{I},w} \cap (U \times U)$ for $r \in \mathsf{N_R}$. An $S5_{FOL}$-formula ϕ with k free variables is *ℓ-local* for $\ell \geq 0$ if for every $S5$-interpretation $\mathcal{I}$, $w \in W^{\mathcal{I}}$, and and $\bar{d} \in (\Delta^{\mathcal{I}})^k$,

$$\mathcal{I}, w, \bar{d} \models \phi \quad \text{iff} \quad \mathcal{I}|_{U^{\ell}(\bar{d})}, w, \bar{d} \models \phi.$$

In these terms, we organize the proof of our main result as follows, following a generic strategy proposed by Otto [2004]:

Proof of Theorem 4.1 (Sketch). For ϕ $\approx$-invariant of rank n, we prove the following steps in order:

- ϕ is ℓ-local, where $\ell = 3^n$ (Lemma 4.7).
- ϕ is $\approx_{2\ell+1}$-invariant (Lemma 4.9).
- ϕ is equivalent to a concept of rank $2\ell + 1$ (Lemma 4.12).

(The locality bound is slightly generous, for simplicity.) □

Standard FOL comes with its own notion of invariance, with respect to *Ehrenfeucht-Fraïssé equivalence* [Libkin, 2004]. This notion has been extended to $S5_{FOL}$ by complementing it with bisimilarity in the world dimension [van Benthem, 2001; Sturm and Wolter, 2001]. Here, we introduce a bounded version of this equivalence, which we phrase in game-theoretic terms; this will be instrumental in the proof of locality:

Definition 4.5 (Bounded Ehrenfeucht-Fraïssé game for $S5_{FOL}$)**.** Let $\mathcal{I}$, $\mathcal{J}$ be $S5$-interpretations, let $(w_0, d_0) \in W^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $(v_0, e_0) \in W^{\mathcal{J}} \times \Delta^{\mathcal{J}}$, and let $n \geq 0$. The *n-round Ehrenfeucht-Fraïssé game for $\mathcal{I}, w_0, d_0$ and $\mathcal{J}, v_0, e_0$* is played by players S and D. The *configurations* are quadruples $((w, \bar{d}), (v, \bar{e}))$, where $w \in W^{\mathcal{I}}$, $v \in W^{\mathcal{J}}$ and $\bar{d}$ and $\bar{e}$ are finite sequences over $\Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{J}}$, respectively. The initial configuration is $((w_0, d_0), (v_0, e_0))$. The possible moves from configuration $((w, \bar{d}), (v, \bar{e}))$ are:

1. S may pick a world $w' \in W^{\mathcal{I}}$, and D then needs to pick a world $v' \in W^{\mathcal{J}}$; the new configuration is $((w', \bar{d}), (v', \bar{e}))$.
2. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.
3. S may pick some $d \in \Delta^{\mathcal{I}}$ and D then needs to pick $e \in \Delta^{\mathcal{J}}$. The new configuration is $((w, \bar{d}d), (v, \bar{e}e))$.
4. Same with the roles of $\mathcal{I}$ and $\mathcal{J}$ interchanged.

The winning conditions are as in the n-round bisimulation game, except that a configuration is now *winning for S* if it fails to be a partial isomorphism. Here, $((w, (d_0, \ldots, d_k)), (v, (e_0, \ldots, e_k)))$ is a *partial isomorphism* if

- for all $0 \leq i, j \leq k$, $d_i = d_j \Leftrightarrow e_i = e_j$; and
- for all $0 \leq i_1, \ldots, i_m \leq k$ and m-ary relation symbols R, $(d_{i_1}, \ldots, d_{i_m}) \in R^{\mathcal{I},w} \Leftrightarrow (e_{i_1}, \ldots, e_{i_m}) \in R^{\mathcal{J},v}$.

We say that $\mathcal{I}, w_0, d_0$ and $\mathcal{J}, v_0, e_0$ are *$S5$-Ehrenfeucht-Fraïssé equivalent up to depth n*, and write

$$\mathcal{I}, w_0, d_0 \cong_n \mathcal{J}, v_0, e_0,$$

if D has a winning strategy in this game.

As announced, $S5_{FOL}$ is invariant under $S5$-Ehrenfeucht-Fraïssé equivalence. For the unbounded variant, this has been shown in earlier work [Sturm and Wolter, 2001]; for our bounded variant, invariance takes the following shape:

Lemma 4.6 (Bounded $S5$-Ehrenfeucht-Fraïssé invariance)**.** *Every $S5_{FOL}$-formula of rank at most n with one free variable is $\cong_n$-invariant.*

We use this to prove locality:

Lemma 4.7. *Let ϕ be a $\approx$-invariant $S5_{FOL}$-formula of rank n. Then ϕ is ℓ-local for $\ell = 3^n$.*

Proof (sketch). Let $\mathcal{I}$ be an $S5$-interpretation and $(w_0, d_0) \in W^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Put $\mathcal{J} = \mathcal{I}|_{U^{\ell}(d_0)}$; we need to show that $\mathcal{I}, w_0, d_0 \models \phi \Leftrightarrow \mathcal{J}, w_0, d_0 \models \phi$. By $\approx$-invariance, we can disjointly extend the domains of $\mathcal{I}$ and $\mathcal{J}$ without affecting satisfaction of ϕ. We thus extend both $\mathcal{I}$ and $\mathcal{J}$ with n copies of both $\mathcal{I}$ and $\mathcal{J}$ each, obtaining $\mathcal{I}'$ and $\mathcal{J}'$, respectively.

By Lemma 4.6, it suffices to show that $\mathcal{I}', w_0, d_0 \cong_n \mathcal{J}', w_0, d_0$. The winning strategy for D is to maintain the following invariant, where we put $\ell_i = 3^{n-i}$ for $0 \leq i \leq n$:

> If $((w, \bar{d}), (v, \bar{e}))$ is the current configuration, with $\bar{d} = (d_0, \ldots, d_i), \bar{e} = (e_0, \ldots, e_i)$, then $w = v$ and there is an isomorphism between $\mathcal{I}'|_{U^{\ell_i}(\bar{d})}$ and $\mathcal{J}'|_{U^{\ell_i}(\bar{e})}$ mapping each d_j to e_j.

D maintains the invariant as follows: Whenever S picks a new world in either interpretation, D can just pick the same world in the other interpretation, as $\mathcal{I}'$ and $\mathcal{J}'$ have the same set of worlds. Whenever S picks a new individual d in $U^{2\ell_{i+1}}(\bar{d})$ or $U^{2\ell_{i+1}}(\bar{e})$ (where $\bar{d} = (d_0, \ldots, d_i)$ and $\bar{e} = (e_0, \ldots, e_i)$), then d is in the domain or range of the isomorphism in the invariant, and D picks his response according to the isomorphism. Otherwise, D picks a 'fresh' copy of the appropriate type ($\mathcal{I}$ or $\mathcal{J}$, depending on where d lies) in the other interpretation and responds with d in that copy. □

Having proved locality of $\approx$-invariant formulas, we next establish invariance even under finite-depth bisimilarity. To this end, we need *tree unravellings* of $S5$-interpretations:

Definition 4.8 (Tree unravelling)**.** Let $\mathcal{I}$ be an interpretation and $d_0 \in \Delta^{\mathcal{I}}$. The *tree unravelling* $\mathcal{I}^*_{d_0}$ of $\mathcal{I}$ is the interpretation with set $W^{\mathcal{I}^*_{d_0}} = W^{\mathcal{I}}$ of worlds; with domain $\Delta^{\mathcal{I}^*_{d_0}}$ consisting of all paths of the form $(d_0, \ldots, d_k)$ such that for each $i \in \{0, \ldots, k-1\}$, $(d_i, d_{i+1}) \in r^{\mathcal{I},w}$ for some role name r and some world w; and with the following interpretations of concept and role names:

$$A^{\mathcal{I}^*_{d_0},w} = \{\bar{d} \in \Delta^{\mathcal{I}^*_{d_0}} \mid \pi(\bar{d}) \in A^{\mathcal{I},w}\}$$
$$r^{\mathcal{I}^*_{d_0},w} = \{(\bar{d}, \bar{d}d) \mid \bar{d} \in \Delta^{\mathcal{I}^*_{d_0}}, (\pi(\bar{d}), d) \in r^{\mathcal{I},w}\}$$

where $\pi : (d_0, \ldots, d_k) \mapsto d_k$ is projection to the last entry.

It is then easy to show that $\mathcal{I}, w, d \approx \mathcal{I}^*_d, w, d$. In fact, a bisimulation is given by the function π (and identity on the set of worlds). Also, $\mathcal{I}^*_d, w, d \approx^{\mathsf{alt}}_{\ell} \mathcal{I}^*_d|_{U^{\ell}(d)}, w, d$.

Lemma 4.9. *Let $\phi = \phi(x)$ be $\approx$-invariant and ℓ-local. Then ϕ is $\approx_{2\ell+1}$-invariant.*

Proof (sketch). Let $\mathcal{I}, w, d \approx_{2\ell+1} \mathcal{J}, v, e$ and $\mathcal{I}, w, d \models \phi$. We need to show that $\mathcal{J}, v, e \models \phi$. By Lemma 3.9, $\mathcal{I}, w, d \approx^{\mathsf{alt}}_{\ell} \mathcal{J}, v, e$. By $\approx$-invariance of ϕ, we may pass from $\mathcal{I}$ and $\mathcal{J}$ to their unravellings, and by ℓ-locality of ϕ, we may then restrict those to the radius ℓ neighbourhoods of d

and e, respectively. The resulting interpretations $\mathcal{I}_d^*|_{U^\ell(d)}$ and $\mathcal{J}_e^*|_{U^\ell(e)}$ then are trees of height at most ℓ in the individual dimension.

Now $\mathcal{I}_d^*|_{U^\ell(d)}, w, d \approx_\ell^{\mathsf{alt}} \mathcal{J}_e^*|_{U^\ell(e)}, v, e$, i.e. D wins the alternating ℓ-round bisimulation game. Due to the tree structure on the domains, D's winning strategy is also winning for the unbounded alternating bisimulation game, as eventually a leaf node will be reached and S will not have a legal move in the second phase of a round. So, using Lemma 3.9 again, D wins the unbounded ordinary bisimulation game, and therefore $\mathcal{J}, v, e \models \phi$ by $\approx$-invariance of ϕ. □

Remark 4.10. In the case of finite interpretations (the 'Rosen' part of the characterization theorem), there is a caveat: the tree unravelling of a finite interpretation is not finite in general, so we cannot use $\approx$-invariance over finite interpretations to pass from interpretations to their unravellings. To remedy this, we work with *partial unravellings* up to level ℓ instead. Such a partial unravelling is constructed by restricting the tree unravelling $\mathcal{I}_{d_0}^*$ to the radius $\ell + 1$ neighbourhood of d_0 and then identifying each leaf node $\bar{d}$ with the corresponding element $\pi(\bar{d})$ in a fresh disjoint copy of $\mathcal{I}$. The resulting interpretation is clearly finite if $\mathcal{I}$ is finite, and readily shown to be bisimilar to $\mathcal{I}$. Also, the radius ℓ neighbourhood of d_0 in the partial unravelling is a tree.

Finally, we construct an equivalent $S5_{\mathcal{ALC}}$-concept for a given formula that is invariant under finite-depth bisimulation. We will make use of normal forms, as introduced by Fine [1975].

Since the formula ϕ is fixed, we can assume w.l.o.g. that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite sets $\mathsf{N_C} = \{A_1, \dots, A_s\}$ and $\mathsf{N_R} = \{r_1, \dots, r_t\}$.

Definition 4.11. The sets nf_k and at_k of *normal forms* and *atoms* of rank $k \geq 0$, respectively, are defined by induction:

$$\mathsf{at}_k = \{A_1, \dots, A_s\} \cup \{\exists r_i.C \mid 1 \leq i \leq t, C \in \mathsf{nf}_{k-1}\} \cup \{\Diamond C \mid C \in \mathsf{nf}_{k-1}\}$$

and nf_k is the set of finite conjunctions of the form $\bigwedge_{B \in \mathsf{at}_k} \varepsilon_B B$ (according to some fixed total ordering on at_k) where each ε_B is either nothing or negation. Moreover, $\mathsf{nf}_{-1} = \emptyset$ for convenience.

These normal forms have the following properties:

- For any $\mathcal{I}, w, d$, there is exactly one normal form $C^k_{\mathcal{I},w,d}$ of rank k such that $\mathcal{I}, w, d \models C^k_{\mathcal{I},w,d}$.
- We have $\mathcal{I}, w, d \approx_k \mathcal{J}, v, e$ iff $C^k_{\mathcal{I},w,d} = C^k_{\mathcal{J},v,e}$.

Lemma 4.12. *Every $\approx_k$-invariant $S5_{FOL}$-formula ϕ with one free variable x can be expressed as an $S5^{loc}_{\mathcal{ALC}}$-concept of rank k, namely*

$$\phi \equiv \mathsf{ST}_x\big(\bigvee_{\mathcal{I},w,d \models \phi} C^k_{\mathcal{I},w,d}\big).$$

Proof. First, note that the above disjunction is finite, even though there may be infinitely many interpretations satisfying ϕ. We denote the arising $S5^{loc}_{\mathcal{ALC}}$-concept by C.

For the implication from ϕ to $\mathsf{ST}_x(C)$, just note that if $\mathcal{I}, w, d \models \phi$, then $C^k_{\mathcal{I},w,d}$ is one of the disjuncts in C.

For the reverse implication, let $\mathcal{I}, w, d \models C$ and let $C^k_{\mathcal{J},v,e}$ be a disjunct in C such that $\mathcal{I}, w, d \models C^k_{\mathcal{J},v,e}$. By the above properties of normal forms, it follows that $C^k_{\mathcal{I},w,d} = C^k_{\mathcal{J},v,e}$ and therefore $\mathcal{I}, w, d \approx_k \mathcal{J}, v, e$. By definition, $\mathcal{J}, v, e \models \phi$, so $\mathcal{I}, w, d \models \phi$ by $\approx_k$-invariance of ϕ, as desired. □

This completes the proof of Theorem 4.1 as outlined above.

Characterization within two-sorted FOL The natural first-order correspondence language for $S5_{FOL}$ [Sturm and Wolter, 2001] is a two-sorted language with sorts *domain* and *world*; for every n-ary predicate R in the $S5_{FOL}$ language, the two-sorted language has an $n+1$-ary predicate R with n arguments of sort *domain* and one additional argument of sort *world*. This language $\mathcal{SL}$ is interpreted in the standard way over two-sorted first-order structures; for the two-sorted language induced by the correspondence language of $S5^{loc}_{\mathcal{ALC}}$, these are just $S5$-interpretations. One has a translation $(-)^{\dagger_v}$ of $S5_{FOL}$ into the two-sorted first-order language, given by $R(x_1, \dots, x_n)^{\dagger_v} = R(x_1, \dots, x_n, v)$ and $(\Box\phi)^{\dagger_v} = \forall v.\, (\phi^{\dagger_v})$, and commutation with all other constructs, where v is a variable of sort *world*. Sturm and Wolter [2001] show that $S5_{FOL}$ is, over unrestricted $S5$-interpretations, precisely the fragment of $\mathcal{SL}$ that is determined by invariance under *potential $S5$-isomorphisms*, i.e. unbounded $S5$-Ehrenfeucht-Fraïssé equivalence, defined as above but without a bound on the number of rounds. Since every potential $S5$-isomorphism is a bisimulation, we can combine this result with Theorem 4.1 to obtain that $S5^{loc}_{\mathcal{ALC}}$ is the bisimulation-invariant fragment of $\mathcal{SL}$:

Corollary 4.13 (Modal characterization within $\mathcal{SL}$)**.** *Let $\phi = \phi(x, v)$ be a $\approx$-invariant formula with one variable x of sort domain and one free variable v of sort world, in the two-sorted first-order language $\mathcal{SL}$. Then there exists an $S5^{loc}_{\mathcal{ALC}}$-concept C such that ϕ is logically equivalent to $(\mathsf{ST}_x(C))^{\dagger_v}$.*

(Unlike for Theorem 4.1, there is as yet no version of Corollary 4.13 for finite $S5$-interpretations, as the characterization of $S5_{FOL}$ within $\mathcal{SL}$ is known only for the unrestricted case.)

5 Conclusions

We have proved a modal characterization theorem for the modal description logic $S5^{loc}_{\mathcal{ALC}}$, i.e. $S5_{\mathcal{ALC}}$ [Gabbay *et al.*, 2003] with only local roles. Specifically, we have shown that $S5^{loc}_{\mathcal{ALC}}$, one of the modal description logics originally introduced by Wolter and Zakharyaschev [1999b], is, both over finite and over unrestricted models, the bisimulation-invariant fragment of $S5$-modal first-order logic. By a result of Sturm and Wolter [2001], it follows moreover that $S5^{loc}_{\mathcal{ALC}}$ is, over unrestricted models, the bisimulation-invariant fragment of two-sorted FOL with explicit worlds. To our knowledge, these are the first modal characterization theorems in modal description logic.

It remains a topic of interest to obtain similar characterization theorems for other modal description logics or many-dimensional modal logics. Notably, this concerns logics whose modal dimension differs from the comparatively simple structure of $S5$, e.g. $K_{\mathcal{ALC}}$. Also, one may investigate the possibility of a modal characterization of full $S5_{\mathcal{ALC}}$, then of course with respect to a different notion of equivalence.

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