A Dichotomy Theorem on the Existence of Efficient or Neutral Sequential Voting Correspondences

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Abstract

Sequential voting rules and correspondences provide a way for agents to make group decisions when the set of available options has a multi-issue structure. One important question about sequential voting rules (correspondences) is whether they satisfy two crucial criteria, namely *neutrality* and *efficiency*. Recently, Benoit and Kornhauser established an important result about seat-by-seat voting rules (which are a special case of sequential voting rules): they proved that if the multi-issue domain satisfies some properties, then the only seat-by-seat rules being either efficient or neutral are dictatorships. However, there are still some cases not covered by their results, including a very important and interesting case—voting correspondences. In this paper, we extend the impossibility theorems by Benoit and Kornhauser to voting correspondences, and obtain a dichotomy theorem on the existence of efficient or neutral sequential (seat-by-seat) voting rules and correspondences. Therefore, the question of whether sequential (seat-by-seat) voting rules (correspondences) can be efficient or neutral is now completely answered.

1 Introduction

In many real-life group decision making problems, the space of alternatives has a multi-issue (or combinatorial) structure. For instance, in multiple referenda [Brams *et al.*, 1998], the inhabitants of some local community are asked to make a common decision on several related issues of local interest, such as building a public facility. As another example, the members of an association may have to elect a steering committee, composed of a president, a vice-president and a treasurer [Benoit and Kornhauser, 1991].

When voting is used in Artificial Intelligence contexts, these situations become even more pronounced. For example, agents may have to vote over a joint plan or an allocation of tasks or resources. These alternative spaces are also combinatorial, and they are generally much larger than those considered in human domains. This is one of the problems that is driving the burgeoning field of *computational social choice* (for an introduction, see [Chevaleyre *et al.*, 2007]).

In classical social choice theory, voters are supposed to submit their preferences as *linear orders* over the set of alternatives, and then a *voting rule* (*voting correspondence*) is applied to select one alternative (multiple alternatives) as the winner. However, problems emerge when the set of alternatives has a multi-issue structure: the number of alternatives becomes exponentially large, so it is unrealistic to ask voters to specify their preferences as (explicit) linear orders; hence, traditional voting rules or correspondences cannot be applied in a straightforward way. A simple idea to cope with this problem consists in decomposing an election into a set of independent elections, each of which bears on a single issue. This process gives rise to a *seat-by-seat*[1] voting rule. It works to some extent when the preferences of voters are *separable* (that is, if voters' preferences on each issue are independent from the values of other issues), but it is impractical when they are not, because in this case a voter cannot specify her preferences on a single issue without knowing the values of the other issues.

Instead of decomposing the election in parallel, it was proposed in [Lang, 2007] to compose local voting rules (or correspondences) *sequentially*: given a fixed linear order O (without loss of generality, $O = \mathbf{x}_1 > \ldots > \mathbf{x}_p$) on the set of issues, these rules work by holding an election for each issue according to a local voting rule, following O – that is, the election on issue issue $\mathbf{x}_i$ takes place after the outcome of the elections on $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ has been decided. For this sequential procedure to be applied, it is sufficient that each voter expresses, for each issue $\mathbf{x}_i$, her local preference relation over the values of $\mathbf{x}_i$, given any vector of values of issues $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$, That is, the voters' preferences are modeled by *acyclic CP-nets* [Boutilier *et al.*, 2004] whose structural part is an acyclic graph respecting O. The voting rules obtained in this way is called *sequential compositions of local voting rules*, or for short, *sequential voting rules*. There are also other approaches for making group decisions over multi-issue domains, including extensions of sequential voting rules [Xia *et al.*, 2007b; 2008], *m*CP-nets [Rossi *et al.*, 2004], and a method to aggregate numerical preferences over multi-issue domains, represented compactly by GAI-networks [Gonzales *et al.*, 2008].

[1]We keep this terminology "seat by seat" used in [Benoit and Kornhauser, 2006] even though our notions and results apply to any kind of voting context on a multi-issue domain, such as multiple referenda.

Sequential voting rules are elicitation-friendly, because they can be executed with an elicitation protocol which asks each voter only polynomially many queries, and easy to compute (provided that the local voting rules are easy to compute). When the preferences of all voters are *separable*, the outcome of a sequential voting rule is independent on the choice of the order O and the rule thus obtained is a seat-by-seat rule in the sense of [Benoit and Kornhauser, 2006]. Therefore, sequential voting rules are a generalization of seat-by-seat voting rules.

As local voting rules, local voting correspondences can also be composed in a similar way, giving rise to sequential voting correspondences. One of the most important questions for seat-by-seat voting rules and correspondences, and more generally for sequential voting rules and correspondences, is the following: are there any efficient or neutral seat-by-seat voting rules (or correspondences) other than dictatorships? Three recent papers partly answer this question. When the multi-issue domain is composed of two binary issues, two positive results are proved—one in [Xia *et al.*, 2007a], in which we showed that the sequential composition of two majority correspondences satisfies many good properties, including neutrality; another in [Özkal-Sanver and Sanver, 2006], in which the authors proved that when there is an odd number of voters, the seat-by-seat voting rule composed of majority rules is efficient. On the other hand, it was also proved in [Özkal-Sanver and Sanver, 2006] that when the domain is composed of at least three issues, or two issues and the number of voters is even, no anonymous seat-by-seat voting rule is efficient. Remarkably, Benoit and Kornhauser [2006] proved the following two elegant impossibility theorems: (a) when the multi-issue domain is not composed of two binary issues, the only efficient seat-by-seat voting rules are dictatorships; (b) when the multi-issue domain is composed of three or more issues, the only neutral and locally efficient seat-by-seat voting rules are dictatorships. We note that any non-existence result about seat-by-seat voting rules (correspondences) can be easily extended to the case of sequential composition of local rules (correspondences), because the latter are an extension of the former. Similarly, any existence result about sequential composition of local rules (correspondences) can be easily extended to the case of seat-by-seat voting rules (correspondences).

However, when the multi-issue domain is composed of two issues, at least one of which is non-binary, the neutrality of seat-by-seat voting rules is still not clear. Furthermore, it seems that the technique used to prove the impossibility theorems (a) and (b) cannot be fully extended to the case of voting correspondences. We argue that correspondences are very important, because of the following reasons. First, commonly studied voting rules are defined by voting correspondences plus tie-breaking mechanisms, and the voting correspondence part is the major part from a theoretical point of view; second, voting correspondences can have more good properties than voting rules. For example, many common voting correspondences satisfy *symmetry*, that is, both *neutrality* (all alternatives are treated equally) and *anonymity* (all voters are treated equally), but no voting rule satisfies symmetry. Third, there are many situations in which the agents want more options, that is, they want a set of "best alternatives" rather than just one winning alternative. In these situations, voting correspondences are better choices than voting rules. Specifically, sequential voting correspondences are extremely interesting for preference aggregation on multi-issue domains, because even though the set of alternatives is exponentially large, the output of a sequential voting correspondences can always be represented in a compact way, making it computationally feasible to output a set of winning alternatives.

In this paper, we prove that except in the case where the multi-issue domain is composed of two binary issues, the only efficient or neutral seat-by-seat voting correspondences are dictatorships, anti-dictatorships, and the trivial correspondence that always outputs the whole set of alternatives. We note that since it is a non-existence result, it can be easily generalized to the case of sequential voting rules and correspondences. We also show that the sequential composition of two majority correspondences is efficient. Finally, combining these results with previous work, we obtain dichotomy theorems on the existence of neutral or efficient sequential (seat-by-seat) voting rules and correspondences. Our dichotomy theorems completely answer the question on the existence of non-dictatorial and non-trivial sequential (seat-by-seat) voting rules and correspondences that are either efficient or neutral.

2 Preliminaries

Basics of voting. Let $\mathcal{X}$ be a finite set of *alternatives*. A *vote* V is a linear preference relation on $\mathcal{X}$, that is, a transitive, antisymmetric, and complete relation. We write $x \succ_V x'$ if an alternative x is preferred to another alternative x' in vote V. In this paper, we assume that the set of alternatives has a *multi-issue* (a.k.a. *multiattribute* or *combinatorial*) structure. That is, let $\mathfrak{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of issues, where each issue $\mathbf{x}_i$ takes values in a *local domain* D_i, with $|D_i| \geq 2$. An alternative is uniquely identified by the combination of values of its issues, that is, $\mathcal{X} = D_1 \times \ldots \times D_p$. A *profile* is a collection of individual linear preference relations over $\mathcal{X}$. A *voting rule* maps each profile to a unique winning alternative, and a *voting correspondence* maps each profile to a non-empty set of winning alternatives. For example, the *plurality* correspondence selects the alternatives that are ranked at the top the most number of times; when there are only two alternatives $\{x, y\}$, the *majority* correspondence maj is defined by $maj(P) = \{x\}$ (resp. $\{y\}$) if more voters in P prefer x to y (resp. y to x), and $maj(P) = \{x, y\}$ in case of a tie. A voting correspondence C is a *dictatorship* (resp. *anti-dictatorship*) if there exists a voter j such that for any profile $P = (V_1, \ldots, V_N)$, $C(P) = \{top(V_j)\}$ (resp. $C(P) = \{bot(V_j)\}$, where $top(V_j)$ ($bot(V_j)$) is the alternative that is ranked at the top (bottom) place in V_j. We note that a voting rule can be seen as a voting correspondence that always output one winner. Therefore, all definitions and results for correspondences *a fortiori* apply to rules, but the reverse does not always holds. For example, no voting rule satisfies both neutrality and anonymity, but many common voting correspondences (e.g. the plurality correspondence) satisfy both of them.

Preferences represented as CP-nets. A CP-net $\mathcal{N}$ over

$\mathcal{X}$ consists of two parts: (a) a directed graph $G = (\mathfrak{A}, E)$ and (b) a set of conditional linear preferences $\succeq^i_{\vec{u}}$ over D_i, for any assignment $\vec{u}$ of the parents of $\mathbf{x}_i$ in G. Let $CPT(\mathbf{x}_i)$ be the set of all conditional linear preferences on D_i, called a *conditional preference table (CPT)*. When G is acyclic, $\mathcal{N}$ is said to be an *acyclic CP-net*. $\mathcal{N}$ induces a partial preorder $\succeq_{\mathcal{N}}$ in the following way: for any $a_i, b_i \in D_i$, any assignment $\vec{u}$ of the parents of $\mathbf{x}_i$ (denoted by $Par_G(\mathbf{x}_i)$), and any assignment $\vec{z}$ of $\mathfrak{A} - Par_G(\mathbf{x}_i) - \{\mathbf{x}_i\}$, $(a_i, \vec{u}, \vec{z}) \succeq_{\mathcal{N}} (b_i, \vec{u}, \vec{z})$ if and only if $a_i \succeq^i_{\vec{u}} b_i$. That is, given the values of all parents of $\mathbf{x}_i$, the preference over $\mathbf{x}_i$ is independent of the values of all issues that are neither the parents of $\mathbf{x}_i$ nor $\mathbf{x}_i$ itself. When $\mathcal{N}$ is acyclic, $\succeq_{\mathcal{N}}$ is transitive and asymmetric, that is, a strict partial order.

A linear order V *extends* a CP-net $\mathcal{N}$, denoted by $V \sim \mathcal{N}$, if it extends the partial order induced by $\mathcal{N}$. V is *separable* if it extends a CP-net whose directed graph has no edge. For any assignment $\vec{u}$ of $Par_G(\mathbf{x}_i)$, let $V|_{\mathbf{x}_i:\vec{u}}$ be the restriction of V to $\mathbf{x}_i$, given $\vec{u}$. That is, $V|_{\mathbf{x}_i:\vec{u}}$ is the linear order $\succeq^i_{\vec{u}}$ over D_i. Given a DAG G on $\mathfrak{A}$, a CP-net $\mathcal{N}$ is *compatible* with G if its graph $G_{\mathcal{N}}$ is compatible with G, which means that $G_{\mathcal{N}} \subseteq G$; V is compatible with G if there exists a CP-net $\mathcal{N}$ such that $V \sim \mathcal{N}$ and $\mathcal{N}$ is compatible with G. If V is compatible with G, we also say that V is G*-legal*; we say V is *legal*, if it is G-legal for some acyclic graph G. The set of all G-legal votes is denoted by $Legal(G)$. A profile is G-legal if all of its votes are G-legal. For any linear order O on $\mathfrak{A}$, we define G_O to be the *graph induced by O*—that is, there is an edge $(\mathbf{x}_i, \mathbf{x}_j)$ in G_O if and only if $\mathbf{x}_i >_O \mathbf{x}_j$. We note that for any DAG G, a linear order O can be found such that $G \subseteq G_O$, which means that any G-legal profile is also G_O-legal.

Sequential voting rules and correspondences. Given an order $O = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ and a set of local correspondences $\{c_1, \ldots, c_p\}$ (that is, for any $i \leq p$, c_i is a voting correspondence on D_i), the *sequential composition* of local correspondences $c_1, \ldots, c_p$ w.r.t. O, denoted by $Seq(c_1, \ldots, c_p)$, is defined for all G_O-legal profiles as follows: $Seq(c_1, \ldots, c_p)(P)$ is characterized by the set of alternatives $(d_1, \ldots, d_p) \in \mathcal{X}$, so that for any $i \leq p$, we have $d_i \in c_i(P|_{\mathbf{x}_i:d_1\ldots d_{i-1}})$. We note that $Seq(c_1, \ldots, c_p)$ is well-defined, because for any G-legal profile, the set of winners is the same for all O such that $G \subseteq G_O$ (see [Lang, 2007]). When G has no edge, $Seq(c_1, \ldots, c_p)$ becomes a *seat-by-seat correspondence*, denoted by $Sbs(c_1, \ldots, c_p)$. It follows that for any separable profile P_s, $Sbs(c_1, \ldots, c_p)(P_s) = W_1 \times \ldots \times W_p$, where for any $i \leq p$, $W_i = c_i(P_s|_{\mathbf{x}_i})$.

Example 1 *Let $\mathfrak{A} = \{\mathbf{x}, \mathbf{y}\}$ with $D_{\mathbf{x}} = \{x_1, x_2, x_3\}$ and $D_{\mathbf{y}} = \{y, \bar{y}\}$, and $P = (V_1, \ldots, V_6)$ be the following 6-voter profile:*

V_1, V_2, V_3: $x_1\bar{y} \succ x_1y \succ x_2\bar{y} \succ x_2y \succ x_3y \succ x_3\bar{y}$
V_4, V_5: $x_2y \succ x_3y \succ x_2\bar{y} \succ x_1y \succ x_3\bar{y} \succ x_1\bar{y}$
V_6: $x_2y \succ x_2\bar{y} \succ x_1\bar{y} \succ x_1y \succ x_3\bar{y} \succ x_3y$

All these preference relations are compatible with the graph G over $\{\mathbf{x}, \mathbf{y}\}$ whose single edge is $(\mathbf{x}, \mathbf{y})$; equivalently, they follow the order $\mathbf{x} > \mathbf{y}$. Hence, $P \in Legal(G)$. The corresponding conditional preference tables are shown in Table 1.

Take $c_{\mathbf{x}}$ to be the plurality rule, and $c_{\mathbf{y}}$ to be the majority rule. The projection of P on $\mathbf{x}$, namely $P|_{\mathbf{x}} = (V_1|_{\mathbf{x}}, \ldots, V_6|_{\mathbf{x}})$, contains three votes $x_1 \succ x_2 \succ x_3$, two votes $x_2 \succ x_3 \succ x_1$, and one vote $x_2 \succ x_1 \succ x_3$. Therefore, the plurality winners for $P|_{\mathbf{x}}$ are $c_{\mathbf{x}}(P|_{\mathbf{x}}) = \{x_1, x_2\}$. Now, the projection of P on $\mathbf{y}$ given $\mathbf{x} = x_1$, namely $P|_{\mathbf{y}:\mathbf{x}=x_1} = (V_1|_{\mathbf{y}:\mathbf{x}=x_1}, \ldots, V_6|_{\mathbf{y}:\mathbf{x}=x_1})$, is composed of 4 votes for $\bar{y}$ and 2 for y, therefore $c_{\mathbf{y}}(P|_{\mathbf{y}:\mathbf{x}=x_1}) = \{\bar{y}\}$; $P|_{\mathbf{y}:\mathbf{x}=x_2}$ is composed of 3 votes for y and 3 votes for $\bar{y}$, therefore $c_{\mathbf{y}}(P|_{\mathbf{y}:\mathbf{x}=x_2}) = \{y, \bar{y}\}$. The sequential winners are now obtained by combining the $\mathbf{x}$–winners and the conditional $\mathbf{y}$-winners given any $\mathbf{x}$-winner, namely $Seq(c_{\mathbf{x}}, c_{\mathbf{y}})(P) = \{x_1\bar{y}, x_2y, x_2\bar{y}\}$.

voters 1,2,3	voters 4,5	voter 6
$x_1 \succ x_2 \succ x_3$	$x_2 \succ x_3 \succ x_1$	$x_2 \succ x_1 \succ x_3$
$x_1 : \bar{y} \succ y$	$x_1 : y \succ \bar{y}$	$x_1 : \bar{y} \succ y$
$x_2 : \bar{y} \succ y$	$x_2 : y \succ \bar{y}$	$x_2 : y \succ \bar{y}$
$x_3 : y \succ \bar{y}$	$x_3 : y \succ \bar{y}$	$x_3 : \bar{y} \succ y$

Table 1: CPTs of the 6 votes.

A sequential voting correspondence $C = Seq(c_1, \ldots, c_p)$ is *neutral*, if for any legal profile P and any permutation M on $\mathcal{X}$ such that $M(P)$ is legal, we have $C(M(P)) = M(C(P))$. C is *efficient* (resp. *anti-efficient*), if for any legal profile P, any $y \in C(P)$, any $x \in \mathcal{X}$, if $x \succ_V y$ (resp. $y \succ_V x$) for all $V \in P$, then, $x \in C(P)$.

Özkal-Sanver and Sanver [2006] proved that if there are at least three binary issues (or two binary issues and an even number of voters), then the seat-by-seat voting rule composed of majority rules is not efficient. Within the same line of research, two impossibility theorems were proved on the neutrality and efficiency of seat-by-seat voting rules [Benoit and Kornhauser, 2006].

Theorem 1 ([Benoit and Kornhauser, 2006]) *When the multi-issue domain is not composed of two binary issues, the only efficient seat-by-seat voting rules are dictatorships.*

Theorem 2 ([Benoit and Kornhauser, 2006]) *When the multi-issue domain is composed of at least three issues, and each local rule is efficient, the only neutral seat-by-seat voting rules are dictatorships.*

When the domain is composed of two binary issues, there are possibility results on the existence of seat-by-seat (sequential) voting rules or correspondences that are efficient, neutral, and nondictatorial. In [Xia *et al.*, 2007a], we proved that the sequential correspondence that is composed of two majority correspondences is neutral. It was also shown in [Özkal-Sanver and Sanver, 2006] that when there is an odd number of voters, the seat-by-seat voting rule that is composed of two majority rules is efficient.

3 Dichotomy theorems

We note that Theorem 2 does not cover the case in which the domain is composed of two issues, where at least one of the issues is not binary. Moreover, both Theorem 1 and Theorem 2 apply only to voting rules (but not correspondences), and the requirement that there is a unique winner plays an important role in their proofs. Therefore, it seems that these impossibility theorems cannot be easily extended to the case of voting correspondences. In this section, we first show that similar impossibility results hold for voting correspondences. Our results also cover the case of voting rules that is not covered in Theorem 2. Then, we show that the sequential composition of two majority correspondences is efficient. Finally, we obtain a dichotomy theorem on the existence of efficient or neutral seat-by-seat voting correspondence. Similar dichotomy theorems hold for sequential voting rules and correspondences.

Theorem 3 *If $\mathcal{X}$ is not composed of two binary issues, then the only efficient or neutral seat-by-seat voting correspondences on $\mathcal{X}$ are dictatorships, anti-dictatorships, and the trivial correspondence that always outputs the whole set of alternatives $\mathcal{X}$.*

Proof: The theorem is proved for two cases. Case 1: $\mathcal{X}$ is composed of three or more issues (Lemma 17). Case 2: $\mathcal{X}$ is composed of two issues, and at least one of the issues is not binary (Lemma 18). Statements and proofs of the lemmas are in the appendix. Due to space constraints, some proofs are omitted. □

Theorem 4 *Let $\mathcal{X} = \{x,\bar{x}\} \times \{y,\bar{y}\}$. $Seq(maj,maj)$ is efficient, where $O = \mathbf{x} > \mathbf{y}$.*

Proof: For the sake of contradiction we assume that the theorem does not hold. Then, there exists a profile P such that $\vec{a} = (a_1,a_2) \in Seq(maj,maj)(P)$, $\vec{b} = (b_1,b_2) \succ_V \vec{a}$ for any $V \in P$, and $\vec{b} \notin Seq(maj,maj)(P)$. If $a_2 = b_2$, then $b_1 \succ_{V|_\mathbf{x}} a_1$ for all $V \in P$, which means that $maj(P|_\mathbf{x}) = \{b_1\}$, $\vec{a} \notin Seq(maj,maj)(P)$, contradiction. Therefore, $a_2 \neq b_2$. Similarly, $a_1 \neq b_1$. Without loss of generality, we let $\vec{a} = (x,y), \vec{b} = (\bar{x},\bar{y})$. For any $V \in P$, because $(\bar{x},\bar{y}) \succ_V (x,y)$, either $[V|_\mathbf{x} = \bar{x} \succ x]$, or $[V|_\mathbf{x} = x \succ \bar{x}$, $V|_{\mathbf{y}:x} = \bar{y} \succ y$, and $V|_{\mathbf{y}:\bar{x}} = \bar{y} \succ y]$. Let n_1 (resp. n_2) be the number of votes of the former (resp. latter) type in P. It follows from $x \in maj(P|_\mathbf{x})$ that $n_2 \geq n_1$; from $y \in maj(P|_{\mathbf{y}:x})$ we must have that $n_1 \geq n_2$. Hence $n_1 = n_2$. Then, we have that $(\bar{x},\bar{y}) \in Seq(maj,maj)(P)$, contradiction. □

Combining Theorem 3, Theorem 4, and the Theorem 5.3 in [Xia *et al.*, 2007a], we obtain the following dichotomy theorem on the existence of efficient or neutral seat-by-seat correspondences and sequential voting correspondences.

Theorem 5 *Except in the case where the domain is composed of two binary issues, the only efficient or neutral seat-by-seat (sequential) correspondences are dictatorships, anti-dictatorships[2], and the trivial correspondence that always outputs the whole set of alternatives; when the domain is composed of two binary issues, and the local correspondences are majority correspondences, the seat-by-seat (sequential) voting correspondence is neutral and efficient.*

The restriction of Theorem 6 to seat-by-seat voting rules, combined with the results in [Özkal-Sanver and Sanver, 2006]), gives a dichotomy theorem that generalizes Theorem 1 and Theorem 2.

Theorem 6 *Except in the case where the domain is composed of two binary issues, the only efficient or neutral seat-by-seat (sequential) voting rules are dictatorships and anti-dictatorships[3]; when the domain is composed of two binary issues, a non-dictatorial anonymous voting rule can be efficient or neutral if and only if the number of voters is odd.*

[2] Not possible if the voting correspondence is efficient.

[3] Again, not possible if the voting rule is efficient.

4 Conclusion

In this paper we focus on characterizing the existence of efficient or neutral sequential (seat-by-seat) voting correspondences by the structure of the multi-issue domain. Our main results are the extensions of the two impossibility theorems in [Benoit and Kornhauser, 2006] to the case of voting correspondences. Based on our impossibility theorems, we obtain dichotomy theorems on the existence of efficient or neutral sequential (seat-by-seat) voting rules and correspondences.

These results have a strong impact in many classes of practical collective decision making situations. Every time a vote on several issues is organized sequentially or in parallel (which typically happens in recruiting committees, when several positions of different kinds are open, or in interest groups on the Internet which often make such decisions by conducting a parallel vote on several issues), there is no way of ensuring efficiency, nor neutrality, as soon as there are more than two issues, or as soon as there are two issues, one of which is not binary. If one wants neutrality and/or efficiency to hold, one has to find different methods for collective decision making, such as asking voters to report their preferences at once in a compact representation language, and then aggregate the individuals' inputs; but then the price to be paid is likely to be a very high communicational and computational cost.

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Appendix: Lemmas and proofs

We let $D_i = \{0_i, 1_i, \ldots, a_i\}$, where $a = |D_i|$. For any $l \leq |\mathcal{X}|$, any linear order V over $\mathcal{X}$, we let $(V)_l$ denote the alternative that is ranked at the l-th position in V. The alternative that is ranked at the top of V is denoted by $top(V)$ and the alternative that is ranked at the bottom of V is denoted by $bot(V)$. Two alternatives $\alpha, \beta \in \mathcal{X}$ are *exchangeable* in a linear order V that extends a CP-net $\mathcal{N}$, if α and β are adjacent, i.e. there is no $\gamma \in \mathcal{X}$ such that $\alpha \succ_V \gamma \succ_V \beta$, and for any V' that is obtained from V by exchanging α and β, $V' \sim_{\mathcal{N}} V$, that is, V' and V both extend the CP-net $\mathcal{N}$. We write $\mathcal{N} \not\models \beta \succ \alpha$, if there exists V that extends $\mathcal{N}$, $\alpha \succ_V \beta$, that is, $\beta \succ \alpha$ not in the partial order that $\mathcal{N}$ induces. The next lemma will be frequently used in later proofs.

Lemma 7 *Let $\alpha, \beta \in \mathcal{X}$, and let $\mathcal{N}$ be a CP-net such that $\mathcal{N} \not\models \beta \succ \alpha$ and $\mathcal{N} \not\models \alpha \succ \beta$, there exists a linear preference V extending $\mathcal{N}$ such that α and β are exchangeable in V, and $\alpha \succ_V \beta$.*

Then, we make an observation on how the dictatorship of seat-by-seat voting correspondence transfers to local correspondences.

Lemma 8 *If a seat-by-seat voting correspondence $Sbs(c_1, \ldots, c_p)$ is efficient or neutral, and it is not a dictatorship, then for any $i \leq p$, c_i is not a dictatorship.*

The next lemma states that if a seat-by-seat correspondence is neutral, then all local correspondences are efficient, or anti-efficient (a correspondence C is anti-efficient, if $\alpha \succ \beta$ in each vote, and α is a winner, then β is also a winner).

Lemma 9 *If $Sbs(c_1, \ldots, c_p)$ is neutral, then one of the following two conditions holds: 1. for all $i \leq p$, c_i satisfies efficiency. 2. for all $i \leq p$, c_i satisfies anti-efficiency.*

For all $k \leq N$, we let $R_i^k = (V_1^i, \ldots, V_N^i)$ be a profile on D_i defined as follows.

$$V_j^i = \begin{cases} 0_i \succ \ldots \succ a_i & \text{if } j \leq k \\ a_i \succ \ldots \succ 0_i & \text{if } j > k \end{cases}$$

For any $\alpha, \beta \in \mathcal{X}$, we say the preference between α and β is *not determined* in a CP-net $\mathcal{N}$, if $\mathcal{N} \not\models \beta \succ \alpha$ and $\mathcal{N} \not\models \alpha \succ \beta$. The next Lemma provides an easy way to check whether the preference between α and β is determined in a CP-net with no edge.

Lemma 10 *Let $\mathcal{N}$ be a CP-net whose graph has no edge. The preference between α and β is not determined in $\mathcal{N}$ if and only if there exist $i, j \leq p$ such that $\mathcal{N}|_{\mathbf{x}_i} \models \alpha_i \succ \beta_i$ and $\mathcal{N}|_{\mathbf{x}_j} \models \beta_j \succ \alpha_j$.*

Lemma 11 *Let $\mathcal{N}_1, \ldots, \mathcal{N}_N$ be N CP-nets such that for all $j \leq N$, the relative preference between α and β is not determined in $\mathcal{N}_j$. For any $Sbs(c_1, \ldots, c_p)$ that is efficient or neutral, and any profile P extending the CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_N$,*

$$\beta \in Sbs(c_1, \ldots, c_p)(P) \Longleftrightarrow \alpha \in Sbs(c_1, \ldots, c_p)(P).$$

Lemma 12 *Let $Sbs(c_1, \ldots, c_p)$ be a seat-by-seat correspondence that satisfies neutrality or efficiency, and there exists $i \leq p$ such that 1. there exists $d_i \in c_i(R_i^N)$, $d_i \notin \{0_i, a_i\}$, or 2. $\{0_i, a_i\} \subseteq c_i(R_i^N)$. Then, $c_i(R_i^N) = D_i$ for all $i \leq p$.*

We recall that R_i^N is a profile on D_i, in which all voters have the same preferences, namely, $0_i \succ \ldots \succ a_i$. The next lemma states that if the a efficient or neutral seat-by-seat correspondence is not trivial, then it must be either $c_i(R_i^N) = \{0_i\}$ for all $i \leq p$, or, $c_i(R_i^N) = \{a_i\}$ for all $i \leq p$.

Lemma 13 *Assume $Sbs(c_1, \ldots, c_p)$ satisfies neutrality or efficiency, and that there exist $i, j \leq p$ such that $0_i \in c_i(R_i^N)$ and $a_j \in c_j(R_j^N)$. Then $c_i(R_i^N) = D_i$ for all $i \leq p$.*

Lemma 14 *Let $Sbs(c_1, \ldots, c_p)$ be a seat-by-seat correspondence that is efficient or neutral. For any $i, j \leq p$, the following two propositions hold.*

1. $0_i \in c_i(R_i^N) \Longleftrightarrow a_j \in c_j(R_j^0)$.
2. $c_i(R_i^N) = \{0_i\} \Longleftrightarrow c_j(R_j^0) = \{a_j\}$.

Lemma 15 *Suppose the domain is not composed of two binary issues, and $Sbs(c_1, \ldots, c_p)$ is efficient. If there exists an N-vote profile P such that $Sbs(c_1, \ldots, c_p)(P) \neq \mathcal{X}$, then $c_i(R_i^N) = \{0_i\}$ and $c_i(R_i^0) = \{a_i\}$ for all $i \leq p$.*

The next lemma discusses a property about winners of a neutral seat-by-seat rule for the two cases separately.

Lemma 16 *When $\mathcal{X}$ is not composed of two binary issues, and $Sbs(c_1, \ldots, c_p)$ is neutral, if there exists an N-vote profile P s.t. $Sbs(c_1, \ldots, c_p)(P) \neq \mathcal{X}$, then for all $i \leq p$, one of the following holds.*

1. $c_i(R_i^N) = \{0_i\}$, *when each r_i is efficient*
2. $c_i(R_i^N) = \{a_i\}$, *when each r_i is anti-efficient.*

We next prove the impossibility theorem when there are at least three issues.

Lemma 17 *For any $p \geq 3$, we have:*

1. *If $Sbs(c_1, \ldots, c_p)$ satisfies efficiency, then it is either a dictatorship or the trivial correspondence.*
2. *If $Sbs(c_1, \ldots, c_p)$ satisfies neutrality, then it is either a dictatorship, an anti-dictatorship, or the trivial correspondence.*

Proof of Lemma 17: By Theorem 4.1 in [Xia *et al.*, 2007a], if $Sbs(c_1, \ldots, c_p)$ is efficient, then c_i is efficient for any $i \leq p$. By Lemma 9, if $Sbs(c_1, \ldots, c_p)$ is neutral, then either c_i is efficient for any $i \leq p$, or c_i is anti-efficient for any $i \leq p$. We next prove the case in which c_i is efficient for all $i \leq p$; the anti-efficient case can be proved similarly.

Suppose $Sbs(c_1,\dots,c_p)$ is not the trivial correspondence. If $Sbs(c_1,\dots,c_p)$ satisfies efficiency, then by Lemma 15 $c_i(R_i^0)=\{a_i\}$ for all $i \leq p$. If $Sbs(c_1,\dots,c_p)$ satisfies neutrality, then we also have, $c_i(R_i^0)=\{a_i\}$, because by Lemma 16 $c_i(R_i^0)=\{a_i\}$ or $c_i(R_i^0)=\{0_i\}$, and the latter case is excluded by the efficiency of c_i. Let k_i be the natural number such that for all $l \leq k_i-1$, $a_i \in c_i(R_i^l)$ and $a_i \notin c_i(R_i^{k_i})$. Since $a_i \notin c_i(R_i^N)=\{0_i\}$, k_i is well defined and $k_i \leq N$ for all $i \leq p$.

First we claim $k_j=k_r$ for all $j,r \leq N$. If not, w.l.o.g. suppose $k_1<k_2$. We let $\mathfrak{N}=(\mathcal{N}_1,\dots,\mathcal{N}_p)$ be a set of CP-nets defined as follow

$$\begin{aligned}\mathcal{N}_1=\mathcal{N}_2=\dots=\mathcal{N}_{k_1}:&(0_1\succ\dots\succ a_1, 0_2\succ\dots\succ a_2,\\ &0_3\succ\dots\succ a_3,\dots,0_p\succ\dots\succ a_p)\\ \mathcal{N}_{k_1+1}=\dots=\mathcal{N}_N:&(a_1\succ\dots\succ 0_1, a_2\succ\dots\succ 0_2,\\ &0_3\succ\dots\succ a_3,\dots,0_p\succ\dots\succ a_p)\end{aligned}$$

It follows that $a_1 \notin c_1(R_1^{k_1})$ and $a_2 \in c_2(R_2^{k_1})$. Let $d_1 \in c_1(R_1^{k_1})$. Because c_1 is efficient, $(d_1,a_2,0_3,\dots,0_p) \in Sbs(c_1,\dots,c_p)(P)$ for any P extending $\mathfrak{N}$. We note that in all $\mathcal{N}_j$, $d_1 \succ_{\mathcal{N}_j|_{\mathbf{x}_1}} a_1$ iff $0_2 \succ_{\mathcal{N}_j|_{\mathbf{x}_2}} a_2$. By Lemma 10, the preference between $(d_1,a_2,0_3,\dots,0_p)$ and $(a_1,0_2,0_3,\dots,0_p)$ is not determined in any $\mathcal{N}_j$, so by Lemma 11 $(a_1,0_2,0_3,\dots,0_p) \in Sbs(c_1,\dots,c_p)(P)$, which means $a_1 \in c_1(R_1^{k_1})$, contradiction. Therefore $k_1=k_2$. We let $k=k_1$.

Now since $Sbs(c_1,\dots,c_p)$ is not a dictatorship, by Lemma 8 c_i is not a dictatorship. Therefore, there exists an N-voter profile $P^i=(V_1^i,\dots,V_N^i)$ on D_i such that $c_i(P^i) \neq \{top(V_k^i)\}$, because voter k is not a dictator for c_i. W.l.o.g. we let $i=3$. We consider the following CP-nets $\mathfrak{N}'=(\mathcal{N}_1',\dots,\mathcal{N}_N')$ where $\mathcal{N}_j'$ is defined as follows:

$$\mathcal{N}_j': \begin{cases} (0_1\succ\dots\succ a_1, 0_2\succ\dots\succ a_2, V_j^3,\dots,0_p\succ\dots\succ a_p) & \text{if } j \leq k-1\\ (0_1\succ\dots\succ a_1, a_2\succ\dots\succ 0_2, V_j^3,\dots,0_p\succ\dots\succ a_p) & \text{if } j = k\\ (a_1\succ\dots\succ 0_1, a_2\succ\dots\succ 0_2, V_j^3,\dots,0_p\succ\dots\succ a_p) & \text{if } j \geq k+1\end{cases}$$

We note that $\mathcal{N}'=(R_1^k,R_2^{k-1},P^3,R_4^N,\dots,R_p^N)$. By the definition of k, $a_1 \notin c_1(R_1^k)$. Therefore, there exists $d_1 \in c_1(\mathfrak{N}'|_{\mathbf{x}_1})$ such that $d_1 \neq a_1$, and for any P extending $\mathfrak{N}'$, any $d_3 \in c_3(P^3)$, we have $(d_1,a_2,d_3,0_4,\dots,0_p) \in Sbs(c_1,\dots,c_p)(P)$. Notice that when $j \neq k$, $d_1 \succ_{\mathcal{N}_j'|_{\mathbf{x}_1}} a_1$ iff $0_2 \succ_{\mathcal{N}_j'|_{\mathbf{x}_2}} a_2$. When $j=k$, $top(V_k^3) \succ_{\mathcal{N}_k'|_{\mathbf{x}_3}} d_3$ for any $d_3 \in c_3(P^3)$. Therefore the relative preference between $(d_1,a_2,d_3,0_4,\dots,0_p)$ and $(a_1,0_2,top(V_k^3),0_4,\dots,0_p)$ is not determined. Then, by Lemma 11, $(a_1,0_2,top(V_k^3),0_4,\dots,0_p) \in Sbs(c_1,\dots,c_p)(P)$, which means $a_1 \in c_1(R_1^k)$, contradicting with the definition of k. This completes the proof. □

Finally we consider the case of two issues, and one of the issues is not binary.

Lemma 18 *If $p=2$ and $|D_1| \geq 3$ or $|D_2| \geq 3$, then the only seat-by-seat correspondence satisfying neutrality or efficiency is a dictatorship, an anti-dictatorship, or the trivial correspondence.*

Proof of Lemma 18: Similar with the proof for Lemma 17, we assume that c_i is efficient for all $i \leq p$. Without loss of generality, let $|D_1| \geq 3$. If the lemma is not true, then there exists a non-trivial non-dictatorial $Sbs(c_1,c_2)$ satisfying neutrality or efficiency. First we make the following claim.

Claim 1 *Given $l \leq N$, let $(V_1^2,\dots,V_N^2)$ be a profile on D_2 s.t. $V_j^2 = \begin{cases} 0_2\succ\dots\succ a_2 & \text{if } j=l\\ a_2\succ\dots\succ 0_2 & \text{if } j\neq l\end{cases}$ Then, $c_2(V_1^2,\dots,V_N^2)=\{a_2\}$.*

The proof for Claim 1 utilizes the following claim. Due to space constraints, the proofs for Claim 1 and 2 are omitted.

Claim 2 *Let $P'^2=(V_1'^2,\dots,V_N'^2)$ such that $V_1'^2=V_1^2=(0_2\succ\dots\succ a_2)$, and for every $j\neq 2$, $V_j'^2 \in \{0_2\succ\dots\succ a_2, a_2\succ\dots\succ 0_2\}$. Denote the natural preorder on D_2 by $\rhd$, namely $0_2 \rhd \dots \rhd a_2$. Then*

(1) if there exists $d_2 \neq a_2$ such that $d_2 \in c_2(P^2)$ then there exists $e_2 \unrhd d_2$ such that $e_2 \in c_2(P'^2)$.

(2) if $c_2(P'^2)=D_2$ then either $c_2(P^2)=\{a_2\}$ or $c_2(P^2)=D_2$.

Claim 1 states that if only one of the voters votes for $0_2\succ\dots\succ a_2$, and the other voters vote for $a_2\succ\dots 0_2$, then, a_i is the only winner. Similarly, it can be proved that if only one of the voters votes for $a_2\succ\dots\succ 0_2$, and the other voters vote for $0_2\succ\dots\succ a_2$, then, 0_i is the only winner. When there are two voters, it follows that $c_2(0_2\succ\dots\succ a_2, a_2\succ\dots\succ 0_2)=\{a_2\}=\{0_2\}$, which is a contradiction.

Then, we consider $N \geq 3$. We make the following claim.

Claim 3 *For any $1 \leq k \leq N$, $c_2(R_2^k)=\{a_2\}$.*

The proof is by induction on k. Again, we omit it due to space constraints.

By Claim 3, and let $k=N$, we have $c_2(R_2^N)=\{a_2\}$. Because the premise of Lemma 15 is satisfied by any profile P whose projection on D_2 is R_2^N, from Lemma 15 we have $c_2(R_2^N)=\{0_2\}$, a contradiction. So the assumption in the beginning of the proof is not true, which means the theorem is true. □