Characterizing the NP-PSPACE Gap in the Satisfiability Problem for Modal Logic*

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Abstract

There has been a great deal of work on characterizing the complexity of the satisfiability and validity problem for modal logics. In particular, Ladner showed that the satisfiability problem for all logics between $\mathbf{K}$ and $\mathbf{S4}$ is *PSPACE*-hard, while for $\mathbf{S5}$ it is *NP*-complete. We show that it is *negative introspection*, the axiom $\neg Kp \Rightarrow K\neg Kp$, that causes the gap: if we add this axiom to any modal logic between $\mathbf{K}$ and $\mathbf{S4}$, then the satisfiability problem becomes *NP*-complete. Indeed, the satisfiability problem is *NP*-complete for any modal logic that includes the negative introspection axiom.

1 Introduction

There has been a great deal of work on characterizing the complexity of the satisfiability and validity problem for modal logics (see [Halpern and Moses, 1992; Ladner, 1977; Spaan, 1993; Vardi, 1989] for some examples). In particular, Ladner 1977 showed that the validity (and satisfiability) problem for every modal logic between $\mathbf{K}$ and $\mathbf{S4}$ is *PSPACE*-hard; and is *PSPACE*-complete for the modal logics $\mathbf{K}$, $\mathbf{T}$, and $\mathbf{S4}$. He also showed that the satisfiability problem for $\mathbf{S5}$ is *NP*-complete.

What causes the gap between *NP* and *PSPACE* here? We show that, in a precise sense, it is the negative introspection axiom: $\neg K\varphi \Rightarrow K\neg K\varphi$. It easily follows from Ladner's proof of *PSPACE*-hardness that for any modal logic L between $\mathbf{K}$ and $\mathbf{S4}$, there exists a family of formulas φ_n, all consistent with L such that such that $|\varphi_n| = O(n)$ but the smallest Kripke structure satisfying φ has at least 2^n states (where $|\varphi|$ is the length of φ viewed as a string of symbols). By way of contrast, we show that for all of the (infinitely many) modal logics L containing $\mathbf{K5}$ (that is, every modal logic containing the axiom K—$K\varphi \wedge K(\varphi \Rightarrow \psi) \Rightarrow K\psi$— and the negative introspection axiom, which has traditionally been called axiom 5), if a formula φ is consistent with L, then it is satisfiable in a Kripke structure of size linear in $|\varphi|$. Using this result and a characterization of the set of finite structures consistent with a logic L containing $\mathbf{K5}$ due to Nagle and Thomason 1985, we can show that the consistency (i.e., satisfiability) problem for L is *NP*-complete. Thus, roughly speaking, adding negative introspection to any logic between $\mathbf{K}$ and $\mathbf{S4}$ lowers the complexity from *PSPACE*-hard to *NP*-complete.

The fact that the consistency problem for specific modal logics containing $\mathbf{K5}$ is *NP*-complete has been observed before. As we said, Ladner already proved it for $\mathbf{S5}$; an easy modification (see [Fagin *et al.*, 1995]) gives the result for $\mathbf{KD45}$ and $\mathbf{K45}$.[1] That the negative introspection axiom plays a significant role has also been observed before; indeed, Nagle 1981 shows that every formula φ consistent with a normal modal logic[2] L containing $\mathbf{K5}$ has a finite model (indeed, a model exponential in $|\varphi|$) and using that, shows that the provability problem for every logic L between $\mathbf{K}$ and $\mathbf{S5}$ is decidable; Nagle and Thomason 1985 extend Nagle's result to all logics containing $\mathbf{K5}$ not just normal logics. Despite all this prior work and the fact that our result follows from a relatively straightforward combination of results of Nagle and Thomason and Ladner's techniques for proving that the consistency problem for $\mathbf{S5}$ is *NP*-complete, our result seems to be new, and is somewhat surprising (at least to us!).

The rest of the paper is organized as follows. In the next section, we review standard notions from modal logic and the key results of Nagle and Thomason 1985 that we use. In Section 3, we prove the main result of the paper. We discuss related work in Section 4.

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[1] Nguyen 2005 also claims the result for $\mathbf{K5}$, referencing Ladner. While the result is certainly true for $\mathbf{K5}$, it is not immediate from Ladner's argument.

[2] A modal logic is *normal* if it satisfies the generalization rule RN: from φ infer $K\varphi$.

2 Modal Logic: A Brief Review

We briefly review basic modal logic, introducing the notation used in the statement and proof of our result. The syntax of the modal logic is as follows: formulas are formed by starting with a set $\Phi = \{p, q, \dots\}$ of primitive propositions, and then closing off under conjunction ($\wedge$), negation ($\neg$), and the modal operator K. Call the resulting language $\mathcal{L}_1^K(\Phi)$. (We often omit the Φ if it is clear from context or does not play a significant role.) As usual, we define $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ as abbreviations of $\neg(\neg\varphi \wedge \neg\psi)$ and $\neg\varphi \vee \psi$, respectively. The intended interpretation of $K\varphi$ varies depending on the context. It typically has been interpreted as knowledge, as belief, or as necessity. Under the epistemic interpretation, $K\varphi$ is read as "the agent *knows* φ"; under the necessity interpretation, $K\varphi$ can be read "φ is necessarily true".

The standard approach to giving semantics to formulas in $\mathcal{L}_1^K(\Phi)$ is by means of Kripke structures. A tuple $F = (S, \mathcal{K})$ is a *(Kripke) frame* if S is a set of states, and $\mathcal{K}$ is a binary relation on S. A *situation* is a pair (F, s), where $F = (S, \mathcal{K})$ is a frame and $s \in S$. A tuple $M = (S, \mathcal{K}, \pi)$ is a *Kripke structure (over Φ)* if $(S, \mathcal{K})$ is a frame and $\pi : S \times \Phi \to \{\mathbf{true}, \mathbf{false}\}$ is an *interpretation (on S)* that determines which primitive propositions are true at each state. Intuitively, $(s, t) \in \mathcal{K}$ if, in state s, state t is considered possible (by the agent, if we are thinking of K as representing an agent's knowledge or belief). For convenience, we define $\mathcal{K}(s) = \{t : (s, t) \in \mathcal{K}\}$.

Depending on the desired interpretation of the formula $K\varphi$, a number of conditions may be imposed on the binary relation $\mathcal{K}$. $\mathcal{K}$ is *reflexive* if for all $s \in S$, $(s, s) \in \mathcal{K}$; it is *transitive* if for all $s, t, u \in S$, if $(s, t) \in \mathcal{K}$ and $(t, u) \in \mathcal{K}$, then $(s, u) \in \mathcal{K}$; it is *serial* if for all $s \in S$ there exists $t \in S$ such that $(s, t) \in \mathcal{K}$; it is *Euclidean* iff for all $s, t, u \in S$, if $(s, t) \in \mathcal{K}$ and $(s, u) \in \mathcal{K}$ then $(t, u) \in \mathcal{K}$. We use the superscripts r, e, t and s to indicate that the $\mathcal{K}$ relation is restricted to being reflexive, Euclidean, transitive, and serial, respectively. Thus, for example, $\mathcal{S}^{rt}$ is the class of all situations where the $\mathcal{K}$ relation is reflexive and transitive.

We write $(M, s) \models \varphi$ if φ is true at state s in the Kripke structure M. The truth relation is defined inductively as follows:

$$(M, s) \models p, \text{ for } p \in \Phi, \text{ if } \pi(s, p) = \mathbf{true}$$
$$(M, s) \models \neg\varphi \text{ if } (M, s) \not\models \varphi$$
$$(M, s) \models \varphi \wedge \psi \text{ if } (M, s) \models \varphi \text{ and } (M, s) \models \psi$$
$$(M, s) \models K\varphi \text{ if } (M, t) \models \varphi \text{ for all } t \in \mathcal{K}(s)$$

A formula φ is said to be *satisfiable in Kripke structure* M if there exists $s \in S$ such that $(M, s) \models \varphi$; φ is said to be *valid in* M, written $M \models \varphi$, if $(M, s) \models \varphi$ for all $s \in S$. A formula is *satisfiable* (resp., *valid*) *in a class* $\mathcal{N}$ of Kripke structures if it is satisfiable in some Kripke structure in $\mathcal{N}$ (resp., valid in all Kripke structures in $\mathcal{N}$). There are analogous definitions for situations. A Kripke structure $M = (S, \mathcal{K}, \pi)$ is *based on* a frame $F = (S', \mathcal{K}')$ if $S' = S$ and $\mathcal{K}' = \mathcal{K}$. The formula φ is *valid in* situation (F, s), written $(F, s) \models \varphi$, where $F = (S, \mathcal{K})$ and $s \in S$, if $(M, s) \models \varphi$ for all Kripke structure M based on F.

Modal logics are typically characterized by axiom systems. Consider the following axioms and inference rules, all of which have been well-studied in the literature [Blackburn *et al.*, 2001; Chellas, 1980; Fagin *et al.*, 1995]. (We use the traditional names for the axioms and rules of inference here.) These are actually *axiom schemes* and *inference schemes*; we consider all instances of these schemes.

Prop. All tautologies of propositional calculus

K. $(K\varphi \wedge K(\varphi \Rightarrow \psi)) \Rightarrow K\psi$ (Distribution Axiom)

T. $K\varphi \Rightarrow \varphi$ (Knowledge Axiom)

4. $K\varphi \Rightarrow KK\varphi$ (Positive Introspection Axiom)

5. $\neg K\varphi \Rightarrow K\neg K\varphi$ (Negative Introspection Axiom)

D. $\neg K(\mathit{false})$ (Consistency Axiom)

MP. From φ and $\varphi \Rightarrow \psi$ infer ψ (Modus Ponens)

RN. From φ infer $K\varphi$ (Knowledge Generalization)

The standard modal logics are characterized by some subset of the axioms above. All are taken to include Prop, MP, and RN; they are then named by the other axioms. For example, $\mathbf{K5}$ consists of all the formulas that are provable using Prop, K, 5, MP, and RN; we can similarly define other systems such as $\mathbf{KD45}$ or $\mathbf{KT5}$. $\mathbf{KT}$ has traditionally been called $\mathbf{T}$; $\mathbf{KT4}$ has traditionally been called $\mathbf{S4}$; and $\mathbf{KT45}$ has traditionally been called $\mathbf{S5}$.

For the purposes of this paper, we take a *modal logic* L to be any collection of formulas that contains all instances of Prop and is closed under modus ponens (MP) and substitution, so that if φ is a formula in L and p is a primitive proposition, then $\varphi[p/\psi] \in L$, where $\varphi[p/\psi]$ is the result of replacing all instances of p in φ by ψ. A logic is *normal* if it contains all instances of the axiom K and is closed under the inference rule RN. In terms of this notation, Ladner 1977 showed that if L is a normal modal logic between $\mathbf{K}$ and $\mathbf{S4}$ (since we are identifying a modal logic with a set of formulas here, that just means that $\mathbf{K} \subseteq L \subseteq \mathbf{S4}$), then determining if $\varphi \in L$ is *PSPACE*-hard. (Of course, if we think of a modal logic as being characterized by an axiom system, then $\varphi \in L$ iff φ is provable from the axioms characterizing L.) We say that φ is *consistent with* L if $\neg\varphi \notin L$. Since consistency is just the dual of provability, it follows from Ladner's result that testing consistency is *PSPACE*-hard for every normal logic between $\mathbf{K}$ and $\mathbf{S4}$. Ladner's proof actually shows more: the proof holds without change for non-normal logics, and it shows that some formulas consistent with logics between $\mathbf{K}$ and $\mathbf{S4}$ are satisfiable only in large models. More precisely, it shows the following:

Theorem 2.1: [Ladner, 1977]

(a) Checking consistency is PSPACE complete for every logic between $\mathbf{K}$ and $\mathbf{S4}$ (even non-normal logics).

(b) For every logic L between $\mathbf{K}$ and $\mathbf{S4}$, there exists a family of formulas φ_n^L, $n = 1, 2, 3, \dots$, such that (i) for all n, φ_n^L is consistent with L, (ii) there exists a constant d such that $|\varphi_n^L| \leq dn$, (iii) the smallest Kripke structure that satisfies φ has at least 2^n states.

There is a well-known correspondence between properties of the $\mathcal{K}$ relation and axioms: reflexivity corresponds to T, transitivity corresponds to 4, the Euclidean property corresponds to 5, and the serial property corresponds to D. This correspondence is made precise in the following well-known theorem (see, for example, [Fagin *et al.*, 1995]).

Theorem 2.2: *Let $\mathcal{C}$ be a (possibly empty) subset of {T, 4, 5, D} and let C be the corresponding subset of $\{r, t, e, s\}$. Then {Prop, K, MP, RN} $\cup\, \mathcal{C}$ is a sound and complete axiomatization of the language $\mathcal{L}_1^K(\Phi)$ with respect to $\mathcal{S}^C(\Phi)$.*[3]

Given a modal logic L, let $\mathcal{S}^L$ consist of all situations (F, s) such that $\varphi \in L$ implies that $(F, s) \models \varphi$. An immediate consequence of Theorem 2.2 is that $\mathcal{S}^e$, the situations where the $\mathcal{K}$ relation is Euclidean, is a subset of $\mathcal{S}^{\mathbf{K5}}$.

Nagle and Thomason 1985 provide a useful semantic characterization of all logics that contain $\mathbf{K5}$. We review the relevant details here. Consider all the finite situations $((S, \mathcal{K}), s)$ such that either

1. S is the disjoint union of S_1, S_2, and $\{s\}$ and $\mathcal{K} = (\{s\} \times S_1) \cup ((S_1 \cup S_2) \times (S_1 \cup S_2))$, where $S_2 = \emptyset$ if $S_1 = \emptyset$; or
2. $\mathcal{K} = S \times S$.

Using (a slight variant of) Nagle and Thomason's notation, let $\mathcal{S}_{m,n}$, with $m \geq 1$ and $n \geq 0$ or $(m, n) = (0, 0)$, denote all situations of the first type where $|S_1| = m$ and $|S_2| = n$, and let $\mathcal{S}_{m,-1}$ denote all situations of the second type where $|S| = m$. (The reason for taking -1 to be the second subscript for situations of the second type will become clearer below.) It is immediate that all situations in $\mathcal{S}_{m,n}$ for fixed m and n are isomorphic—they differ only in the names assigned to states. Thus, the same formulas are valid in any two situations in $\mathcal{S}_{m,n}$. Moreover, it is easy to check that the $\mathcal{K}$ relation in each of the situations above in Euclidean, so each of these situations is in $\mathcal{S}^{\mathbf{K5}}$. It is well known that the situations in $\mathcal{S}_{m,-1}$ are all in $\mathcal{S}^{\mathbf{K5}}$ and the situations in $\mathcal{S}_{m,-1} \cup \mathcal{S}_{m,0}$ are all in $\mathcal{S}^{\mathbf{KD45}}$. In fact, $\mathbf{S5}$ (resp., $\mathbf{KD45}$) is sound and complete with respect to the situations in $\mathcal{S}_{m,-1}$ (resp., $\mathcal{S}_{m,-1} \cup \mathcal{S}_{m,0}$). Nagle and Thomason show that much more is true. Let $\mathcal{T}^L = (\cup\{\mathcal{S}_{m,n} : m \geq 1, n \geq -1 \text{ or } (m, n) = (0, 0)\}) \cap \mathcal{S}^L$.

Theorem 2.3: [Nagle and Thomason, 1985] *For every logic L containing $\mathbf{K5}$, L is sound and complete with respect to the situations in $\mathcal{T}^L$.*

The key result of this paper shows that if a formula φ is consistent with a logic L containing $\mathbf{K5}$, then there exists m, n, a Kripke structure $M = (S, \mathcal{K}, \pi)$, and a state $s \in S$ such that $((S, \mathcal{K}), s) \in \mathcal{S}_{m,n}$, $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$, and $m + n < |\varphi|$.

[3] We remark that soundness and completeness is usually stated with respect to the appropriate class $\mathcal{M}^C$ of structures, rather than the class $\mathcal{S}^C$ of situations. However, the same proof applies without change to show completeness with respect to $\mathcal{S}^C$, and using $\mathcal{S}^C$ allows us to relate this result to our later results. While for normal logics it suffices to consider only validity with respect to structures, for non-normal logics, we need to consider validity with respect to situations.

That is, if φ is satisfiable at all, it is satisfiable in a situation with a number of states that is linear in $|\varphi|$.

One more observation made by Nagle and Thomason will be important in the sequel.

Definition 2.4 : A *p-morphism* (short for *pseudo-epimorphism*) from situation $((S', \mathcal{K}'), s')$ to situation $((S, \mathcal{K}), s)$ is a function $f : S' \to S$ such that

- $f(s') = s$;
- if $(s_1, s_2) \in \mathcal{K}$, then $(f(s_1), f(s_2)) \in \mathcal{K}'$;
- if $(f(s_1), s_3) \in \mathcal{K}$, then there exists some $s_2 \in S'$ such that $(s_1, s_2) \in \mathcal{K}'$ and $f(s_2) = s_3$.

∎

This notion of p-morphism of situations is a variant of standard notions of p-morphism of frames and structures [Blackburn *et al.*, 2001]. It is well known that if there is a p-morphism from one structure to another, then the two structures satisfy the same formulas. An analogous result, which is proved in the full paper, holds for situations.

Theorem 2.5: *If there is a p-morphism from situation (F', s') to (F, s), then for every modal logic L, if $(F', s') \in \mathcal{S}^L$ then $(F, s) \in \mathcal{S}^L$.*

Now consider a partial order on pairs of numbers, so that $(m, n) \leq (m', n')$ iff $m \leq m'$ and $n \leq n'$. Nagle and Thomason observed that if $(F, s) \in \mathcal{S}_{m,n}$, $(F', s') \in \mathcal{S}_{m',n'}$, and $(1, -1) \leq (m, n) \leq (m', n')$, then there is an obvious p-morphism from (F', s') to (F, s): if $F = (S, \mathcal{K})$, $S = S_1 \cup S_2$, $F' = (S', \mathcal{K}')$, $S' = S'_1 \cup S'_2$ (where S_i and S'_i for $i = 1, 2$ are as in the definition of $\mathcal{S}_{m,n}$), then define $f : S' \to S$ so that $f(s') = s$, f maps S'_1 onto S_1, and, if $S_2 \neq \emptyset$, then f maps S'_2 onto S_2; otherwise, f maps S'_2 to S_1 arbitrarily. The following result (which motivates the subscript -1 in $\mathcal{S}_{m,-1}$) is immediate from this observation and Theorem 2.5.

Theorem 2.6: *If $(F, s) \in \mathcal{S}_{m,n}$, $(F', s') \in \mathcal{S}_{m',n'}$, and $(1, -1) \leq (m, n) \leq (m', n')$, then for every modal logic L, if $(F', s') \in \mathcal{T}^L$ then $(F, s) \in \mathcal{T}^L$.*

3 The Main Results

We can now state our key technical result.

Theorem 3.1: *If L is a modal logic containing $\mathbf{K5}$ and $\neg\varphi \notin L$, then there exist m, n such that $m + n < |\varphi|$, a situation $(F, s) \in \mathcal{S}^L \cap \mathcal{S}_{m,n}$, and structure M based on F such that $(M, s) \models \varphi$.*

Proof: By Theorem 2.3, if $\neg\varphi \notin L$, there is a situation $(F', s_0) \in \mathcal{T}^L$ such that $(F', s_0) \not\models \neg\varphi$. Thus, there exists a Kripke structure M' based on F' such that $(M', s_0) \models \varphi$. Suppose that $F' \in \mathcal{S}_{m',n'}$. If $m' + n' < |\varphi|$, we are done, so suppose that $m' + n' \geq |\varphi|$. Note that this means $m' \geq 1$. We now construct a a situation $(F, s) \in \mathcal{S}_{m.n}$ such that $(1, -1) \leq (m, n) \leq (m', n')$, $m + n < |\varphi|$, and $(M, s) \models \varphi$ for some Kripke structure based on F. This gives the desired result. The construction of M is similar in spirit to Ladner's 1977 proof of the analogous result for the case of $\mathbf{S5}$.

Let C_1 be the set of subformulas of φ of the form $K\psi$ such that $(M', s_0) \models \neg K\psi$, and let C_2 be the set of subformulas of φ of the form $K\psi$ such that $KK\psi$ is a subformula of φ and $(M', s_0) \models \neg KK\psi \wedge K\psi$. (We remark that it is not hard to show that if $\mathcal{K}$ is either reflexive or transitive, then $C_2 = \emptyset$.)

Suppose that $M' = (S', \mathcal{K}', \pi')$. For each formula $K\psi \in C_1$, there must exist a state $s_\psi^{C_1} \in \mathcal{K}'(s_0)$ such that $(M', s_\psi^{C_1}) \models \neg\psi$. Note that if $C_1 \neq \emptyset$ then $\mathcal{K}'(s_0) \neq \emptyset$. Define $I(s_0) = \{s_0\}$ if $s_0 \in \mathcal{K}'(s_0)$, and $I(s_0) = \emptyset$ otherwise. Let $S_1 = \{s_\psi^{C_1} : K\psi \in C_1\} \cup I(s_0)$. Note that $S_1 \subseteq \mathcal{K}'(s_0) = S_1'$, so $|S_1| \leq |S_1'|$. If $K\psi \in C_2$ then $KK\psi \in C_1$, so there must exist a state $s_\psi^{C_2} \in \mathcal{K}'(s_{K\psi}^{C_1})$ such that $(M', s_\psi^{C_2}) \models \neg\psi$. Moreover, since $(M', s_0) \models K\psi$, it must be the case that $s_\psi^{C_2} \notin \mathcal{K}'(s_0)$. Let $S_2 = \{s_\psi^{C_2} : K\psi \in C_2\}$. By construction, $S_2 \subseteq S_2'$, so $|S_2| \leq |S_2'|$, and S_1 and S_2 are disjoint. Moreover, if $S_1 = \emptyset$, then $C_1 = \emptyset$, so $C_2 = \emptyset$ and $S_2 = \emptyset$.

Let $S = \{s_0\} \cup S_1 \cup S_2$. Define the binary relation $\mathcal{K}$ on S by taking $\mathcal{K}(s_0) = S_1$ and $\mathcal{K}(t) = S_1 \cup S_2$ for $t \in S_1 \cup S_2$. In the full paper, we show that $\mathcal{K}$ is well defined; in particular, we show that (a) $s_0 \notin S_2$ and (b) if $s_0 \in S_1$, then $S_2 = \emptyset$. We also show that $\mathcal{K}$ is the restriction of $\mathcal{K}'$ to S.

Let $M = (S, \mathcal{K}, \pi)$, where π is the restriction of π' to $\{s_0\} \cup S_1 \cup S_2$. It is well known [Fagin *et al.*, 1995] (and easy to prove by induction on φ) that there are at most $|\varphi|$ subformulas of φ. Since C_1 and C_2 are disjoint sets of subformulas of φ, all of the form $K\psi$, and at least one subformula of φ is a primitive proposition (and thus not of the form $K\psi$), it must be the case that $|C_1| + |C_2| \leq |\varphi| - 1$, giving us the desired bound on the number of states.

In the full paper we show, by a relatively straightforward induction on the structure of formulas, that for all states $s \in S$ and for all subformulas ψ of φ (including φ itself), $(M, s) \models \psi$ iff $(M', s) \models \psi$. The proof proceeds by induction on the structure of φ.

By construction, $(F, s) \in \mathcal{S}_{m,n}$, where $m = |S_1|$ and $n = |S_2|$. We have already observed that $m+n < |\varphi|$, $|S_1| \leq |S_1'|$, and $|S_2| \leq |S_2'|$. Thus, $(m, n) \leq (m', n')$. It follows from Theorem 2.6 that $(F, s) \in \mathcal{T}^L \subseteq \mathcal{S}^L$. This completes the proof. ∎

The idea for showing that the consistency problem for a logic L that contains **K5** is *NP*-complete is straightforward. Given a formula φ that we want to show is consistent with L, we simply guess a frame $F = (S, \mathcal{K})$, structure M based on F, and state $s \in S$ such that $(F, s) \in \mathcal{S}_{m,n}$ with $m+n < |\varphi|$, and verify that $(M, s) \models \varphi$ and $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$. Verifying that $(M, s) \models \varphi$ is the *model-checking problem*. It is well known that this can be done in time polynomial in the number of states of M, which in this case is linear in $|\varphi|$. So it remains to show that, given a logic L containing **K5**, checking whether $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$ can be done efficiently. This follows from observations made by Nagle and Thomason 1985 showing that that, although $\mathcal{T}^L$ may include $\mathcal{S}_{m',n'}$ for infinitely many pairs (m', n'), $\mathcal{T}^L$ has a finite representation that makes it easy to check whether $\mathcal{S}_{m.n} \subseteq \mathcal{T}^L$.

Say that (m, n) is a *maximal index* of $\mathcal{T}^L$ if $m \geq 1$, $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$, and it is not the case that $\mathcal{S}_{m',n'} \subseteq \mathcal{T}^L$ for some (m', n') with $(m, n) < (m', n')$. It is easy to see that $\mathcal{T}^L$ can have at most finitely many maximal indices. Indeed, if (m, n) is a maximal index, then there can be at most $m+n-1$ maximal indices, for if (m', n') is another maximal index, then either $m' < m$ or $n' < n$ (for otherwise $(m, n) \leq (m', n')$, contradicting the maximality of (m, n)). Say that $m \geq 1$ is an *infinitary first index* of $\mathcal{T}^L$ if $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$ for all $n \geq -1$. Similarly, say that $n \geq -1$ is an *infinitary second index* of $\mathcal{T}^L$ if $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$ for all $m \geq 1$. Note that it follows from Theorem 2.6 that if $(1, -1) \leq (m, n) \leq (m', n')$, then if m' is an infinitary first index of $\mathcal{T}^L$, then so is m, and if n' is an infinitary second index of $\mathcal{T}^L$, then so is n. Suppose that m^* is the largest infinitary first index of $\mathcal{T}^L$ and n^* is the largest infinitary second index of $\mathcal{T}^L$, where we take $m^* = n^* = \infty$ if all first indices are infinitary (or, equivalently, if all second indices are infinitary), we take $m^* = 0$ if no first indices are infinitary, and we take $n^* = -2$ if no second indices are infinitary. It follows from all this that $\mathcal{T}^L$ can be represented by the tuple $(i, m^*, n^*, (m_1, n_1), \ldots, (m_k, n_k))$, where

- i is either 0 or 1, depending on whether $\mathcal{S}_{0,0} \in \mathcal{T}^L$;
- m^* is the largest infinitary first index;
- n^* is the largest infinitary second index; and
- $((m_1, n_1), \ldots, (m_k, n_k)$ are the maximal indices.

Given this representation of $\mathcal{T}^L$, it is immediate that $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$ iff one of the following conditions holds:

- $(m, n) = (0, 0)$ and $i = 1$;
- $1 \leq m \leq m^*$;
- $-1 \leq n \leq n^*$; or
- $(m, n) \leq (m_k, n_k)$.

We can assume that the algorithm for checking whether a formula is consistent with L is "hardwired" with this description of L. It follows that only a constant number of checks (independent of φ) are required to see if $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$.[4]

Putting all this together, we get our main result.

Theorem 3.2: *For all logics L containing* **K5***, checking whether φ is consistent with L is an NP-complete problem.*

4 Discussion and Related Work

We have shown all that, in a precise sense, adding the negative introspection axiom pushes the complexity of a logic between **K** and **S4** down from *PSPACE*-hard to *NP*-complete. This is not the only attempt to pin down the *NP-PSPACE* gap and to understand the effect of the negative introspection axiom. We discuss some of the related work here.

A number of results showing that large classes of logics have an *NP*-complete satisfiability problem have been proved

[4]Here we have implicitly assumed that checking whether inequalities such as $(m, n) \leq (m', n')$ hold can be done in one time step. If we assume instead that it requires time logarithmic in the inputs, then checking if $\mathcal{S}_{m,n} \subseteq \mathcal{T}^L$ requires time logarithmic in $m+n$, since we can take all of $m^*, n^*, m_1, \ldots m_k, n_1, \ldots, n_k$ to be constants.

recently. For example, Litak and Wolter 2005 show that the satisfiability for all finitely axiomatizatble tense logics of linear time is *NP*-complete, and Bezhanishvili and Hodkinson 2004 show that every normal modal logic that properly extends $\mathbf{S5}^2$ (where $\mathbf{S5}^2$ is the modal logic that contains two modal operators K_1 and K_2, each of which satisfies the axioms and rules of inference of $\mathbf{S5}$ as well as the axiom $K_1K_2p \Leftrightarrow K_2K_1p$) has a satisfiability problem that is *NP*-complete. Perhaps the most closely related result is that of Hemaspaandra 1993, who showed that the consistency problem for any normal logic containing $\mathbf{S4}.3$ is also *NP*-complete. $\mathbf{S4}.3$ is the logic that results from adding the following axiom, known in the literature as D1, to $\mathbf{S4}$:

D1. $K(K\varphi \Rightarrow \psi) \vee K(K\psi \Rightarrow \varphi)$

D1 is characterized by the *connectedness* property: it is valid in a situation $((S, \mathcal{K}), s)$ if for all $s_1, s_2, s_3 \in S$, if $(s_1, s_2) \in \mathcal{K}$ and $(s_1, s_3) \in \mathcal{K}$, then either $(s_2, s_3) \in \mathcal{K}$ or $(s_3, s_2) \in \mathcal{K}$. Note that connectedness is somewhat weaker than the Euclidean property; the latter would require that *both* (s_2, s_3) and (s_3, s_2) be in $\mathcal{K}$.

As it stands, our result is incomparable to Hemspaandra's. To make the relationship clearer, we can restate her result as saying that the consistency property for any normal logic that contains $\mathbf{K}$ and the axioms T, 4, and D1 is *NP*-complete. We do not require either 4 or T for our result. However, although the Euclidean property does not imply either transitivity or reflexivity, it does imply *secondary reflexivity* and *secondary transitivity*. That is, if $\mathcal{K}$ satisfies the Euclidean property, then for all states s_1, s_2, s_3, s_4, if $(s_1, s_2) \in \mathcal{K}$, then $(s_2, s_2) \in \mathcal{K}$ and if (s_2, s_3) and $(s_3, s_4) \in \mathcal{K}$, then $(s_2, s_4) \in \mathcal{K}$; roughly speaking, reflexivity and transitivity hold for all states s_2 in the range of $\mathcal{K}$. Secondary reflexivity and secondary transitivity can be captured by the following two axioms:

T$'$. $K(K\varphi \Rightarrow \varphi)$

4$'$. $K(K\varphi \Rightarrow KK\varphi)$

Both T$'$ and 4$'$ follow from 5, and thus both are sound in any logic that extends $\mathbf{K5}$. Clearly T$'$ and 4$'$ also both hold in any logic that extends $\mathbf{S4}.3$, since $\mathbf{S4}.3$ contains T, 4, and the inference rule RN. We conjecture that the consistency property for every logic that extends $\mathbf{K}$ and includes the axioms T$'$, 4$'$, and D1 is *NP*-complete. If this result were true, it would generalize both our result and Spaan's result.

Vardi 1989 used a difference approach to understand the semantics, rather than relational semantics. This allowed him to consider logics that do not satisfy the K axiom. He showed that some of these logics have a consistency problem that is *NP*-complete (for example, the minimal normal logic, which characterized by Prop, MP, and RN), while others are *PSPACE*-hard. In particular, he showed that adding the axiom $K\varphi \wedge K\psi \Rightarrow K(\varphi \wedge \psi)$ (which is valid in $\mathbf{K}$) to Prop, MP, and RN gives a logic that is *PSPACE*-hard. He then conjectured that this ability to "combine" information is what leads to *PSPACE*-hardness. However, this conjecture has been shown to be false. There are logics that lack this axiom and, nevertheless, the consistency problem for these logics is *PSPACE*-complete (see [Allen, 2005] for a recent discussion and pointers to the relevant literature).

All the results for this paper are for single-agent logics. Halpern and Moses 1992 showed that the consistency problem for a logic with two modal operators K_1 and K_2, each of which satisfies the $\mathbf{S5}$ axioms, is *PSPACE*-complete. Indeed, it is easy to see that if K_i satisfies the axioms of L_i for some normal modal logic L_i containing $\mathbf{K5}$, then the consistency problem for the logic that includes K_1 and K_2 must be *PSPACE*-hard. This actually follows immediately from Ladner's 1977 result; then it is easy to see that K_1K_2, viewed as a single operator, satisfies the axioms of $\mathbf{K}$. We conjecture that this result continues to hold even for non-normal logics.

We have shown that somewhat similar results hold when we add awareness to the logic (in the spirit of Fagin and Halpern 1988), but allow awareness of unawareness [Halpern and Rêgo, 2006]. In the single-agent case, if the K operator satisfies the axioms K, 5, and some (possibly empty) subset of $\{T, 4\}$, then the validity problem for the logic is decidable; on other hand, if K does not satisfy 5, then the validity problem for the logic is undecidable. With at least two agents, the validity problem is undecidable no matter which subset of axioms K satisfies. We conjecture that, more generally, if the K operator satisfies the axioms of any logic L containing $\mathbf{K5}$, the logic of awareness of unawareness is decidable, while if K satisfies the axioms of any logic between $\mathbf{K}$ and $\mathbf{S4}$, the logic is undecidable.

All these results strongly suggest that there is something about the Euclidean property (or, equivalently, the negative introspection axiom) that simplifies things. However, they do not quite make precise exactly what that something is. More generally, it may be worth understanding more deeply what is about properties of the $\mathcal{K}$ relation that makes things easy or hard. We leave this problem for future work.

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