Asymptotic Conditional Probability in Modal Logic: A Probabilistic Reconstruction of Nonmonotonic Logic

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Abstract

We analyze the asymptotic conditional validity of modal formulas, i.e., the probability that a formula ψ is valid in the finite Kripke structures in which a given modal formula φ is valid, when the size of these Kripke structures grows to infinity. We characterize the formulas ψ that are almost surely valid (i.e., with probability 1) in case φ is a flat, S5-consistent formula, and show that these formulas ψ are exactly those which follow from φ according to the nonmonotonic modal logic $\mathsf{S5}_G$. Our results provide – for the first time – a probabilistic semantics to a well-known nonmonotonic modal logic, establishing a new bridge between nonmonotonic and probabilistic reasoning, and give a computational account of the asymptotic conditional validity problem in Kripke structures.

1 Introduction

Asymptotic Probabilities in Modal Logic. Asymptotic (or limit) probabilities of classical logic formulas have been investigated in various contexts [Glebskii *et al.*, 1969; Fagin, 1976; Compton, 1988; Kolaitis and Vardi, 1990; Le Bars, 1998]. Halpern and Kapron have analyzed asymptotic probability in modal logic [Halpern and Kapron, 1994], where, instead of relational structures, Kripke structures are considered, and where the size of a structure is measured in terms of the number of its worlds. Among various results, [1] they have shown that every modal formula is either almost surely true or almost surely false in finite Kripke structures. Thus, there is a *0-1 law* for modal logic K, analogous to the 0-1 law for function-free first-order logic [Fagin, 1976; Glebskii *et al.*, 1969].

To give a simple example, if p is a propositional letter of a considered finite alphabet $\mathcal{A}$, then the modal formula[2] Mp is almost surely true (w.r.t. modal logic K). In fact, it can be seen that if $n \rightarrow \infty$, then the cardinality of the set A_n of Kripke structures for $\mathcal{A}$ of size $\leq n$ satisfying Mp (i.e., where at least one world satisfying p is reachable from each world) grows much faster than the cardinality of the set B_n of Kripke structures which do not satisfy Mp, and thus $\lim_{n\rightarrow\infty} |A_n|/|C_n| = 1$ where C_n is the set of *all* Kripke structures with n worlds over the considered alphabet.

Assigning asymptotic probabilities to modal formulas provides an interesting nonstandard semantics to modal logics and has important connections to philosophy and to artificial intelligence:

PHILOSOPHY: It has been observed [Halpern and Kapron, 1994; Gottlob, 1999] that the modal formulas that are almost surely true in all Kripke models are exactly those formulas which are valid in Carnap's modal logic, exposed in his well-known foundational treatise *Meaning and Necessity* [Carnap, 1947]. Carnap argued that precisely these formulas are those to be considered logically true (*L-true*). The same logic has since be considered by various philosophers and logicians [Gottlob, 1999].

ARTIFICIAL INTELLIGENCE: Various *nonmonotonic modal logics* have been defined in the literature. Examples are autoepistemic logic [Moore, 1985], nonmonotonic logics K, S4, etc. [Marek and Truszczyński, 1993], the logic MBNF [Lifschitz, 1994] and the logic of minimal knowledge $\mathsf{S5}_\mathsf{G}$ [Halpern and Moses, 1985]. In these logics, the modal operator K is interpreted as an epistemic operator of knowledge or of belief. As observed in [Gottlob, 1999], the formulas that can be derived from an empty set of premises in *all these logics* precisely coincide with those formulas which are almost surely true in all Kripke structures. This set of formulas is furthermore identical to the *stable set* [Stalnaker, 1993] based on the empty set of formulas, thus, in a sense, to the "absolute" stable set, containing only those formulas that are epistemic consequences of the empty theory, i.e., that can be assumed in the case of total factual ignorance (for a definition of stable sets, see Section 4). We thus retain that in nonmonotonic and epistemic logics, in absence of further knowledge, a modal formula is considered true if and only if this formula is almost surely valid.

Reasoning and Asymptotic Conditional Truth. While almost sure validity provides an appealing probabilistic semantics of truth in the case of total factual ignorance, this does not yet allow us to *reason* on the basis of *premises*, which is the the most important goal of all logical formalisms, and, in particular, of nonmonotonic and epistemic logics. In the setting of limit probabilities, the inference $\varphi \models \psi$ of a (pos-

[1] Some of the results of [Halpern and Kapron, 1994] have been corrected in [Le Bars, 2002], but those referred-to in the present paper are perfectly correct in [Halpern and Kapron, 1994].

[2] We here use K as the symbol for the necessity (= knowledge) operator and M ($\equiv \neg K \neg$) for the possibility operator.

sibly modal) formula ψ from a knowledge base (theory) φ would most intuitively correspond to the statement that the conditional probability $P(\psi \mid \varphi)$ is asymptotically equal to 1. Assuming a uniform probability distribution of Kripke structures, this means that for the sets A_n and B_n of Kripke models of size n satisfying $\psi \wedge \varphi$ and φ, respectively, the limit $\lim_{n\to\infty} |A_n|/|B_n|$ exists and is equal to 1.

There is, in general, no 0-1 law for conditional probabilities in our setting, which immediately follows from previous results (see e.g. [Grove *et al.*, 1996b]) and from the fact that modal logic corresponds to a fragment of function-free first-order logic. The existence and determination of conditional probabilities in first and higher order logics has been the subject of several studies [Grove *et al.*, 1996b; 1996a]. However, to our best knowledge, conditional probabilities for modal logics have never been studied.

Observe that reasoning via conditional limit probabilities as explained above clearly constitutes a form of nonmonotonic reasoning. For example, if p is a propositional letter, then $\models_{\lim} Mp$ but $K\neg p \not\models_{\lim} Mp$, thus, adding a premise may invalidate a consequence (here $\models_{\lim}$ denotes inference of almost sure formulas under conditional limit probabilities). The nonmonotonic behaviour of conditional inference has been pointed out in the context of FO and higher order classical logic (see e.g. [Grove *et al.*, 1996b]). We deem the context of modal logic particularly interesting, because most nonmonotonic logics that have been defined are modal logics. It would thus be very interesting to know how the nonmonotonic modal logic obtained from conditional limit probabilities of Kripke structures relates to well-known and well-studied nonmonotonic modal logics.

Nonmonotonic Modal Logics. For every "classical" monotonic system of modal logic S, a nonmonotonic version NM-S is obtained by the following definition: A set E of modal formulas is an NM-S-expansion of a knowledge base φ iff $E = Cn_{\mathsf{S}}(\varphi \cup \{\neg K\gamma | \gamma \in \mathcal{L}_K(\mathcal{A}) - E\})$, where Cn_{S} is the consequence operator according to modal logic S and $\mathcal{L}_K(\mathcal{A})$ is the underlying modal language, i.e., the set of all possible modal formulas over the alphabet of propositions $\mathcal{A}$[3]. In particular, Moore's autoepistemic logic corresponds to NM-KD45. For more background, consult [Marek and Truszczyński, 1993].

It was often criticized that the above fixed-point equation is somewhat too liberal, because it allows a theory to have expansions that are not sufficiently "grounded" in the premises and contain positive knowledge that a rational agent should never conclude from the premises. For this reason, *ground nonmonotonic modal logics* have been defined by restricting the introspection of the agent to non-modal sentences. The notion of groundedness has a rather intuitive motivation: in fact, it corresponds to discarding any reasoning based on epistemic assumptions, which, for example, would enable the agent to conclude that something is true in the world, by assuming to know it. The fixed-point equation defining a S_G-*expansion* of φ is: $E = Cn_{\mathsf{S}}(\varphi \cup \{\neg K\gamma \mid \gamma \in \mathcal{L}(\mathcal{A}) - E\})$, where $\mathcal{L}(\mathcal{A})$ denotes the set of all objective (i.e., non-modal) formulas of the underlying propositional language. According to this equation, we can associate a logic S_{G} to each modal logic S. Given $\varphi \in \mathcal{L}_K(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, we say that ψ is entailed by φ in S_{G} (and write $\varphi \models_{\mathsf{S}_{\mathsf{G}}} \psi$) iff ψ belongs to all S_G-expansions for φ.

Among all ground nonmonotonic logics, the logic $\mathsf{S5}_{\mathsf{G}}$ (which, unlike NM-S5, is a true nonmonotonic logic) has received considerable attention in the literature and is generally referred to as the *logic of minimal knowledge* [Halpern and Moses, 1985] (or the *logic of maximal ignorance*). In fact, independently from its fixed-point characterization, $\mathsf{S5}_{\mathsf{G}}$ was characterized on a semantic basis, by means of a preference criterion among the models of an agent's knowledge selecting just those models in which objective knowledge (i.e, the set of formulas of type $K\varphi$ such that $\varphi \in \mathcal{L}(\mathcal{A})$) is minimal [Halpern and Moses, 1985; Shoham, 1987; Lifschitz, 1994]. As shown in [Shoham, 1987], $\mathsf{S5}_{\mathsf{G}}$ has a simple, elegant model theory: The $\mathsf{S5}_{\mathsf{G}}$ models of a theory T are precisely those Kripke structures which are universal (i.e., totally connected) and are maximal set of worlds (w.r.t. set-containment). For details on ground nonmonotonic modal logics and $\mathsf{S5}_{\mathsf{G}}$, see [Halpern and Moses, 1985; Lifschitz, 1994; Donini *et al.*, 1997].

Main Problems Studied. The investigations reported in this paper were motivated by the following questions:

1. Given a formula φ, which formulas ψ are almost surely true in the Kripke models of φ, i.e., how can we characterize the formulas ψ that are true with limit probability one in the Kripke models of φ, when the size of these Kripke models grows towards infinity?
2. Given that modal inference under almost sure validity constitutes a form of nonmonotonic reasoning, to which of the nonmonotonic modal logics from the abovementioned plethora of logics does this form of reasoning best correspond?
3. Can we characterize the set of formulas φ that guarantee a 0-1 law for the asymptotic probability $P(\psi|\varphi)$ for arbitrary modal formulas ψ?
4. What is the complexity of modal reasoning based on asymptotic conditional probabilities?

For studying these questions, we make a very weak assumption on φ, which assures that φ does not contradict some principles of knowledge: we assume that φ is S5-consistent, i.e., we assume that φ is consistent with the axioms of (monotonic) S5, which means that φ admits at least one S5-model. This assumption is indeed very weak. It does neither mean that φ contains the S5 axioms, or that these axioms should follow from φ, nor that φ has to be interpreted under S5 Kripke structures only. Our assumption merely requires that formulas such as $Kp \wedge K\neg p$ which bluntly contradict some axioms of (monotonic) S5 cannot be deduced from φ. If φ satisfies this requirement, we say that φ is *knowledge-consistent*. We make this assumption because knowledge-inconsistent theories are well-known to be *inconsistent* for all currently known nonmonotonic modal logics anyway, i.e., in each such nonmonotonic logic, a knowledge-inconsistent formula φ entails a contradiction and thus all formulas of the modal language $\mathcal{L}_K(\mathcal{A})$. In particular, with regard to our goal of comparing asymptotic conditional reasoning to

[3] This definition is applicable to a large range of modal logics, but a collapse happens at S5 because NM-S5=S5 [Marek and Truszczyński, 1993].

nonmonotonic modal logics, there is no point to consider knowledge-inconsistent modal premises. Moreover, we characterize asymptotic conditional reasoning when the premise is a *flat* modal formula, i.e., a formula without nested occurrences of the modal operator. The flat fragment of a modal logic of knowledge is certainly a very important (if not the most important) syntactically restricted fragment. It consists of the Boolean closure of *knowledge bases*, i.e., of objective theories under the K operator. In particular, the flat fragment in $\mathsf{S5_G}$ is extremely powerful and expressive. As shown in [Rosati, 1998], via appropriate translations, this fragment captures the well-known formalisms of default logic and logic programming under the stable model semantics. In the present paper we limit our attention to this flat fragment. However, we think that (by slightly more involved methods) our results carry over to the fully nested fragment (i.e., the fragment with nesting depth greater than 1).

Results. By first answering the second of the above-stated research questions, we show the following main result:

Answer to Question 2: φ almost surely entails ψ iff ψ is entailed by φ in $\mathsf{S5_G}$.

This result gives a fresh probabilistic semantics to the well-known nonmonotonic modal logic $\mathsf{S5_G}$, providing a new justification for $\mathsf{S5_G}$ based on probabilistic rationality. At the same time, it provides the answers to our questions 1 and 4, too. In fact, as already mentioned, reasoning in $\mathsf{S5_G}$ has precise characterizations in terms of model-theory and complexity, hence the same characterizations now apply to asymptotic conditional reasoning over Kripke models:

Answer to Question 1: φ almost surely entails ψ iff ψ is satisfied by all Kripke models of φ that are universal and have a maximal set of worlds.

Answer to Question 4: Deciding whether φ almost surely entails ψ is Π_2^p-complete.[4]

A theory is *honest* [Halpern and Moses, 1985] iff it has exactly one $\mathsf{S5_G}$-model. It has been argued that the epistemic state of a perfectly rational agent is necessarily honest, e.g., it could not be of the form $(KT_1) \vee (KT_2)$ where T_1 and T_2 are theories that contradict each other. Honest premises drastically simplify asymptotic conditional reasoning:

Answer to Question 3: The class of knowledge-consistent premises φ that imply a 0-1 law is exactly the class of honest formulas. In other words, the honest formulas are precisely the formulas φ such that, for all ψ, the asymptotic probability that ψ holds in the structures in which φ is valid is either 0 or 1. Moreover, asymptotic conditional reasoning based on honest theories is only Θ_2^p-complete.

Due to space limitations, we can only include the sketches of some results in the present version of the paper.

2 Preliminaries

We assume familiarity with modal logics K and $\mathsf{S5}$. We deal with a propositional alphabet $\mathcal{A}$ such that either $\mathcal{A}$ is finite and fixed, i.e., it is the same for every problem instance, or $\mathcal{A}$ is not bounded, but each problem instance comes along with a finite alphabet $\mathcal{A}$ as part of the input. We denote by $\mathcal{L}(\mathcal{A})$ the set of propositional (or objective) formulas over $\mathcal{A}$, denote by $\mathcal{L}_K(\mathcal{A})$ the set of modal formulas over $\mathcal{A}$, and denote by $\mathcal{L}_K^F(\mathcal{A})$ the set of *flat* modal formulas, i.e., the subset of $\mathcal{L}_K(\mathcal{A})$ of formulas over $\mathcal{A}$ satisfying the following abstract syntax:

$$\varphi ::= Kf \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2, \text{ where } f \in \mathcal{L}(\mathcal{A})$$

We also use the symbol true to denote the formula $a \vee \neg a$, and the symbol false to denote the formula $a \wedge \neg a$.

We now recall some auxiliary definitions that we will use in the following sections [Marek and Truszczyński, 1993; Donini *et al.*, 1997]. Given $\varphi \in \mathcal{L}_K(\mathcal{A})$, we denote by $MA(\varphi)$ the set of *modal atoms* from φ, i.e., the set of subformulas of the form $K\psi$ occurring in φ. In the following, we say that an occurrence of a modal atom $K\psi$ in a formula $\varphi \in \mathcal{L}_K(\mathcal{A})$ is *strict* if it does not lie within the scope of a modal operator.

Given a partition (P, N) of the set $MA(\varphi)$ and a formula $\psi \in \mathcal{L}_K(\mathcal{A})$, we denote by $\psi(P, N)$ the formula obtained from ψ by substituting each strict occurrence in ψ of a modal atom in P with true, and each strict occurrence in ψ of a modal atom in N with false. Notice that, if $P \cup N$ contains $MA(\psi)$, then $\psi(P, N)$ is a propositional formula.

Let $\varphi \in \mathcal{L}_K(\mathcal{A})$ and let (P, N) be a partition of $MA(\varphi)$. We denote by $obj_\varphi(P, N)$ the propositional formula $obj_\varphi(P, N) = \varphi(P, N) \wedge \bigwedge_{K\psi \in P} \psi(P, N)$.

Given a partition (P, N) of the set $MA(\varphi)$, we say that (P, N) is $\mathsf{S5}$-*consistent with* φ if (P, N) satisfies the following conditions: (1) the propositional formula $obj_\varphi(P, N)$ is satisfiable; (2) for each $K\psi \in N$, the propositional formula $obj_\varphi(P, N) \wedge \neg\psi(P, N)$ is satisfiable. It is immediate to see that there exists a partition of $MA(\varphi)$ $\mathsf{S5}$-consistent with φ iff φ is knowledge-consistent, i.e., there exists an $\mathsf{S5}$-structure $\mathcal{S}$ such that $(w, \mathcal{S}) \models K\varphi$ where w is a world of $\mathcal{S}$. Finally, given a structure $\mathcal{S} = \langle W, R, V \rangle$ and a world $w \in W$, we say that (P, N) is the partition of $MA(\varphi)$ *satisfied by* $(w, \mathcal{S})$ if, for each $K\psi \in MA(\varphi)$, $(w, \mathcal{S}) \models K\psi$ iff $K\psi \in P$.

3 Strong almost-sure conditional validity

In the work of Halpern and Kapron [Halpern and Kapron, 1994], almost sure structure validity is studied by considering all possible Kripke structures equally likely, i.e., uniformly distributed. This amounts to assume that that every propositional variable is true with probability 1/2 in a randomly chosen world. Under such an assumption, the asymptotic probability of φ w.r.t. a propositional alphabet $\mathcal{A}$ corresponds to the limit $\lim_{n\to\infty} |\mathcal{W}_n^\varphi|/|\mathcal{W}_n|$, where: (i) $\mathcal{W}_n$ denotes the set of all n-*structures* over $\mathcal{A}$, i.e., the structures with n worlds of the form $\langle W, R, V \rangle$, where $W = \{1, \ldots, n\}$, the accessibility relation R is a binary relation over W, and V is a function mapping each world into a propositional interpretation over $\mathcal{A}$; (ii) $\mathcal{W}_n^\varphi$ denotes the set of all n-structures in which φ holds; (iii) $|S|$ represents the cardinality of a set S. Moreover, under the above uniform probability assumption, asymptotic conditional probability of ψ given φ corresponds to the limit $\lim_{n\to\infty} |\mathcal{W}_n^{(\varphi\wedge\psi)}|/|\mathcal{W}_n^\varphi|$.

It would be more appealing to consider the notion of *strong* almost sure validity, where the asymptotic probability of a formula ψ is required to be equal to 1 *for every possible*

[4] Π_2^p is the complement of Σ_2^p=$\mathrm{NP}^{\mathrm{NP}}$; Θ_2^p is the class of problems solved in PTIME by a logarithmic number of calls to an NP-oracle.

probability distribution assigning rational truth probabilities to the propositional variables. It turns out that in the context of [Halpern and Kapron, 1994] both concepts are equivalent, thus all relevant results of [Halpern and Kapron, 1994] extend to strong almost sure validity. In the context of conditional probabilities, these concepts differ, however. In this paper we choose to characterize the notion of *strong* almost sure conditional validity, which is independent of a particular fixed probability distribution. However, we will use (Section 4) the concept of almost sure conditional validity (i.e., the one corresponding to the uniform distribution of Kripke structures) as a tool for establishing (Section 5) our main result on strong almost sure validity.

More formally, we associate to each propositional atom $a \in \mathcal{A}$ a rational probability m such that $0 < m < 1$, which is interpreted as the probability that the proposition a is true. Such an assignment is part of the input. Assignments giving probability 1 (resp. 0) to a are not considered, since in such cases the proposition a is certainly true (resp. false) and all its occurrences in a formula can be eliminated in a simple way. We assume without loss of generality that m is a finite sum of the negative powers of two. Hence, we say that $\mathcal{P}$ is a *probability assignment* over an alphabet $\mathcal{A}$ if $\mathcal{P}$ is a function mapping each propositional symbol from $\mathcal{A}$ to a rational number m such that there exists a finite binary sequence $m_1 \ldots m_p$ such that $m = \sum_{i=1}^{p} \frac{m_i}{2^i}$.

It is immediate to verify the following relationship between the cardinality of a set of interpetations and the number of different n-structures defined over such interpretations.[5]

Lemma 3.1 *Let $\mathcal{I}$ be a set of propositional interpretations, and let $\mathcal{W}_n$ be the set of n-structures defined using the set of interpretations $\mathcal{I}$. Then, $|\mathcal{W}_n| = |\mathcal{I}|^n \cdot 2^{n^2}$.*

Let $\varphi, \psi \in \mathcal{L}_K(\mathcal{A})$, and let $\mathcal{P}$ be a probability assignment over $\mathcal{A}$. Then, ${p_n}^{\mathcal{P}}(\psi|\varphi)$ denotes the probability that ψ is valid in the n-structures in which φ is valid, under the probability assignment $\mathcal{P}$. We are now ready to define almost-sure and strong almost-sure conditional validity.

Definition 3.2 ***($\mathcal{P}$-almost-sure validity)*** *Let $\mathcal{P}$ be a probability assignment over $\mathcal{A}$. We say that ψ conditioned by φ is* $\mathcal{P}$-almost-surely valid *if $\lim_{n\to\infty} {p_n}^{\mathcal{P}}(\psi|\varphi) = 1$, i.e., the asymptotic probability that ψ conditioned by φ is valid is 1 under the probability assignment $\mathcal{P}$. Conversely, if $\lim_{n\to\infty} {p_n}^{\mathcal{P}}(\psi|\varphi) < 1$, then we say that ψ conditioned by φ is not $\mathcal{P}$-almost-surely valid.*

Definition 3.3 ***(strong-almost-sure validity)*** *ψ conditioned by φ is* strongly almost-surely valid *if $\lim_{n\to\infty} {p_n}^{\mathcal{P}}(\psi|\varphi) = 1$ for each probability assignment $\mathcal{P}$ over $\mathcal{A}$. Conversely, if there exists a probability assignment $\mathcal{P}$ over $\mathcal{A}$ such that $\lim_{n\to\infty} {p_n}^{\mathcal{P}}(\psi|\varphi) < 1$, then we say that ψ conditioned by φ is not strongly almost-surely valid.*

We denote by *SAS*$(\varphi, \mathcal{A})$ the set of formulas ψ from $\mathcal{L}_K(\mathcal{A})$ such that ψ conditioned by φ is strongly almost-surely valid.

4 Counting structures

In this section we prove the correspondence between the notion of stable set in nonmonotonic modal logics and strong almost-sure conditional validity with respect to objective formulas, and establish a first correspondence between $\mathsf{S5_G}$ and strong almost-sure conditional validity in modal logic.

From now on, we denote by h the number of propositional interpretations of $\mathcal{A}$, i.e., $h = 2^{|\mathcal{A}|}$. Moreover, given a formula $f \in \mathcal{L}(\mathcal{A})$, we denote by h_f the number of interpretations of $\mathcal{A}$ satisfying f.

We start our analysis by studying the properties of the set of n-structures for f, i.e., the n-structures in which a propositional formula f is valid. First of all, from Lemma 3.1, it follows that the number of n-structures for f is $h_f \cdot 2^{n^2}$.

Then, we recall the definition of *stable set* of modal formulas. Let f be a satisfiable propositional formula over the propositional alphabet $\mathcal{A}$. The *stable set of f in $\mathcal{A}$* (denoted by $Stable(f, \mathcal{A})$) is the unique set of modal formulas $\mathcal{T} \subset \mathcal{L}_K(\mathcal{A})$ that satisfies the following conditions: (i) for each $\psi \in \mathcal{L}(\mathcal{A})$, $\psi \in \mathcal{T}$ iff $f \supset \psi$ is a tautology; (ii) if $\psi \in \mathcal{T}$ then $K\psi \in \mathcal{T}$; (iii) if $\psi \in \mathcal{L}_K(\mathcal{A}) - \mathcal{T}$ then $\neg K\psi \in \mathcal{T}$; (iv) $\mathcal{T}$ is closed under propositional consequence [Marek and Truszczyński, 1993].

Next, we prove the correspondence between formulas strongly almost-surely valid with respect to an objective formula f and the formulas in $Stable(f, \mathcal{A})$. The proof is easily obtained by extending an analogous result in [Halpern and Kapron, 1994].

Lemma 4.1 *Let $f \in \mathcal{L}(\mathcal{A})$ be a satisfiable objective formula. Then, SAS$(f, \mathcal{A}) = Stable(f, \mathcal{A})$.*

We now introduce two auxiliary lemmas.

Lemma 4.2 *Let φ be a flat and knowledge-consistent formula, let $\mathcal{W}_n^{\varphi}$ be the set of n-structures for φ and let $\mathcal{D}_n$ be the set of n-structures for φ in which all worlds satisfy the same partition (P, N) of $MA(\varphi)$ $\mathsf{S5}$-consistent with φ. Then, $\lim_{n\to\infty} |\mathcal{D}_n|/|\mathcal{W}_n^{\varphi}| = 1$.*

Proof sketch. The proof is divided in two steps: First, we prove that, if φ is a flat and knowledge-consistent formula, $\lim_{n\to\infty} |\mathcal{D}'_n|/|\mathcal{W}_n^{\varphi}| = 1$, where $\mathcal{D}'_n$ denotes the set of n-structures for φ in which no world satisfies an $\mathsf{S5}$-inconsistent partition of $MA(\varphi)$. Then, we concentrate on the set $\mathcal{D}'_n$: let $\mathcal{D}''_n$ denote the subset of $\mathcal{D}'_n$ in which two worlds satisfy two different $\mathsf{S5}$-consistent partitions of $MA(\varphi)$ (P_1, N_1), (P_2, N_2), let $\mathcal{D}_n^1$ be the subset of $\mathcal{D}'_n$ in which all worlds satisfy (P_1, N_1) and let $\mathcal{D}_n^2$ be the subset of $\mathcal{D}'_n$ in which all worlds satisfy (P_2, N_2). We prove that either $\lim_{n\to\infty} |\mathcal{D}''_n|/|\mathcal{D}_n^1| = 0$ or $\lim_{n\to\infty} |\mathcal{D}''_n|/|\mathcal{D}_n^2| = 0$. □

The following property can be derived by an argument analogous to the proof of Lemma 4.1.

Lemma 4.3 *Let φ be a flat and knowledge-consistent formula and let (P, N) be a partition of $MA(\varphi)$ consistent with φ. Let $\mathcal{W}_n^{\varphi}(P, N)$ denote the set of n-structures for φ in which all worlds satisfy the partition (P, N), and let $\mathcal{D}_n$ be the set of n-structures for $obj_{\varphi}(P, N) \wedge \mu(N)$, where $\mu(N) = \bigwedge_{Kf \in \mathcal{N}} \neg Kf$. Then, $\lim_{n\to\infty} |\mathcal{D}_n|/|\mathcal{W}_n^{\varphi}(P, N)| = 1$. Moreover, $\lim_{n\to\infty} |\mathcal{W}_n^{\varphi}(P, N)| = (h_{obj_{\varphi}(P,N)})^n \cdot 2^{n^2}$.*

We now define $\mathsf{S5_G}$-preferred partitions of modal atoms.

Definition 4.4 ***($\mathsf{S5_G}$-preferred partition)*** *Let $\varphi \in \mathcal{L}_K(\mathcal{A})$. A partition (P, N) of $MA(\varphi)$ is* $\mathsf{S5_G}$-preferred *for φ if (P, N)*

[5] We adopt the well-known *random worlds* method [Grove *et al.*, 1996b; 1996a].

is S5*-consistent with* φ *and there exists no other partition* $(P', N') \neq (P, N)$ *of* $MA(\varphi)$ *such that (i)* (P', N') *is* S5*-consistent with* φ*; and (ii) the propositional formula* $obj_\varphi(P, N) \wedge \neg obj_\varphi(P', N')$ *is not satisfiable.*

It is immediate to verify that $\mathsf{S5_G}$-preferred partitions of $MA(\varphi)$ are in one-to-one correspondence with $\mathsf{S5_G}$-expansions of φ: in particular, each such partition (P, N) identifies the $\mathsf{S5_G}$-expansion corresponding to $Stable(obj_\varphi(P, N), \mathcal{A})$.

We are now ready to show a first fundamental step towards the correspondence between $\mathsf{S5_G}$ and strong almost-sure conditional validity: For each knowledge-consistent modal formula φ, to compute (strong) almost sure conditional validity we can safely consider only the set of n-structures in which all worlds satisfy one of the partitions of $MA(\varphi)$ that are $\mathsf{S5_G}$-preferred for φ.

Theorem 4.5 *Let* φ *be a flat, knowledge-consistent formula, let* $\mathcal{W}_n^\varphi$ *denote the set of* n*-structures for* φ*, and let* $\mathcal{D}_n$ *be the union of the sets of* n*-structures for* φ *in which all worlds satisfy the same partition* (P, N) *of* $MA(\varphi)$*, where* (P, N) *is* $\mathsf{S5_G}$*-preferred for* φ*. Then,* $\lim_{n\to\infty} |\mathcal{D}_n|/|\mathcal{W}_n^\varphi| = 1$.

Proof. Let $\mathcal{C}'_n$ be the union of the sets of n-structures for $obj_\varphi(P, N) \wedge \mu(N)$ for each partition (P, N) of $MA(\varphi)$ that is S5-consistent with φ. First, by Lemma 4.2 and Lemma 4.3 it follows that $\lim_{n\to\infty} |\mathcal{C}'_n|/|\mathcal{W}_n^\varphi| = 1$. Moreover, let $\mathcal{C}_n$ be the union of the sets of n-structures for $obj_\varphi(P, N) \wedge \mu(N)$ for each partition (P, N) of $MA(\varphi)$ that is $\mathsf{S5_G}$-preferred for φ: By the same lemmas it follows that $\lim_{n\to\infty} |\mathcal{C}_n|/|\mathcal{D}_n| = 1$. Thus, we have to prove that $\lim_{n\to\infty} |\mathcal{C}_n|/|\mathcal{C}'_n| = 1$.

Let (P, N) be a partition (P, N) of $MA(\varphi)$ $\mathsf{S5_G}$-preferred for φ, and consider all the partitions (P', N') of $MA(\varphi)$ such that $obj_\varphi(P, N)$ is satisfied by all propositional interpretations satisfying $obj_\varphi(P', N')$. Let $\mathcal{C}_n^{(P,N)}$ be the set of n-structures for $obj_\varphi(P, N) \wedge \mu(N)$, and let $\mathcal{C}_n^{(P',N')}$ be the union of the set of n-structures for $obj_\varphi(P', N') \wedge \mu(N')$ for each such partition (P', N'). Obviously, $\mathcal{C}_n^{(P,N)} \subseteq \mathcal{C}_n^{(P',N')}$ (since (P, N) is one of such partitions (P', N')). We now prove that $\lim_{n\to\infty} \frac{|\mathcal{C}_n^{(P,N)}|}{|\mathcal{C}_n^{(P',N')}|} = 1$. Let $\mathcal{C}''_n = \mathcal{C}_n^{(P',N')} - \mathcal{C}_n^{(P,N)}$. That is, $\mathcal{C}''_n$ is the set of n-structures for $obj_\varphi(P', N') \wedge \mu(N)$ for each partition (P', N') that is S5-consistent with φ and such that $obj_\varphi(P, N)$ is satisfied by all propositional interpretations satisfying $obj_\varphi(P', N')$, and there exists at least an interpretation satisfying $obj_\varphi(P, N) \wedge \neg obj_\varphi(P', N')$. Now let k be the number of interpretations of $\mathcal{A}$ satisfying $obj_\varphi(P, N)$: It is immediate to verify that there can be at most k such different partitions (P', N'). Moreover, for each such partition (P', N'), there exists at least a propositional interpretation $\mathcal{I}$ that satisfies $obj_\varphi(P, N)$ and does not satisfy $obj_\varphi(P', N')$. Therefore, from Lemma 4.3 we have that $\frac{|\mathcal{C}''_n|}{|\mathcal{C}_n^{(P,N)}|} \leq \frac{k\cdot(k-1)^n \cdot 2^{n^2}}{k^n \cdot 2^{n^2}} = k \cdot ((k-1)/k)^n$. Consequently, $\lim_{n\to\infty} \frac{|\mathcal{C}''_n|}{|\mathcal{C}_n^{(P,N)}|} = 0$, which proves that $\lim_{n\to\infty} \frac{|\mathcal{C}_n^{(P,N)}|}{|\mathcal{C}_n^{(P',N')}|} = 1$. Hence, $\lim_{n\to\infty} \frac{|\mathcal{C}_n|}{|\mathcal{C}'_n|} = 1$. □

5 Asymptotic conditional probability and $\mathsf{S5_G}$

In this section we prove the main result of the paper, which establishes the correspondence between strong almost-sure conditioned validity and entailment in the logic $\mathsf{S5_G}$. To this aim, we need some preliminary definitions and properties.

Let $\mathcal{A}'$ be the set of propositional atoms a^i such that $a \in \mathcal{A}$ and $1 \leq i \leq m$. We define the *canonical probability assignment* $\mathcal{P}_c$ over $\mathcal{A}'$ as follows: $\mathcal{P}_c(a) = \frac{1}{2}$ for each $a \in \mathcal{A}'$. Let $\mathcal{A}$ be a propositional alphabet and let $\mathcal{I}_f$ be the set of propositional interpretations over $\mathcal{A}$ that satisfy f. The *canonical probability* of a propositional formula f, denoted by $cp(f)$, is defined as $cp(f) = \frac{|\mathcal{I}_f|}{2^{|\mathcal{A}|}}$.

The following auxiliary lemma establishes a sufficient condition over the partitions of $MA(\varphi)$ which implies that $\lim_{n\to\infty} {p_n}^{\mathcal{P}_c}(\psi|\varphi) < 1$.

Lemma 5.1 *Let* $\varphi \in \mathcal{L}_K^F(\mathcal{A})$*. If there exists a partition* (P, N) *of* $MA(\varphi)$ *such that: (1)* (P, N) *is* $\mathsf{S5_G}$*-preferred for* φ*; (2) for each partition* (P', N') *that is* $\mathsf{S5_G}$*-preferred for* φ*,* $cp(obj_\varphi(P, N)) \geq cp(obj_\varphi(P', N'))$*; (3)* ψ *does not hold in the structures for* $obj_\varphi(P, N)$*, then* $\lim_{n\to\infty} {p_n}^{\mathcal{P}_c}(\psi|\varphi) < 1$.

Let $a \in \mathcal{A}$ and let $\mathcal{P}(a) = k$ where k is a rational number satisfying the definition of probability assignment. Then, we can construct a propositional formula $f^k(a)$ defined over an alphabet $\mathcal{A}^k = \{a^1, \ldots, a^{m_k}\}$ (where m_k is a number depending on k) such that $\mathcal{P}(a) = cp(f^k(a))$. Then, given a probability assignment $\mathcal{P}$ over $\mathcal{A}$, we define $\tau_\mathcal{P}(\varphi)$ as the formula obtained from φ by replacing, for each $a \in \mathcal{A}$, each occurrence of a in φ with $f^{\mathcal{P}(a)}(a)$. Now, in order to prove our main result, we need some auxiliary lemmas.

Lemma 5.2 *Let* $\psi \in \mathcal{L}_K(\mathcal{A})$*, and let* $f \in \mathcal{L}(\mathcal{A})$*. Then,* $\psi \in SAS(f, \mathcal{A})$ *iff* $\tau_\mathcal{P}(\psi) \in SAS(\tau_\mathcal{P}(f), \mathcal{A}^k)$*.*

Lemma 5.3 *Let* $\varphi, \psi \in \mathcal{L}_K(\mathcal{A})$*. then, for each partition* (P, N) *that is* $\mathsf{S5_G}$*-preferred for* φ*, the partition* $(\tau_\mathcal{P}(P), \tau_\mathcal{P}(N))$ *is* $\mathsf{S5_G}$*-preferred for* $\tau_\mathcal{P}(\varphi)$*, while for each partition* $(\tau_\mathcal{P}(P), \tau_\mathcal{P}(N))$ *that is* $\mathsf{S5_G}$*-preferred for* $\tau_\mathcal{P}(\varphi)$*, the partition* (P, N) *is* $\mathsf{S5_G}$*-preferred for* φ*.*

Lemma 5.4 *Let* $\varphi \in \mathcal{L}_K(\mathcal{A})$*. If a partition* (P, N) *of* $MA(\varphi)$ *is* $\mathsf{S5_G}$*-preferred for* φ*, then the stable set of* $obj_\varphi(P, N)$ *in* $\mathcal{A}$ *is an* $\mathsf{S5_G}$*-expansion for* φ*. Moreover, if a set* $\mathcal{T} \subset \mathcal{L}_K(\mathcal{A})$ *is an* $\mathsf{S5_G}$*-expansion for* φ*, then there exists a partition* (P, N) *of* $MA(\varphi)$ *such that* (P, N) *is* $\mathsf{S5_G}$*-preferred for* φ *and* $Stable(obj_\varphi(P, N), \mathcal{A}) = \mathcal{T}$*.*

Theorem 5.5 *Let* $\varphi \in \mathcal{L}_K^F(\mathcal{A})$*,* $\psi \in \mathcal{L}_K(\mathcal{A})$*, and let* φ *be a knowledge-consistent formula. Then,* ψ *conditioned by* φ *is strongly almost-surely valid iff* $\varphi \models_{\mathsf{S5_G}} \psi$*.*

Proof. First, if $\varphi \models_{\mathsf{S5_G}} \psi$, then, from definition of entailment in $\mathsf{S5_G}$ and from Lemma 5.4 it follows that, for each partition (P, N) that is $\mathsf{S5_G}$-preferred for φ, ψ belongs to $Stable(obj_\varphi(P, N), \mathcal{A})$, consequently, by Lemma 4.1, $\psi \in SAS(obj_\varphi(P, N), \mathcal{A})$. Hence, by Lemma 5.2 and Lemma 5.3, for each probability assignment $\mathcal{P}$ over $\mathcal{A}$, $\tau_\mathcal{P}(\psi) \in SAS(\tau_\mathcal{P}(obj_\varphi(P, N)), \mathcal{A}^k)$. Therefore, by Theorem 4.5, $\lim_{n\to\infty} {p_n}^{\mathcal{P}_c}(\tau_\mathcal{P}(\psi)|\tau_\mathcal{P}(\varphi)) = 1$, which implies that ψ conditioned by φ is strongly almost-surely valid.

Conversely, if $\varphi \not\models_{\mathsf{S5_G}} \psi$, then by Lemma 5.4 it follows that there exists a partition (P, N) that is $\mathsf{S5_G}$-preferred for φ and such that ψ does not belong to $Stable(obj_\varphi(P, N), \mathcal{A})$, consequently, by Lemma 4.1, $\psi \notin SAS(obj_\varphi(P, N), \mathcal{A})$. Therefore, by Lemma 5.2, $\tau_\mathcal{P}(\psi) \notin SAS(\tau_\mathcal{P}(obj_\varphi(P, N)), \mathcal{A}^k)$. Moreover, it is immediate to verify the existence a probability assignment $\mathcal{P}'$ over $\mathcal{A}$ such that, for each partition (P', N') that is $\mathsf{S5_G}$-preferred for φ, $cp(\tau_{\mathcal{P}'}(obj_\varphi(P, N))) \geq cp(\tau_{\mathcal{P}'}(obj_\varphi(P', N')))$. Consequently, by Lemma 5.1, it follows that $\lim_{n\to\infty} p_n{}^{\mathcal{P}_c}(\tau_\mathcal{P}(\psi)|\tau_\mathcal{P}(\varphi)) < 1$, i.e., ψ conditioned by φ is not strongly almost-surely valid. □

6 Complexity results

As explained above, we consider both the case in which $\mathcal{A}$ is a fixed finite alphabet and the case in which it is considered as finite but not fixed in advance. If $\mathcal{A}$ is finite and fixed, then, based on the algorithm for entailment in $\mathsf{S5_G}$ reported in [Donini *et al.*, 1997] it can be easily shown that entailment in $\mathsf{S5_G}$ can be decided in polynomial time, and therefore, by Theorem 5.5, strong almost-sure conditional validity can be decided in polynomial time as well. Consider now the case of a finite, non-fixed alphabet $\mathcal{A}$.

Theorem 6.1 *Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, and let φ be a knowledge-consistent formula. Deciding whether ψ conditioned by φ is strongly almost-surely valid is a Π_2^p-complete problem.*

Proof. Follows immediately from Theorem 5.5 and from the fact that the entailment problem $\varphi \models_{\mathsf{S5_G}} \psi$ when $\varphi \in \mathcal{L}_K^F(\mathcal{A})$ is Π_2^p-complete, which is an immediate consequence of Corollary 4.12 of [Donini *et al.*, 1997] and of Theorem 11 of [Rosati, 1998]). □

Next, we study a subclass of flat modal formulas for which deciding strong almost-sure conditional validity is computationally easier than in the general case: the class of *honest* formulas, introduced by [Halpern and Moses, 1985]. A flat formula φ is honest iff it has exactly one $\mathsf{S5_G}$-model. It is known that deciding whether φ is honest is a Θ_2^p-complete problem. Moreover, if φ is honest, then deciding the entailment $\varphi \models_{\mathsf{S5_G}} \psi$ is also Θ_2^p-complete. Therefore, from Theorem 5.5 the following property holds.

Theorem 6.2 *Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, and let φ be a honest formula. Deciding whether ψ conditioned by φ is strongly almost-surely valid is Θ_2^p-complete.*

Furthermore, the notion of honesty precisely characterizes the subclass of formulas, among the formulas φ such that $K\varphi$ is S5-consistent, that condition according to a 0-1 law, i.e., such that either $\lim_{n\to\infty} p_n{}^{\mathcal{P}}(\psi|\varphi) = 1$ for each probability assignment $\mathcal{P}$ or $\lim_{n\to\infty} p_n{}^{\mathcal{P}}(\psi|\varphi) = 0$ for each probability assignment $\mathcal{P}$, which is immediately implied by Theorem 5.5 and by the fact that a honest formula has a single $\mathsf{S5_G}$-model.

7 Conclusions

The present work can be extended in several directions. In particular, we are currently investigating the following issues: (i) strong almost-sure validity for (classes of) conditioning formulas with nested occurrences of modalities; (ii) extending the study of asymptotic conditional validity to the framework of multimodal logic; (iii) computing $\mathcal{P}$-almost-sure conditional validity, i.e., the value of the asymptotic conditional validity for a given probability distribution $\mathcal{P}$ of the truth of primitive propositions.

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