Relations Between Spatial Calculi
About Directions and Orientations (Extended Abstract)

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Abstract

A qualitative representation of space and/or time provides mechanisms that characterize the essential properties of objects or configurations. The advantages over quantitative representations can be: (1) a better match with human concepts related to natural language, and (2) better efficiency for reasoning. The two main trends in qualitative spatial constraint reasoning are topological reasoning about regions and reasoning about directions between points and straight lines and orientations of straight lines or configurations derived from points. In this work, we apply universal algebraic tools to binary qualitative calculi and their relations.

1 Introduction

In constraint-based reasoning about spatial configurations, typically a partial initial knowledge of a scene is represented in terms of qualitative constraints between spatial objects. Implicit knowledge about spatial relations is then derived by constraint propagation. See [Ligozat, 2011] for an overview and references. In this work, we apply universal algebraic tools to binary qualitative calculi and demonstrate that two calculi expressing related features but on different levels of granularity can often be connected via homomorphisms.

2 Relation Algebras for Spatial Reasoning

Standard methods developed for finite domains generally do not apply to constraint reasoning over infinite domains. The theory of relation algebras [Ladkin and Maddux, 1994; Maddux, 2006] allows for a purely symbolic treatment of constraint satisfaction problems involving relations over infinite domains. The corresponding constraint reasoning techniques were originally introduced by [Montanari, 1974], applied for temporal reasoning [Allen, 1983] and later proved to be valuable for spatial reasoning [Renz and Nebel, 1999; Isli and Cohn, 2000]. The central data for a binary calculus is given by:

- a list of (symbolic names for) base-relations, which are interpreted as relations over some domain, having the crucial “JEPD” properties of joint exhaustiveness and pairwise disjointness (a general relation is then simply a union of base-relations).
- a table for the computation of the inverses of relations.
- a table for the computation of the compositions of relations.

Then, the path consistency algorithm [Montanari, 1974] and backtracking techniques [van Beek and Manchak, 1996] are the tools used to tackle the problem of consistency of constraint networks and related problems. The mathematical background of composition in table-based reasoning is given by the theory of non-associative algebras [Maddux, 2006; Ligozat and Renz, 2004], where relations algebras are generalised by dropping associativity. These algebras treat spatial relations as abstract entities (independently of any domain) that can be combined by certain operations and governed by certain equations.

When providing examples, it is easier to start with partition schemes [Ligozat and Renz, 2004; Mossakowski et al., 2006]. A partition scheme partitions the set of binary relations over a given set U. It leads to a non-associative algebra, using weak composition, a symbolic approximation of set-theoretic composition.

Example 1. The most prominent temporal calculus is Allen’s interval calculus, which describes possible relations between intervals in linear flows of time. An interval is a pair (s, t) of real numbers such that s < t. This leads to 13 basic relations between such intervals.

Example 2. The CYC_b calculus [Isli and Cohn, 2000] is based on the domain CYC = {ϕ | −π < ϕ ≤ π} of cyclic orientations. Equivalently, these angles can be represented as oriented straight lines containing the origin of the 2D Euclidean plane associated with a reference system. Using this latter representation, Fig. 1 depicts the four base-relations r, l, o, e (e.g. “right”, “left”, “opposite”, “equal”) of CYC_b.

Example 3. The ORA_c calculus [Moratz, 2006; ?] is based on the domain of Patt = {(p, ϕ) | p ∈ ℝ^2, −π < ϕ ≤ π}
of oriented points in Euclidean plane. An oriented point consists of a point and an angle as its orientation. The full angle is divided using $n$ axes, leading to $4^n$ regions, see Fig. 2. If the points of $A$ and $B$ differ, the relation $A \sim_{l_i} B$ expresses that the difference between $B$’s and $A$’s orientations (angles) is in region $i$.

**Example 4.** The $\text{OPRA}_m^i$ calculus [Dylla, 2008] is similar to $\text{OPRA}_m$. Here, we concentrate on $\text{OPRA}_1$. The important extension is a refinement that is applied to the relations $\sim_0$, $\sim_1$, $\sim_2$, and $\sim_3$. These relations are refined by marking them with letters ’+’, ’−’, ’P’ or ’A’, according to whether the two orientations of the oriented points are positive, negative, parallel or anti-parallel. Altogether, we obtain a set of 28 base-relations.

**Example 5.** A dipole is a pair of distinct points in the Euclidean plane. Before explaining dipole-dipole relations, we first study dipole-point relations. We distinguish between whether a point lies to the left, to the right, or at one of five different locations on the straight line that passes through the corresponding dipole. Using these seven possible relations between a dipole and a point, the relations between two dipoles may be specified according to the following conjunction of four relationships:

$$A R_1 s_B \land A R_2 e_B \land B R_3 s_A \land B R_4 e_A,$$

where $R_i \in \{l, r, b, s, i, e, f\}$ with $1 \leq i \leq 4$. The formal combination gives us 2401 relations, out of which 72 relations are geometrically possible. These constitute the $\text{DRA}_f$.

$Z_{4m}$ is the residue ring; for simplicity, we set $Z_{4m} = \{0, \ldots, 4m - 1\}$.

Note that e.g. $A \sim_{l_B} B$ reads “$B$ is to the right of $A$”.

**3 Granularity and Homomorphisms**

The presented calculi offer the possibility to describe scenes on different levels of granularity. The granularity of a description is the context-dependent selection of an adequate level of detail in the description [Hobbs, 1985]. In this paper we deal with granularity in the form of modality changes. In previous work we also dealt with granularity parameters dealing with scale of resolution even enabling the representation of qualitative shape [Dorr et al., 2015; Moratz and Wallgrün, 2014; Dorr and Moratz, 2017]. Granularity plays a key role in human strategies to deal with the complexity of the spatial features of the real world. This is demonstrated nicely by an example from Hobbs (1985). In his example he points out that humans conceptualize streets as one-dimensional entities when they plan a trip, they use a two-dimensional conception when they cross a street. And in contexts where the pavement has to be dug up the street becomes a three-dimensional volume. The key importance of mechanisms to flexibly switch and translate between granularities for successful reasoning about the world is highlighted by the following quote from Hobbs (1985, p. 432):

“...Our ability to conceptualize the world at different granularities and to switch among these granularities is fundamental to our intelligence and flexibility. It enables us to map the complexities of the world around us into simple theories that are computationally tractable to reason in.

Imagine a scenario involving ships and their relative positions in the open sea (see Fig. 3). Ships can be modeled as elongated, directed entities neglecting their width or any other shape property. The resulting $\text{DRA}_{fp}$ representation uses a single dipole for each ship to be represented (see left part of Fig. 3). In the $\text{OPRA}_1$ representation in addition even the lengths of the ships are neglected (see middle part of Fig. 3). The $\text{CYC}_b$ representation abstracts away the different locations of the ships and only focuses on their relative orientation (see right part of Fig. 3).

In another example ships are represented with $\text{DRA}_{fp}$ in such a way that the start point corresponds to the position of the ship and the end point represents its current speed. More specifically, the end point denotes the future position after one minute travel (if speed and heading were constant). Then longer arrows represent faster ships in a diagram. When we have an alternative representation in $\text{OPRA}_1$, in this representation we might only focus on location and heading of the ships and abstract away from their speed. Then several $\text{DRA}_{fp}$ relations in one representation map onto a single $\text{OPRA}_1$ relation in the alternative representation. For example the three relations $\{\text{fill}, \text{ells}, \text{illr}\}$ are mapped to $\sim_{l_2}^1$ (see Fig. 5).

If different spatial calculi can be used to represent a given spatial situation at different levels of granularity, the relation between the calculi can typically be formalized as a quotient
Homomorphisms have been studied in [Ligozat and Renz, 2004; Ligozat, 2005; 2011] (mainly under the name of representations). We here introduce a more systematic treatment of homomorphisms. For non-associative algebras, we recall and refine the weaker notion of lax homomorphisms [Moratz et al., 2009; Lücke, 2012], which allow for both the embedding of a calculus into its domain, as well as relating several calculi to each other. Dually to lax homomorphisms, we can define oplax homomorphisms, which enable us to define projections from one calculus to another.

**Definition 6** (Lax homomorphism, [Moratz et al., 2009; Lücke, 2012]). Given non-associative algebras $A$ and $B$, a lax homomorphism is a homomorphism $h : A \rightarrow B$ on the underlying Boolean algebras such that:

- $h(\Delta_A) \geq \Delta_B$
- $h(a^\sim) = h(a)^\sim$ for all $a \in A$
- $h(a \circ b) \geq h(a) \circ h(b)$ for all $a, b \in A$

A lax homomorphism between complete atomic non-associative algebras is called semi-strong [Mossakowski et al., 2006] if for atoms $a, b$

$$a \circ b = \bigvee \{ c \mid (h(a) \circ h(b)) \wedge h(c) \neq 0 \}$$

This notion has been inspired by the definition of weak composition and will be used for representation homomorphisms of qualitative calculi.

**Definition 7** (Oplax homomorphism, [Moratz et al., 2009; Lücke, 2012]). Given non-associative algebras $A$ and $B$, an oplax homomorphism is a homomorphism $h : A \rightarrow B$ on the underlying Boolean algebras such that:

- $h(\Delta_A) \leq \Delta_B$
- $h(a^\sim) = h(a)^\sim$ for all $a \in A$
- $h(a \circ b) \leq h(a) \circ h(b)$ for all $a, b \in A$
A homomorphism is full iff it fully induces the structure on its direct image.

A proper homomorphism (sometimes just called a homomorphism) of non-associative algebras is a homomorphism that is lax and oplax at the same time; the above inequalities then turn into equations. Each proper homomorphism is also full. A proper injective homomorphism is also semi-strong.

An oplax homomorphism of non-associative algebras is said to be a quotient homomorphism if it is full and surjective. We have then the following facts:

**Proposition 8.** Proper quotient homomorphisms preserve the holding of equations, in particular, associativity.

**Proposition 9.** Given a quotient homomorphism \( q : A \rightarrow B, \) \( B \)'s converse and composition tables can be computed from those for \( A, \) using \( q \).

A first sample use of homomorphism is the embedding of Allen's interval relations [Allen, 1983] into \( \mathcal{DRA}_p \) via a homomorphism.

**Proposition 10** ([Moratz et al., 2011]). There is a proper homomorphism from Allen's interval algebra to \( \mathcal{DRA}_p \).

Another important application of homomorphisms is their use in the definition of a qualitative calculus. [Ligozat and Renz, 2004] define a qualitative calculus in terms of a so-called weak representation [Ligozat, 2005; 2011]: A weak representation \( \varphi : A \rightarrow \mathcal{P}(U \times U) \) is an identity-preserving and converse-preserving lax homomorphism \( \varphi \) from a complete atomic non-associative algebra \( A \) into the powerset-relation algebra of a domain \( U \).

**Definition 11.** Given weak representations \( \varphi : A \rightarrow \mathcal{P}(U \times U) \) and \( \psi : B \rightarrow \mathcal{P}(V \times V), a \in \{\text{lax, oplax, full, proper}\} \) and \( b \in \{\text{lax, oplax, proper}\} \), an \((a, b)\)-homomorphism of weak representations \( (h, i) : \varphi \rightarrow \psi \) is given by

- an \( a \)-homomorphism of non-associative algebras

\[ h : A \rightarrow B, \]

and

- a map \( i : U \rightarrow V, \) such that \( \psi \circ h = \mathcal{P}(i \times i) \circ \varphi \) if \( b \) is proper (or \( \leq, \geq \) instead of \( = \) if \( b \) is lax, oplax).

**Example 12.** The homomorphism from Prop. 10 can be extended to a (proper, proper) homomorphism of weak representations by letting \( i \) be the embedding of time intervals to dipoles on the \( x \)-axis.

A quotient homomorphism of weak representations is a (full, oplax) homomorphism of weak representations that is surjective in both components.

Given a weak representation \( \varphi : A \rightarrow \mathcal{P}(U \times U) \) and an equivalence relation \( \sim \) on \( U \) that is a congruence for \( \sim \), we obtain a quotient representation \( \varphi / \sim \). Under certain conditions, we can show that the quotient algebra indeed is a non-associative algebra. Under suitable conditions, we obtain a (full, oplax) quotient homomorphism of semi-strong representations.

**Example 13.** \( \text{Cyc}_1 \) is a quotient of \( \mathcal{OPRA}_n \). At the level of domains, it acts as follows: an oriented point \((p, \phi)\) is mapped to the orientation \( \phi \) (the point \( p \) is forgotten).

**Example 14.** \( \mathcal{DRA}_p \) (as a weak representation) is a quotient of \( \mathcal{DRA}_p \). It is obtained by forgetting the labels '+' '-' 'P' and 'A'.

**Example 15.** \( \mathcal{OPRA}_n \) is a quotient of \( \mathcal{OPRA}_n \).

In [Dylla et al., 2013], we show that \( \mathcal{OPRA}_1 \) to \( \mathcal{OPRA}_8 \) are not associative. By Prop. 8 and Ex. 15, this carries over to any \( \mathcal{OPRA}_n \).

**Example 16.** \( \mathcal{OPRA}_1 \) is a quotient of \( \mathcal{OPRA}_1^* \). It forgets the labels '+' '-' 'P' and 'A'.

**Example 17.** \( \mathcal{OPRA}_1^* \) is a quotient of \( \mathcal{DRA}_p \), by keeping start point and direction, but forgetting the end point.

By Prop. 9, the construction of \( \mathcal{OPRA}_1^* \) as a quotient allows us the computation of the converse and composition tables by applying the congruence relations to the tables for \( \mathcal{DRA}_p \). Actually, we have compared the result of this procedure with the composition table for \( \mathcal{OPRA}_1^* \) published by [Dylla, 2008] and with the tool SparQ [Wallgrün et al., 2009]. In the course of checking the full oplaxness property of the quotient homomorphism from \( \mathcal{DRA}_p \) to \( \mathcal{OPRA}_1^* \), we discovered errors in 197 entries of the composition table of \( \mathcal{OPRA}_1^* \) as it was shipped with the qualitative reasoner SparQ. The table has been corrected accordingly in the meantime.5

**Proposition 18.** Quotient homomorphisms of weak representations that are bijective in the second component preserve strength of composition.

**Corollary 19** ([Moratz et al., 2009; Lücke, 2012]). Composition in \( \mathcal{OPRA}_1^* \) is strong.

**Corollary 20.** Composition in \( \text{Cyc}_1 \) is strong.

Altogether, we get the diagram of calculi (semi-strong representations) and homomorphisms in Fig. 4.

Finally, important properties of qualitative calculi can be transferred along suitable homomorphisms:

**Proposition 21.** Given non-associative algebras \( A \) and \( B \), an oplax homomorphism \( h : A \rightarrow B \) preserves algebraic closure. An injective lax homomorphism reflects algebraic closure.

**Proposition 22.** \((\_,\oplus,\) oplax\) homomorphisms of weak representations preserve solutions for scenarios.

**Proposition 23.** Atomic (lax, oplax) homomorphisms \((h, i)\) of weak representations with injective \( h \) preserve the following property to the image of \( h \):

Algebraic closure decides scenario-consistency.

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References


