An Epistemic Logic for Arbitration (Extended Abstract)

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Arbitration is the process of settling a conflict between two or more persons[Liberatore and Schaerf, 1995; Revesz, 1993; 1997]. The first version of the arbitration operator between knowledge bases is proposed in [Revesz, 1993] via the so-called model-fitting operators. The postulates for model-fitting operators and the corresponding semantic characterization are presented. Arbitration is defined as a special kind of model-fitting operators.

In [Revesz, 1997], the arbitration operator is further generalized to make it applicable to weighted knowledge bases. A set of postulates is also directly used in characterizing the arbitration between a weighted knowledge base and a regular knowledge base. A weighted knowledge base in [Revesz, 1997] is defined as a mapping $\tilde{K}$ from model sets to non-negative real numbers and a regular knowledge base is just a finite set of prepositional sentences. A generalized loyal assignment is then defined as a function that assigns to each weighted knowledge base $\tilde{K}$, a pre-order $\leq_{\tilde{K}}$ between prepositional sentences so that some conditions are satisfied for the pre-orders. Finally, the arbitration of a weighted knowledge base $\tilde{K}$ by a regular knowledge base K' is defined as

$$\tilde{K}\Delta K' = \min(K', \leq_{\tilde{K}}),$$

where $\min(K', \leq_{\tilde{K}})$ is the set of $\leq_{\tilde{K}}$-minimal sentences in K'. However, this kind of arbitration is obviously syntax-dependent. For example, if φ_1 and φ_2 are two prepositional sentences such that $\varphi_1 <_{\tilde{K}} \varphi_2$, then $\tilde{K}\Delta\{\varphi_1, \varphi_2\} = \{\varphi_1\} \neq \tilde{K}\Delta\{\varphi_1 \wedge \varphi_2\} = \{\varphi_1 \wedge \varphi_2\}$ even though the two knowledge bases $\{\varphi_1, \varphi_2\}$ and $\{\varphi_1 \wedge \varphi_2\}$ are semantically equivalent.

An alternative, seemingly more natural characterization for arbitration is given in [Liberatore and Schaerf, 1995] without resorting to model-fitting operators. A knowledge base in that work is identified with a set of prepositional models. Thus the semantic characterization for this kind of arbitration is given by assigning to each subset of models *A* a binary relation $\leq_A$ over the set of model sets that satisfies the following conditions (the subscript is omitted when it means all binary relations of the form $\leq_A$):

1. **transitivity:** if $A \leq B$ and $B \leq C$ then $A \leq C$,
2. if $A \subseteq B$ then $B \leq A$,
3. $A \leq A \cup B$ or $B \leq A \cup B$,
4. $B \leq_A C$ for every C iff $A \cap C \neq \emptyset$,
5. $A \leq_{C \cup D} B \Leftrightarrow \begin{cases} C \leq_{A \cup B} D \text{ and } A \leq_C B \text{ or} \\ D \leq_{A \cup B} C \text{ and } A \leq_D B. \end{cases}$

Then the arbitration between two sets of models *A* and *B* is defined as

$$A\Delta B = \min(A, \leq_B) \cup \min(B, \leq_A) \quad (1)$$

Note that although the relationship $\leq_A$ is defined between sets of models, in the definition for arbitration, only $\leq_A$ between singletons is used. Thus by slightly abusing the notation, $\leq_A$ may also denote an ordering between models.

While arbitration is originally proposed for the merging of knowledge bases, it is also applicable in multi-agent systems. In particular, we can apply it to the reasoning about belief fusion of multiple agents. Epistemic logic is an important framework for reasoning about knowledge or belief of agentslFagin *et* al., 1996], so the incorporation of arbitration operators into epistemic logic will facilitate its application to belief fusion.

To incorporate the arbitration operator of [Liberatore and Schaerf, 1995] into epistemic logic, we must first note that according to (1), the arbitration is commutative but not necessarily associative. Therefore, the arbitration operator should be a binary one between two agents. We can directly add a class of modal operators for arbitration into epistemic logic. However, to be more expressive, we will also consider the interaction between arbitration and other epistemic operators, so we define the set of *arbitration expressions* over the agents recursively as the smallest set containing $\{1, 2, \cdots, n\}$ and closed under the binary operators +, ., and Δ. Here + and • correspond respectively to the distributed belief and the so-called "everybody knows" operators in multi-agent epistemic logic[Fagin *et* al., 1996].

Let *C* denote the language of our logic. The alphabet of £ consists of a countable set Φ_0 = {p, *q*, r,...} of atomic propositions, the prepositional constants $\perp$ (falsum or falsity constant) and $\top$ (verum or truth constant), the binary Boolean operator V (or), the unary Boolean operator ¬(not), a set *Ag* = {1,2,..., *n}* of agents, the set of arbitration expressions over *Ag,* and the auxiliary symbols "[" ,"]", "(" , T . and ",".

The set of well-formed formulas(wffs) of *C* is the smallest set containing $\Phi_0 \cup \{\perp, \top\}$ and closed under Boolean operators and the following rule:

- if φ is a wff and *a* is an arbitration expression, then $[a]\varphi$ is a wff.

As usual, other classical Boolean connectives $\wedge$ (and), $\supset$ (implication), and $\equiv$ (equivalence) can be defined as abbreviations. Also, we will write $\langle a\rangle\varphi$ as the abbreviation for $\neg[a]\neg\varphi$. Note that it has been shown that the only associative arbitration satisfying postulates 7 and 8 of [Liberatore and Schaerf, 1995] is $A\Delta B = A \cup B$, so if Δ is an associative arbitration satisfying those postulates, then $[a\Delta b]\varphi$ is reduced to $[a\cdot b]\varphi$, which is, in turn, equivalent to $[a]\varphi\wedge[b]\varphi$.

For the semantics, a possible model is a tuple

$$(W, (\mathcal{R}_i)_{1\le i\le n}, V, \le),$$

where

- W is a set of possible worlds,
- each $\mathcal{R}_i$ is a serial binary relation over W[1],
- $V : \Phi_0 \times W \rightarrow \{0,1\}$ is a truth assignment, and
- $\le$ is a function assigning to each subset of possible worlds A a binary relation $\le_A \subseteq 2^W \times 2^W$ satisfying the five conditions for the definition of arbitration operator.

Note that the first two conditions imply that $\le_A$ is a pre-order over 2^W. For each arbitration expression, we can define the binary relations $\mathcal{R}_{a\Delta b}$, $\mathcal{R}_{a\cdot b}$ and $\mathcal{R}_{a+b}$ over W recursively by

$$\mathcal{R}_{a\Delta b}(w) = \min(\mathcal{R}_a(w), \le_{\mathcal{R}_b(w)}) \cup \min(\mathcal{R}_b(w), \le_{\mathcal{R}_a(w)}) \quad (2)$$

$$\mathcal{R}_{a+b} = \mathcal{R}_a \cap \mathcal{R}_b \quad (3)$$

$$\mathcal{R}_{a\cdot b} = \mathcal{R}_a \cup \mathcal{R}_b \quad (4)$$

Thus the satisfaction for the wff $[a]\varphi$ is defined as

$$u \models [a]\varphi \text{ iff for all } w \in \mathcal{R}_a(u), w \models \varphi.$$

When the arbitration expression a is a single agent index, the formula $[a]\varphi$ corresponds to $K_a\varphi$ in epistemic logic, which means that the agent *a* knows or believes φ. We can also abbreviate $[i_1 + (i_2 + \cdots (i_{k-1} + i_k))]$ as *[G]* where $G = \{i_1, i_2, \cdots, i_k\}$. $[G]\varphi$ corresponds to the distributed knowledge formula $K_G\varphi$ of [Fagin *et al.,* 1996], which means that the combined knowledge of agents in *G* implies *ip.*

Since the set of possible worlds *W* may be infinite in our logic, the minimal models in (2) may not exist, so we define *coherent models* as those satisfying the limit assumption[Arlo-Costa and Shapiro, 1992] for each binary relations $\le_A$ such that $A \subseteq W$:

for any nonempty $U \subseteq$ IV, $\min(U, \le_A)$ is nonempty.

An axiomatic system for coherent models is presented in figure 1, where a, *b,* and *c* are meta-variables for arbitration expressions, *i* is for agent, φ and ψ are for wffs, and *G1* and *G2* are for groups of agents. Axioms (f)-(l) correspond to the postulates 2-8 of [Liberatore and Schaerf, 1995] and the limit assumption is needed for the soundness of axioms (i) and (1). Though the completeness of such a system is not yet proved, this brief presentation has shown that the modal logic approach can provide a framework for integrating epistemic reasoning and different knowledge merging operators at the object logic level.

[1]*A* binary relation R over *W* is called serial if it satisfies: $\forall w\exists u.R(w,u)$.

1. Axioms:
   - **(a) P: all tautologies of the propositional calculus**
   - **(b)** $([a]\varphi \wedge [a](\varphi \supset \psi)) \supset [a]\psi$
   - **(c)** $\neg[\{i\}]\bot$
   - **(d)** $[G_1]\varphi \supset [G_2]\varphi$ if $G_1 \subset G_2$
   - **(e)** $[a\cdot b]\varphi \equiv ([a]\varphi \wedge [b]\varphi)$
   - **(f)** $[a\Delta b]\varphi \equiv [b\Delta a]\varphi$
   - **(g)** $[a\Delta b]\varphi \supset [a+b]\varphi$
   - **(h)** $\neg[a+b]\bot \supset ([a+b]\varphi \supset [a\Delta b]\varphi)$
   - **(i)** $[a\Delta b]\bot \supset [a]\bot \wedge [b]\bot$
   - **(j)** $([a\Delta(b\cdot c)]\varphi \equiv [a\Delta b]\varphi) \vee ([a\Delta(b\cdot c)]\varphi \equiv [a\Delta c]\varphi) \vee ([a\Delta(b\cdot c)]\varphi \equiv [(a\Delta b)\cdot(a\Delta c)]\varphi)$
   - **(k)** $[a]\varphi \wedge [b]\varphi \supset [a\Delta b]\varphi$
   - **(l)** $\neg[a]\bot \supset \neg[a + (a\Delta b)]\bot$
2. Rules of Inference:
   (a) Modus ponens(MP):
   $$\frac{\varphi \quad \varphi\supset\psi}{\psi}$$
   (b) Necessitation(Nec):
   $$\frac{\varphi}{[a]\varphi}$$

Figure 1: An axiomatic system for the logic of arbitration

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