On the Undecidability of Description and Dynamic Logics with Recursion and Counting

Piero A. Bonatti
Univcrsita di Napoli "Federico II"
bonatti@na.infn.it

Abstract

The evolution of Description Logics (DLs) and Propositional Dynamic Logics produced a hierarchy of decidable logics with multiple maximal elements. It would be desirable to combine different maximal logics into one super-logic, but then inference may turn out to be undecidable. Then it is important to characterize the decidability threshold for these logics. In this perspective, an interesting open question pointed out by Sattler and Vardi [Sattler and Vardi, 1999] is whether inference in a hybrid μ-calculus with restricted forms of graded modalities is decidable, and which complexity class it belongs to. In this paper we prove that this calculus and the corresponding DL $\mu\mathcal{ALCIO}_f$ are undecidable. Second, we prove undecidability results for logics that support both a transitive closure operator over roles and number restrictions.

Keywords: Description logics, hybrid μ-calculus, regular roles, graded modalities, number restrictions.

1 Introduction

Description logics are popular knowledge representation languages, with important applications to the semantic web, software engineering and heterogeneous databases. Description logics (DLs) are strictly related to propositional dynamic logics (PDLs), that play an important role in software and protocol verification based on automated reasoning techniques. The analogies between the two frameworks are so tight that DLs and PDLs can be regarded as syntactic variants of the same family of logics.

The simplest DLs and PDLs can be easily embedded into a fragment of L^2, that is, first-order logic with two variables. Application requirements led researchers to extend these basic logics with more expressive constructs, such as *fixpoints, nominals* (that represent individuals in DLs), transitive closure operators similar to Kleene's star, and equivalents of generalized quantification called *number restrictions* (or *counting)* in DLs and *graded modalities* in PDLs. At the same time, applications require these logics to be decidable and have acceptable computational complexity.

The evolution of DLs and PDLs produced—and keeps on extending—a hierarchy of decidable logics with multiple maximal elements. Currently, two of the maximal decidable DLs are $\mu\mathcal{ALCIO}$ (featuring fixpoints and nominals [Sattler and Vardi, 1999; Bonatti, 2002]) and $\mu\mathcal{ALCIQ}$ (featuring fixpoints and number restrictions). The corresponding PDLs are the *hybrid μ-calculus* and the *μ-calculus with graded modalities,* respectively [Sattler and Vardi, 1999; Kupferman *et al,* 2002].

Of course, it would be desirable to combine the features of different maximal logics into one super-logic. A combination o f $\mu\mathcal{ALCIO}$ a $\mu\mathcal{ALCIQ}$ u l d help—for example—in describing the functional behavior of e-Services (cf. [Bonatti, 2002] and related comments on *SVC(X)* in Section 5). However, in the super-logic inference may turn out to be too complex, and in particular undecidable.

A related, interesting open question pointed out by Sattler and Vardi [Sattler and Vardi, 1999] is whether inference in the union of the hybrid μ-calculus and the μ-calculus with graded modalities is decidable, and which complexity class it belongs to. More precisely, Sattler and Vardi mention a slightly simpler logic: a hybrid μ-calculus with *deterministic programs.* Deterministic programs are a special case of graded modality, whose counterpart in DLs *arc features,* i.e., functional roles. The DL corresponding to the hybrid μ-calculus with deterministic programs is called $\mu\mathcal{ALCIO}_f$.

The main contribution of this paper is a negative answer to the above open question. We prove that the hybrid μ-calculus with deterministic programs and the correponding DL $\mu\mathcal{ALCIO}_f$ are undecidable. For this purpose, we use a novel approach based on nested fixpoints.

The second contribution is an undecidability result for logics that support number restrictions together with regular role expressions. These results show that transitive role closure can be more expressive than fixpoints in some contexts.

In the next section we recall the basic notions about DLs and t h μ-calculi. Sectin 3 is devoted to the undecidability proof for $\mu\mathcal{ALCIO}_f$ and the hybrid μ-calculus with deterministic programs. Section 4 briefly illustrates the undecidability result for the combination of transitive closure and number restrictions. Finally, Section 5 concludes the paper with a brief discussion of the results and some directions for further research. Some proofs will be omitted because of space limitations.

2 Preliminaries

The vocabulary of the description logics we deal with in this paper is specified by the following disjoint sets of symbols: a set of atomic concepts At, a set of nominals Nom, a set of concept variables Var, and a set of atomic roles AR.

The set of *roles* is the smallest superset of AR such that if R, R' are roles then R^-, $R \sqcup R'$, and R^* are roles.

Let R be a role, A" $\in$ Var and $n \in$ N. The set of *concepts* is the smallest superset of At U Nom U Var such that if C, C", D arc concepts, then $\neg C$, $C \sqcap D$, $\exists^{\leqslant n} R.C$, and $\mu X.C'$ are concepts, provided that all the free occurrences of A" in C" lie within the scope of an even number of operators -> and $\exists^{\leqslant n}$.

Semantics is based on interpretations of the form X = $(\Delta^{\mathcal{I}}$, $^7)$ where $\Delta^{\mathcal{I}}$ is a set of *individuals* and ^{I} is an interpretation function mapping each $A \in$ At U Nom onto some $A^7 \subseteq \Delta^{\mathcal{I}}$, and each $R \in$ AR onto some $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Furthermore, nominals must be mapped onto singletons. A *valuation on I* is a f u n ρ : Var $\to \wp(\Delta^{\mathcal{I}})$. As usual, $\rho[X/S]$ denotes the valuation such that $\rho[X/S](X) = S$ and for all $\neq X$, $\rho[X/S](Y) = \rho(Y)$. The meaning of inverse roles is

$$(R^-)^{\mathcal{I}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R^{\mathcal{I}}\},$$

while $(R \sqcup R')^{\mathcal{I}} = R^{\mathcal{I}} \cup R'^{\mathcal{I}}$, and $(R^*)^{\mathcal{I}}$ denotes the reflexive transitive closure of R^7.

The meaning of compound concepts is determined by pairs (I, P). By $\sharp S$ we denote the cardinality of a set S.

$$\begin{aligned}
(\neg C)^{\mathcal{I}}_{\rho} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}_{\rho} \qquad A^{\mathcal{I}}_{\rho} = A^{\mathcal{I}} \quad (A \in \mathrm{At})\\
(C \sqcap D)^{\mathcal{I}}_{\rho} &= C^{\mathcal{I}}_{\rho} \cap D^{\mathcal{I}}_{\rho} \qquad X^{\mathcal{I}}_{\rho} = \rho(X) \quad (X \in \mathrm{Var})\\
(\exists^{\leqslant n} R.C)^{\mathcal{I}}_{\rho} &= \{x \mid \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}_{\rho}\} \leqslant n\}\\
(\mu X.C)^{\mathcal{I}}_{\rho} &= \bigcap \left\{S \subseteq \Delta^{\mathcal{I}} \mid C^{\mathcal{I}}_{\rho[X/S]} \subseteq S\right\}.
\end{aligned}$$

Subscript p will be sometimes omitted when it applies to a closed concept (i.e., such that all variables are bound by p).

Other standard constructs can be derived from the above concepts. We use the symbol $\triangleq$ to define abbreviations.

$$\begin{aligned}
C \sqcup D &\triangleq \neg(\neg C \sqcap \neg D) & \top &\triangleq A \sqcup \neg A \ (A \in \mathrm{At})\\
\forall R.C &\triangleq \exists^{\leqslant 0} R.\neg C & \exists R.C &\triangleq \neg\forall R.\neg C\\
\exists^{\geqslant n} R.C &\triangleq \neg\exists^{\leqslant n-1} R.C & \nu X.C &\triangleq \neg\mu X.\neg C[X/\neg X]
\end{aligned}$$

Here $C[X/\neg X]$ is the concept obtained from C by replacing all free occurrences of X with -X".

The syntactic restrictions on concept variables make every concept C monotonic with respect to its free variables. Then $\mu X.C(X)$ and $\nu X.C(X)$ denote exactly the least and the greatest fixpoints of C(X), that can be characterized with the standard iterative constructions.

An *assertion* has the form $C \sqsubseteq D$ where C and D are closed concepts. It is satisfied by X (equivalently, X is a model of the assertion) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. A *TBox* is a finite set of assertions. A TBox is satisfiable iff it has a model, that is, an X that satisfies all the assertions in the TBox. Symmetric pairs of assertions such as $C \sqsubseteq D$ and $D \sqsubseteq C$ will be abbreviated by $C \equiv D$. A TBox T entails $C \sqsubseteq D$ if every model of T satisfies $C \sqsubseteq D$

The description logic *ACC* is a fragment of the logic described so far, supporting only atomic roles, $\neg$, $\sqcap$ and $\exists R.C$ (plus all the constructs definable from these). In *ALC,* Nom $= \emptyset$.

By convention, the name of a description logic contains *ACC* if the logic extends *ACC.* Moreover, the name contains an X if inverse roles (R^-) are supported, an O if Nom $\neq \emptyset$ a Q if number restrictions $(\exists^{\leqslant n} R.C)$ are supported, and a V if fixpoints are supported. For example, *ACCIO* denotes the extension of *ACC* with inverse roles and nominals. A subscript / indicates that roles may be declared to be functions. Note that such assertions are a special case of number restriction, as they can be expressed with axioms of the form $\top \sqsubseteq \exists^{\leqslant 1} R.\top$. If the logic supports role operators besides inversion, we list those operators as superscripts. For example $\mathcal{ALCI}^{\sqcup,*}$ denotes the extension of *ACCX* with role union and reflexive transitive closure.

Description logics can be regarded as variants of the propositional μ-calculi. Individuals correspond to possible worlds and roles correspond to accessibility relations. Atomic concepts play the role of propositional symbols. In particular, $\mu\mathcal{ALCIO}_f$ can be embedded into the hybrid μ-calculus with deterministic programs and graded modalities ((n, *P)F* and $[n, P]F$) via the following satisfiability-preserving translation. For all propositions p, and for all n > 0,

$$\begin{aligned}
\varepsilon(p) &= p & \varepsilon(F \sqcap G) &= \varepsilon(F) \wedge \varepsilon(G)\\
\varepsilon(\neg F) &= \neg\varepsilon(F) & \varepsilon(\exists^{\geqslant n} P.F) &= \langle n-1, P\rangle\varepsilon(F)\\
\varepsilon(\mu X.F) &= \mu X.\varepsilon(F) & \varepsilon(C \sqsubseteq D) &= [o]\varepsilon(\neg C \sqcup D).
\end{aligned}$$

Moreover, functional roles are mapped onto deterministic programs (whose accessibility relation is the graph of a function), and nominals are mapped onto their equivalents (called nominals, too), that in PDL terms arc propositional symbols that are true in exactly one world. Program o denotes the universal program whose accessibility relation consists of all pairs of possible worlds. The reader is referred to [Sattler and Vardi, 1999; Kupferman *et a*/., 2002; De Giacomo, 1995] for further details.

3 Undecidability of $\mu\mathcal{ALCIO}_f$

This section is devoted to the proof of the following theorem.

Theorem 3.1 *In $\mu\mathcal{ALCIO}_f$, concept satisfiability, TBox satisfiability and entailment are all undecidable.*

We find it convenient to prove this theorem by first reducing domino problems to TBox satisfiability, and then extending this result to the other decision problems.

Recall that domino problems consist in placing tiles on an infinite grid, satisfying a given set of constraints on adjacent tiles. Formally, a domino problem is a structure V — (T, *H, V)*, where T is a set of *tile types* and $H, V \subseteq T^2$ specify which tiles can be adjacent horizontally and vertically, respectively. A *solution to V* is a *tiling,* that is, a function r : $\mathbb{N}^2 \to T$, such that

1. if $\tau(x, y) = t$ and $\tau(x+1, y) = t'$ then $(t, t') \in H$, and
2. if $\tau(x, y) = t$ and $\tau(x, y+1) = t'$ then $(t, t') \in V$.

The existence of a solution for a given domino problem is known to be undecidable (cf. [Gradel *et al,* 1999]).

Domino problems are reduced to reasoning problems by characterizing (i) the grid and (ii) correct tilings. Formally, the grid is a structure $\mathcal{G} = \langle \mathbb{N}^2, h^{\mathcal{G}}, v^{\mathcal{G}} \rangle$, where $h^{\mathcal{G}}(x,y) = (x+1, y)$, and $v^{\mathcal{G}}(x,y) = (x, y+1)$, for all $x, y \in \mathbb{N}$.

In description logics, $h^{\mathcal{G}}$ and $v^{\mathcal{G}}$ can be denoted by two roles, *h* and *v*. If the two roles characterize the grid correctly (see Figure 1(a)), then characterizing the solutions of a domino problem is easy, even within simple (and decidable) description logics such as *ACC,* by means of the following assertion:

$$\top \sqsubseteq \Big(\bigsqcup_{t \in T} C_t\Big) \sqcap \Big(\bigsqcap_{t \in T} \bigsqcap_{t' \in T \setminus \{t\}} \neg(C_t \sqcap C_{t'})\Big) \sqcap \quad (1)$$
$$\bigsqcap_{t \in T} \Big[\neg C_t \sqcup \Big(\exists h. \bigsqcup_{(t,t') \in H} C_{t'} \sqcap \exists v. \bigsqcup_{(t,t') \in V} C_{t'}\Big)\Big].$$

Here for each tile type t, a distinct concept name C_f is introduced. Assertion (1) basically states that each individual is a tile (first term), that distinct tile types contain different tiles (rest of the first line), and that the tiling preserves the constraints specified by *H* and *V* (second line).

The real problem is characterizing the grid, because there is no direct way to force *h* and *v* to commute. Here we shall provide a projective characterization of *G,* that is, we shall capture exactly the class of interpretations isomorphic to the expansion $\mathcal{G}^* = \langle \mathbb{N}^2, h^{\mathcal{G}}, v^{\mathcal{G}}, O^{\mathcal{G}^*} \rangle$ of $\mathcal{G}$, where O is a unary relation and $O^{\mathcal{G}^*} = \{(0,0)\}$.

We proceed in three steps. First, *h* and *v* must be forced to be injective functions. For this purpose, we declare all roles and their converse to be functional, which is equivalent to adopting the following assertion, for all roles *R.*

$$\top \sqsubseteq (\exists^{\leqslant 1} R.\top) \sqcap (\exists^{\leqslant 1} R^-.\top) \quad (2)$$

Second, all nodes are classified with respect to their incoming and outcoming edges. Note that the domain and the range of a role *B* can be defined as follows.

$$\mathrm{dom}(R) \triangleq \exists R.\top$$
$$\mathrm{range}(R) \triangleq \exists R^-.\top$$

Now we can define the vertical and the horizontal borders of the grid *(B_v* and *B_h,* respectively), and the internal nodes *(C). O* is a nominal that represents the origin of the grid.

$$B_v \triangleq O \sqcup (\mathrm{dom}(h) \sqcap \mathrm{dom}(v) \sqcap \neg\mathrm{range}(h) \sqcap \mathrm{range}(v))$$
$$B_h \triangleq O \sqcup (\mathrm{dom}(h) \sqcap \mathrm{dom}(v) \sqcap \mathrm{range}(h) \sqcap \neg\mathrm{range}(v))$$
$$C \triangleq \mathrm{dom}(h) \sqcap \mathrm{dom}(v) \sqcap \mathrm{range}(h) \sqcap \mathrm{range}(v)$$

The following assertions state the properties of *O* and force the above concepts to cover all the domain.

$$O \sqsubseteq \mathrm{dom}(h) \sqcap \mathrm{dom}(v) \sqcap \neg\mathrm{range}(h) \sqcap \neg\mathrm{range}(v) \quad (3)$$
$$\top \sqsubseteq B_v \sqcup B_h \sqcup C \quad (4)$$

Third, we characterize the global structure of the grid. The next assertions ensure that the vertical border B_v and the horizontal border B_h have the desired structure (i.e., they should be isomorphic to N).

$$B_v \equiv \mu X\,[O \sqcup \exists v^-.X] \quad (5)$$
$$B_h \equiv \mu X\,[O \sqcup \exists h^-.X] \quad (6)$$

Note that B_v and B_h contain no cycles, because of (2) and (3), so the two fixpoints induce infinite linear sequences of nodes.

Finally, we introduce an assertion that forces h, and *v* to commute everywhere.

$$\top \sqsubseteq \mu X.\xi(X), \text{ where} \quad (7)$$
$$\xi(X) \triangleq O \sqcup \varphi_0(X) \sqcup \varphi_1(X)$$
$$\varphi_0(X) \triangleq (\exists h.\exists v^-.X) \sqcap (\exists v^-.\exists h.X)$$
$$\varphi_1(X) \triangleq B_h \sqcap \exists h^-.\mu Y.\varphi_2(X,Y)$$
$$\varphi_2(X,Y) \triangleq (B_v \sqcup \exists v.\exists h^-.Y) \sqcap X$$

Informally speaking, the constructive characterization of the fixpoint in (7) corresponds to a visit of the grid along diagonals directed north-west (Figure 1(b)). At each iteration, a new node x_0 is considered. Subformula $\varphi_0(X)$ ensures that xo is connected to the last visited element *x* in such a way that h and *v* commute. Actually, *h* and *v* are not explicitly required to commute. They actually do (equation (9)) because each visited node *x but the last one* (i.e., *x\)* must be connected *by vo h* and h~ o *v* to another visited node (Proposition 3.3.e), therefore, by (2), there exist no further links *voh~* and // o *v* connecting *x* to (the not yet visited node) X_0. It follows that only x_1 can be connected to X_0 as specified by $\varphi_0(X)$. This makes *h* and *v* commute and ensures that x_0 is unique. Every time the vertical border B_v is reached, subformula $\varphi_1(X)$ adds the first element of the next diagonal. For this purpose, subformula $\mu Y.\varphi_2(X,Y)$ looks for diagonals entirely contained in X (the set of nodes visited so far) so that a new diagonal is not entered before the previous one has been completely visited. Figure 1(c) illustrates this phase. Let A', be the current set of visited nodes. The black circles are the elements of $\varphi_1(X_i)$. In this example, $\mu Y.\varphi_2(X_i, Y)$ equals X_1, because the latter contains precisely the first 3 diagonals. In the following we formalize all the above intuitions.

Let *T* consist of the assertions (2), the local constraints (3) and (4), and the fixpoint assertions (5), (6) and (7). In the rest of this section, let I be a model of T.

First we introduce some notation related to the domain elements of *X.* Let $o \in O^{\mathcal{I}}$. For all $d, j \in \mathbb{N}$, let

$$e_{d,j} = (h^{-\mathcal{I}} \circ v^{\mathcal{I}})^j \circ (h^{\mathcal{I}})^d(o).$$

Note that e_{dj} is not necessarily defined, because $h\text{~}^J$ is partial. Informally speaking, *d* counts diagonals and *j* is the displacement within a diagonal (cf. Figure 1(b)). Indexes are ordered lexicographically. Define

$$(d_1, j_1) \preceq (d_2, j_2) \text{ iff } d_1 < d_2 \text{ or } (d_1 = d_2 \text{ and } j_1 \leqslant j_2).$$

For all sets $S \subseteq \Delta^{\mathcal{I}}$, let $\max(S)$ be the element $e_{d,j} \in S$ with $\preceq$-maximal index (if any). The restriction of $\preceq$ to $\mathbb{N}^2_{\geqslant} = \{(x,y) \in \mathbb{N}^2 \mid x \geqslant y\}$ admits a successor relation, namely:

$$\mathrm{succ}_{\preceq}(d,j) = \begin{cases} (d, j+1) & \text{if } j < d \\ (d+1, 0) & \text{if } j = d\,. \end{cases}$$

With a slight abuse of notation, $\mathrm{succ}_{\preceq}(e_{d,j})$ will denote $e_{\mathrm{succ}_{\preceq}(d,j)}$.

Next we need a constructive characterization of the fixpoint in assertion (7). To improve readability, in the following we

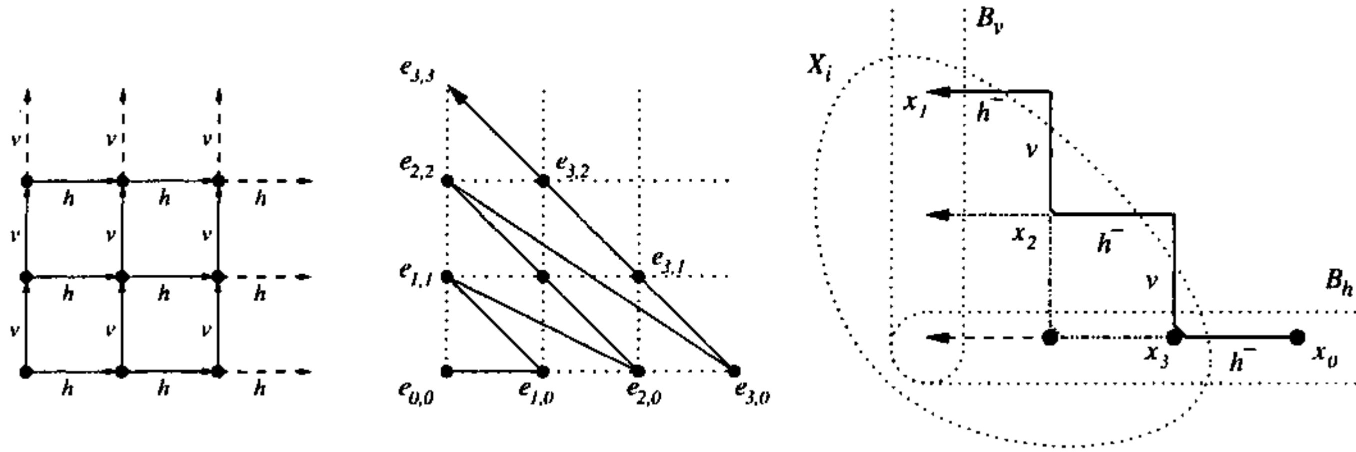

Figure 1: Modeling and visiting the grid $\mathcal{G}$

abbreviate $\xi(X)^{\mathcal{I}}_{\rho[X/S]}$ to $\xi^{\mathcal{I}}(S)$, where $S \subseteq \Delta^{\mathcal{I}}$. I a e φ_i $(i = 0, 1, 2)$ will be treated in a similar way. Then the fixpoint can be characterized as follows.

$$\begin{aligned} \mu X.\xi(X) &= \bigcup_{i<\omega} X_i, \text{ where} \qquad (8) \\ X_0 &= \xi^{\mathcal{I}}(\emptyset) \\ X_i &= \xi^{\mathcal{I}}(X_{i-1}) \quad (i > 0) \end{aligned}$$

To ensure that the fixpoint is actually reached within w steps, as implicitly claimed by the above construction, it suffices to show that $\xi^{\mathcal{I}}_i$ is continuous.

Proposition 3.2 *$\xi^{\mathcal{I}}_i$ is continuous, that is, for all non-decreasing sequences $\{X_i\}_{i\leqslant\alpha}$, where α is an ordinal, $\xi^{\mathcal{I}}(\bigcup_{i<\alpha} X_i) = \bigcup_{i<\alpha} \xi^{\mathcal{I}}(X_i)$.*

The following technical result formalizes the intuitive explanation of the nested fixpoint construction.

Proposition 3.3 *For all integers $i \geqslant 0$ such that $X_i \subset X_{i+1}$, there exists $e_{d,j} \in X_{i+1}$ such that:*

a. $X_{i+1} = X_i \cup \{e_{d,j}\}$.

b. $e_{d,j} = \text{succ}_{\preceq}(\max(X_i)) = \max(X_{i+1})$.

c. If $j > 0$ then $v^{-\mathcal{I}}(e_{d,j}) = e_{d-1,j-1}$.

d. If $d > j$ then $h^{-\mathcal{I}}(e_{d,j}) = e_{d-1,j}$.

e. For all $x \in X_{i+1} \setminus \{e_{d,j}\}$, if $h^{-\mathcal{I}}(x)$ is defined then $v^{\mathcal{I}}(h^{-\mathcal{I}}(x)) \in X_{i+1}$ and $h^{-\mathcal{I}}(v^{\mathcal{I}}(x)) \in X_{i+1}$.

f. For all $e_{k,l} \in X_{i+1}$, if $e_{k,l} \in B^{\mathcal{I}}_v$ then $k = l$.

Proof. We prove a–f simultaneously, by induction on i. For $i = 0$, $X_i = O^{\mathcal{I}} = \{o\} = \{e_{0,0}\}$ and $X_{i+1} = \varphi^{\mathcal{I}}_1(X_0) = X_i \cup \{e_{1,0}\} \subset B^{\mathcal{I}}_h$. Then a–f are trivially satisfied with $(d, j) = (1, 0)$.

Now assume $i > 0$. Let $x_0 \in X_{i+1} \setminus X_i$. Then there are two possibilities: either $x_0 \in \varphi^{\mathcal{I}}_0(X_i)$ or $x_0 \in \varphi^{\mathcal{I}}_1(X_i)$. Note that these two cases are mutually exclusive, because the elements of $\varphi^{\mathcal{I}}_0(X_i)$ are in the range of $v^{\mathcal{I}}$, while the elements of $\varphi^{\mathcal{I}}_1(X_i)$ are not, because they belong to $B^{\mathcal{I}}_h$.

Case 1: $x_0 \in \varphi^{\mathcal{I}}_0(X_i)$. By definition of φ_0, there must be $x_1, x_2 \in X_i$ such that $h^{\mathcal{I}}(v^{-\mathcal{I}}(x_0)) = x_1$ and $v^{-\mathcal{I}}(h^{\mathcal{I}}(x_0)) = x_2$. Since $h^{\mathcal{I}}$ and $v^{-\mathcal{I}}$ are injective functions (by the assertions (2)), $v^{\mathcal{I}}(h^{-\mathcal{I}}(x_1)) = h^{-\mathcal{I}}(v^{\mathcal{I}}(x_2)) = x_0 \notin X_i$, and hence, by induction hypothesis c (I.H.e for short), $x_1 = x_2 = \max(X_i)$. Now, by choosing d and j such that $\max(X_i) = e_{d,j-1}$, we have $x_0 = h^{-\mathcal{I}}(v^{\mathcal{I}}(x_2)) = h^{-\mathcal{I}}(v^{\mathcal{I}}(e_{d,j-1})) = e_{d,j}$. The above equations can be rewritten as

$$\begin{aligned} v^{\mathcal{I}}(h^{-\mathcal{I}}(\max(X_i))) &= h^{-\mathcal{I}}(v^{\mathcal{I}}(\max(X_i))) = \\ &= x_0 = e_{d,j}\,. \qquad (9) \end{aligned}$$

Since (9) holds for arbitrary $x_0 \in X_{i+1} \setminus X_i$, we have that x_0 is the unique member of $X_{i+1} \setminus X_i$, and hence point *a* holds.

Now, since $\max(X_i) = e_{d,j-1}$ and $\text{succ}_{\preceq}(d, j-1) = (d, j)$, we immediately get point *b*.

To prove c, note that

$$\begin{aligned} v^{-\mathcal{I}}(e_{d,j}) &\overset{(9)}{=} v^{-\mathcal{I}}(v^{\mathcal{I}}(h^{-\mathcal{I}}(\max(X_i)))) = \\ &= h^{-\mathcal{I}}(e_{d,j-1}) \overset{\text{I.H.}d}{=} e_{d-1,j-1}\,. \qquad (10) \end{aligned}$$

To prove d, note that

$$\begin{aligned} h^{-\mathcal{I}}(e_{d,j}) &= h^{-\mathcal{I}}(v^{\mathcal{I}}(v^{-\mathcal{I}}(e_{d,j}))) \overset{(10)}{=} \\ &= h^{-\mathcal{I}}(v^{\mathcal{I}}(e_{d-1,j-1})) \overset{\text{by def.}}{=} e_{d-1,j}\,. \end{aligned}$$

Point e follows from the corresponding I.H.e and (9).

We are only left to show that / holds. The induction hypothesis *I.H.J* covers all cases but (k,l) = (d,/). If $e_{d,l} \in B^{\mathcal{I}}_v$, then $v^{-\mathcal{I}}(e_{d,j}) \in B^{\mathcal{I}}_v$, by assertion (5). Moreover, $v^{-\mathcal{I}}(e_{d,j}) = e_{d-1,j-1}$, by (10). Then $e_{d-1,j-1} \in B^{\mathcal{I}}_v$, and hence, by I.H.$f$, $d - 1 = j - 1$. Point f follows immediately.

This concludes the proof for Case 1.

Case 2: $x_0 \in \varphi_1^{\mathcal{I}}(X_i)$. By definition of φ_1, the following facts hold: (i) $x_0 \in B_h^{\mathcal{I}}$, (ii) there exist $\{x_1, \ldots, x_n\} \in X_i$ ($n > 0$) such that

$$x_1 \in B_v^{\mathcal{I}} \quad (11)$$
$$x_n = h^{-\mathcal{I}}(x_0) \quad (12)$$
$$x_k = h^{-\mathcal{I}}(v^{\mathcal{I}}(x_{k+1})) \quad (1 \leqslant k \leqslant n-1). \quad (13)$$

(cf. Figure 1(c)). By I.H.f and (11), $x_1 = e_{d-1,d-1}$ for some $d \in \mathbb{N}$. Then, by (13), $\{x_1, \ldots, x_n\}$ must equal $\{e_{d-1,d-1}, e_{d-1,d-2}, \ldots, e_{d-1,0}\}$.

Now, if $e_{d-1,d-1} \neq \max(X_i)$, then the set $\{e_{d-1,d-1}, e_{d-1,d-2}, \ldots, e_{d-1,0}\}$ would be entirely contained in X_{i-1} and hence $x_0 \in \varphi_1^{\mathcal{I}}(X_{i-1}) \subseteq X_i$, a contradiction. Consequently,

$$e_{d-1,d-1} = \max(X_i). \quad (14)$$

Note also that

$$x_0 \stackrel{(12)}{=} h^{\mathcal{I}}(x_n) = h^{\mathcal{I}}(e_{d-1,0}) = e_{d,0}. \quad (15)$$

Now we are ready to prove a–f. Let $j = 0$.

Since $x_0 = e_{d,0}$ for an arbitrary $x_0 \in X_{i+1} \setminus X_i$, point a holds.

Moreover, $\mathrm{succ}_{\preceq}(d-1, d-1) = (d, 0)$ so point b is entailed by (14).

Point c holds vacuously as $j = 0$.

Point d holds, as $h^{-\mathcal{I}}(e_{d,0}) = e_{d-1,0}$ by definition of $e_{d,0}$.

Point e is covered by I.H.e with the exception of $\max(X_i) = e_{d-1,d-1}$, that satisfies e vacuously because it is a member of $B_v^{\mathcal{I}}$ and hence $h^{-\mathcal{I}}(e_{d-1,d-1})$ is undefined.

Finally, to prove point f, note that $e_{d,0} \notin B_v^{\mathcal{I}}$ (because $e_{d,0} \in B_h^{\mathcal{I}} \setminus O^{\mathcal{I}}$), so f is satisfied vacuously when $(k,l) = (d,0)$. All the other cases are implied by I.H.f. This completes the proof. ∎

We are now ready to prove the two propositions that confirm that T characterizes the grid.

Proposition 3.4 *Every model of $\mathcal{T}$ is isomorphic to $\mathcal{G}^*$.*

Proof. Suppose $\mathcal{I}$ is a model of $\mathcal{T}$. Then (5) and (6) cause $\Delta^{\mathcal{I}}$ to be infinite. On the contrary, each X_i ($i < \omega$) is finite, by Proposition 3.3.a. It follows by assertion (7) that no X_i can be a fixpoint of $\xi^{\mathcal{I}}$ for $i < \omega$, and hence the hypothesis of Proposition 3.3—that is, $X_i \subset X_{i+1}$—is satisfied for all $i < \omega$.

As a first consequence of Propositions 3.3.a,b, we have that for all $(d,j) \in \mathbb{N}^2_{\geqslant}$, $e_{d,j}$ is defined and $\Delta^{\mathcal{I}} = \{e_{d,j} \mid (d,j) \in \mathbb{N}^2_{\geqslant}\}$. Since $X_i \subset X_{i+1}$, Propositions 3.3.a,b imply also that $e_{d,j}$ and $e_{k,l}$ are always distinct when $(d,j) \neq (k,l)$.

Then the function $f : \Delta^{\mathcal{I}} \to \mathbb{N}^2$ specified by

$$f(e_{d,j}) = (d-j, j)$$

is well defined and total. Moreover, the total function $g : \mathbb{N}^2 \to \Delta^{\mathcal{I}}$ defined by $g(i,j) = e_{i+j,j}$ satisfies $g \circ f = f \circ g = id$, so f is a bijection.

We are left to show that f is a morphism. By Proposition 3.3.c,d, we have that for all $(d,j) \in \mathbb{N}^2_{\geqslant}$, $h^{\mathcal{I}}(e_{d,j}) = e_{d+1,j}$ and $v^{\mathcal{I}}(e_{d,j}) = e_{d+1,j+1}$. Then

$$h^{\mathcal{G}}(f(e_{d,j})) = (d-j+1, j) = f(h^{\mathcal{I}}(e_{d,j})), \text{ and}$$
$$v^{\mathcal{G}}(f(e_{d,j})) = (d-j, j+1) = f(v^{\mathcal{I}}(e_{d,j})).$$

Moreover, $f(O^{\mathcal{I}}) = \{f(e_{0,0})\} = \{(0,0)\} = O^{\mathcal{G}^*}$. We conclude that / is an isomorphism.

Proposition 3.5 G* *is a model of T.*

Since the $\mu\mathcal{ALCIO}_f$ TBox T is a projective characterization of G (by Propositions 3.4 and 3.5) and the $\mu\mathcal{ALCIO}_f$ assertion (1) is satisfied only by correct tilings, we derive the following lemma.

Lemma 3.6 *Satisfiability of $\mu\mathcal{ALCIO}_f$ TBoxes is undecidable.*

We are left to extend this lemma to concept satisfiability and entailment. This is done through the following reductions.

Lemma 3.7 *In all extensions of $\mu\mathcal{ALCI}$,*

a. TBox satisfiability can be reduced to concept satisfiability in polynomial time.

b. Concept unsatisfiability can be reduced to entailment in polynomial time.

By Lemma 3.6 and Lemma 3.7, we conclude that the main result of this section. Theorem 3.1, holds.

Finally, with Theorem 3.1 and the standard embedding of description logics into propositional dynamic logics, we immediately obtain the following result.

Corollary 3.8 *Formula satisfiability in the hybrid μ-calculus with deterministic programs is undecidable.*

4 Regular roles and counting

Description logics with regular role expressions introduce a form of recursion (Kleene's star, or reflexive transitive closure) different from fixpoints. It is interesting to investigate the interplay of this form of recursion and counting.

Consider $\mathcal{ALCIQ}^{\sqcup,*}$ (the extension of *ALCIO* with role union and reflexive transitive closure). We prove its undecidability by characterizing domino problems with unbounded grids (i.e., grids without borders) in $\mathcal{ALCIQ}^{\sqcup,*}$. The grid is modelled by splitting the injective functional role *h* (resp. *v*) into the disjoint union of two roles h_0 and h_1 (resp. v_0 and v_1), alternated as shown in Figure 2(a). Nodes are then partioned into four c l ;$\eta_{i,j}$, $0 \leq i,j \leq 1$, according to their incoming and outcoming edges ($(\exists R)$ abbreviates $\exists R.\top$)

$$\eta_{i,j} \triangleq (\exists h_i) \sqcap (\exists v_j) \sqcap (\exists h^-_{1-i}) \sqcap (\exists v^-_{1-j}) \sqcap$$
$$\neg(\exists h_{1-i}) \sqcap \neg(\exists v_{1-j}) \sqcap \neg(\exists h^-_i) \sqcap \neg(\exists v^-_j)$$

All nodes are forced to belong to one of these four classes by an assertion $\top \sqsubseteq \eta_{0,0} \sqcup \eta_{0,1} \sqcup \eta_{1,0} \sqcup \eta_{1,1}$.

Furthermore, the four assertions

$$\eta_{i,j} \sqsubseteq (\forall h_i.\eta_{1-i,j}) \sqcap (\forall v_j.\eta_{i,1-j}) \quad (16)$$

(where $0 \leq i,j \leq 1$) force the desired alternation of h_0, h_1, V_0 and v_1 (cf. Figure 2(a)).

Now the compound role $(h_i \sqcup v_j)^*$ applied to a node C_0 can reach at most five different nodes c_0-c_4 (cf. Figure 2(b)), because the alternation of edges with index 0 and 1 (more precisely, assertions (16)) guarantees that c_1, c_3, c_4 have no

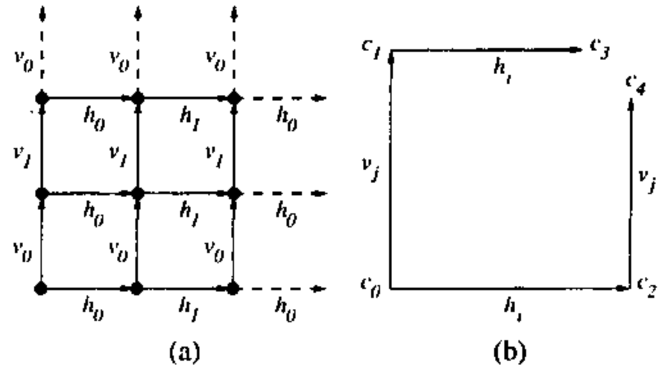

Figure 2: Modeling an even grid G

outcoming v_j edges, and that C_2, C_3 C_4 have no outcoming h_1, edges.

Then the following assertions (where $0 \leq i, j \leq 1$) imply that for each node in $n_{i,j}$ h_1 and v_j commute.

$$n_{i,j} \sqsubseteq \exists^{\leq 4}(h_i \sqcup v_j)^* . \quad (17)$$

In fact—with reference to Figure 2(b)—for all C_0 in $n_{i,j}$, assertion (16) forces nodes c_0-c_4 to belong to different (mutually disjoint) concepts $n_{x,y}$, with the exception of c_3 and C_4, both of which belong to $n_{1-i,1-j}$. Now if $c_3 \neq c_4$ then (17) would be violated (there would be 5 nodes reachable with $(h_i \sqcup v_j)^*$), so h_1 and $v_j$commute (i.e., h_0, h_1 v_0, v_1 characterize the unbounded grid).

We are only left to model correct tilings. It suffices to adapt assertion (1) by replacing h and v with $h_0 \sqcup h_1$ and $v_0 \sqcup V_1$, respectively. From the above discussion we derive the following theorem.

Theorem 4.1 $\mathcal{ALCIQ}^{\sqcup,*}$ *is undecidable.*

On the c ctrary, $\mu\mathcal{ALCIQ}^{\sqcup}$ s decidable. In fact, every expression i $\mu\mathcal{ALCIQ}^{\sqcup}$ is eqalent to an expression in the decidable logic $\mu\mathcal{ALCIQ}$ thanks to the equivalence

$$\exists^{\leq n}(R \sqcup R').C \equiv \bigsqcup_{0 \leq k \leq n} (\exists^{\leq k} R.C \sqcap \exists^{\leq n-k} R'.C).$$

Then Theorem 4.1 suggests that transitive closure is more powerful than fixpoints in this context (the extension of $\mathcal{ALCIQ}^{\sqcup}$ with fixpoints is decidable, while the extension with * is not).

5 Discussion and conclusions

Description logics evolved into a hierarchy of decidable logics with multiple maximal elements. Some support fixpoints, inverse roles, and either nominals or number restrictions (but not both) iSattler and Vardi, 1999; Kupferman *et al,* 2002]. Others support rich sets of role operators, including union and transitive closure.

The results of this paper show that the above features cannot be easily combined into one decidable logic. In particular, no decidable extension of *ALCI* can simultaneously support fixpoints, nominals and number restrictions, even in the very special case where number restrictions are confined into the functionality assertions (2).[1] As a corollary, the hybrid μ-calculus with deterministic programs is proved to be undecidable. Moreover, Theorem 4.1 shows that role union and transitive closure cannot occur together within number restrictions (it provides also evidence that * is more expressive than fixpoints).

These results have immediate implications on *VCR* [Calvanese *et al,* 1999], a rich DL with n-ary relations. Recall that *µ ALCTQ* can be embedded into $\mathcal{DLR}_\mu$ [Calvanese *et al,* 1999]. Similarly, *ALCIQ* can be embedded into *VCR* (the fragment of $\mathcal{DLR}_\mu$ without fixpoints). Then Theorem 3.1 and Theorem 4.1 imply that decidability is preserved neither by extending $\mathcal{DLR}_u$ with nominals,[2] nor by extending DL*R* with role operators U and *.

An interesting question arising from these results concerns the family of *service description logics SVC(X)* [Bonatti, 2002]. These logics are analogous to $\mathcal{DLR}_\mu$, in the sense that they model mappings (that can be regarded as n-ary relations). *SVC(X)* differs from $\mathcal{DLR}_\mu$ because the former features set abstraction and composition, while DLR_μ supports number restrictions. Service descriptions in *SVC(X)* are supposed to extend an underlying ontology written in a standard description logic *X* (modelling concepts and roles only). The main reasoning tasks for *SDL(X)* are proved to be decidable by embedding *SVC(X)* into decidable extensions of both $\mu\mathcal{ALCIO}$ and X. By the undecidability of $\mu\mathcal{ALCIO}_f$, it follows that this technique cannot be applied when *X* supports number restrictions, or simply functional roles. Then the (un)decidability of *SVC(X),* when .V supports number restrictions of some sort, remains an interesting open issue.

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'We conjecture this result still holds when functionality assertions are restricted to atomic roles only.

[2]This application of our results constitutes an alternative proof of a known result (De Giacomo, personal communication).