RCC8 Is Polynomial on Networks of Bounded Treewidth

Manuel Bodirsky
Laboratoire d'Informatique
(LIX, CNRS UMR 6171)
École Polytechnique, France
bodirsky@lix.polytechnique.fr

Stefan Wölfl
Department of Computer Science
University of Freiburg, Germany
woelfl@informatik.uni-freiburg.de

Abstract

We construct an homogeneous (and ω-categorical) representation of the relation algebra RCC8, which is one of the fundamental formalisms for spatial reasoning. As a consequence we obtain that the network consistency problem for RCC8 can be solved in polynomial time for networks of bounded treewidth.

Qualitative spatial reasoning (QSR) is concerned with representation formalisms that are considered close to conceptual schemata used by humans for reasoning about their physical environment—in particular, about processes or events and about the spatial environment in which they are situated. The approach in qualitative reasoning is to develop relational schemas that abstract from concrete metrical data of entities (for example, time points, coordinate positions, distances) by subsuming similar (geo-) metric or topological configurations of entities into one qualitative representation.

RCC8 [Randell *et al.*, 1992b; Cohn and Hazarika, 2001] is a prominent relation algebra studied in QSR that deals with extended regions. The relation algebra is derived from a multi-sorted first-order theory called *region connection calculus*. Its network consistency problem is one of the most fundamental tasks in qualitative spatial reasoning [Renz, 2002; Renz and Nebel, 2007; Bennett, 1998]. There have been numerous papers on the network consistency problem for RCC8, in particular about its mathematical foundations [Renz and Ligozat, 2005; Li and Ying, 2003], its computational complexity [Renz, 2002; Renz and Nebel, 2007; Bennett, 1998], and its cognitive adequacy as a model for human spatial reasoning [Renz *et al.*, 2000; Ragni *et al.*, 2007].

We show that the network consistency problem for RCC8 can be formulated as a constraint satisfaction problem (CSP) with a countably infinite ω-categorical template $\mathfrak{R}$. Using recent results from [Bodirsky and Dalmau, 2008], this shows that the network consistency problem for RCC8 can be solved in polynomial time on networks of bounded treewidth [Bodirsky and Dalmau, 2008].

The graph-theoretic notions of tree decomposition and treewidth have proven valuable concepts in order to investigate algorithmic properties of problems that can be parametrized by structural properties of graphs (see, e.g., [Bodlaender, 1993]). In the context of constraint satisfaction, these concepts come into play as constraint network can be cast as labeled graphs in a natural way. A tree decomposition of a constraint network is a tree decomposition of its constraint graph: roughly speaking, a decomposition defines a set of subnetworks that can be glued together in a tree-like manner. The width of such a decomposition, then, is the size of the largest subnetwork in the decomposition (in terms of the variables in the network). The treewidth of a constraint network is the minimal width for any potential tree decomposition of the network.

It is well-known that a finite domain CSP is tractable when the input is restricted to networks of bounded treewidth; for a recent survey on related results, see e.g. [Samer and Szeider, 2010]. One algorithm to solve arbitrary finite domain CSPs for input instances of treewidth k is the k-consistency algorithm. In fact, it has been shown that finite domain CSPs, restricted to a class of networks $\mathcal{C}$, can be solved in polynomial time by k-consistency if and only if the *cores* of the structures in $\mathcal{C}$ have bounded treewidth [Atserias *et al.*, 2007].

The situation when the domain of the CSP is infinite, and this is the case for the spatial reasoning problems that we study in this paper, is less well-studied. The key concept in the proof of our RCC8 tractability result is the so-called amalgamation property, which allows us to use a general result for infinite-domain CSPs of bounded treewidth shown in [Bodirsky and Dalmau, 2008]. We show that the class of all finite RCC8 structures has that property that if $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ are structures in that class such that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are extensions of $\mathfrak{A}$ by finitely many distinct new elements, then there exists a structure in that class that is an extension of both $\mathfrak{B}_1$ and $\mathfrak{B}_2$. As in model theory [Hodges, 1997], the amalgamation property is defined here in term of models, i.e., in terms of *solutions* of constraint networks. Concepts of amalgamation tailored to the amalgamation of constraint networks themselves have been studied in [Li *et al.*, 2008; 2009]. In [Li *et al.*, 2008] a notion of *network amalgamation property* is introduced: a relation algebra A is said to have the amalgamation property if for each triple of path-consistent constraint networks N_0, N_1 and N_2 (with constraint relations from A) such that N_0 is a subnetwork of N_1 and N_2, there exists a path-consistent constraint network over A that has N_1 and N_2 as subnetworks. More similar to the model-theoretic concept of amalgamation is the notion of *atomic network amalgamation* introduced in [Li *et al.*, 2009].

1 RCC8

RCC8 is a spatial representation formalism derived from the *region connection calculus* [Randell *et al.*, 1992b; Cohn and Hazarika, 2001], a multi-sorted first order theory based on a primitive binary relation, C, expressing *connectedness* of spatially extended entities (referred to as *regions*). Since different versions of the RCC theory can be found in the literature, we introduce RCC models in terms of specific *Boolean contact algebras* [Bennett and Düntsch, 2007]. For the purposes of this paper, an *RCC model* is a model of the theory of non-trivial Boolean algebras ($0 \neq 1$) and the following axioms (where $x \leq y$ is defined as $x \cdot y = x$): (a) $\forall xy(C(x,y) \Rightarrow x,y \neq 0)$, (b) $\forall x(x \neq 0 \Rightarrow C(x,x))$, (c) $\forall xy(C(x,y) \Rightarrow C(y,x))$, (d) $\forall xy(x \leq y \Leftrightarrow \forall z(C(x,z) \Rightarrow C(y,z)))$, (e) $\forall xyz(C(x,y+z) \Rightarrow C(x,y) \vee C(x,z))$, and (f) $\forall x(x \neq 0,1 \Rightarrow C(x,-x))$.

Examples of RCC models can be generated from regular, connected topological spaces when regular closed subsets are conceived of as regions and the contact relation holds between such sets if and only if they interesect. It should be mentioned that RCC models can even be characterized in a topological sense [Bennett and Düntsch, 2007]: each RCC model is isomorphic to a substructure of the RCC model induced (via the standard interpretation) by some weakly regular and connected T1 space (here weakly regular refers to a pointless variant of the regularity separation axiom).

In terms of RCC models the RCC8 relations DC (*disconnected*), EC (*externally connected*), PO (*partial overlap*), EQ (*equals*), TPP (*tangential proper part*), and NTPP (*nontangential proper part*) (as well as the converse relations of TPP and NTPP) may be introduced as follows:

$$\begin{array}{rl}
\mathrm{EQ}(x,y) & \text{iff } \mathrm{P}(x,y) \wedge \mathrm{P}(y,x) \\
\mathrm{DC}(x,y) & \text{iff } x,y \neq 0 \wedge \neg \mathrm{C}(x,y) \\
\mathrm{EC}(x,y) & \text{iff } \mathrm{C}(x,y) \wedge \neg \exists z(\mathrm{P}(z,x) \wedge \mathrm{P}(z,y)) \\
\mathrm{PO}(x,y) & \text{iff } \neg \mathrm{P}(x,y) \wedge \neg \mathrm{P}(y,x) \wedge \exists z(\mathrm{P}(z,x) \wedge \mathrm{P}(z,y)) \\
\mathrm{TPP}(x,y) & \text{iff } \mathrm{PP}(x,y) \wedge \exists z(\mathrm{EC}(z,x) \wedge \mathrm{EC}(z,y)) \\
\mathrm{NTPP}(x,y) & \text{iff } \mathrm{PP}(x,y) \wedge \neg \mathrm{TPP}(x,y) \\
\mathrm{TPPi}(x,y) & \text{iff } \mathrm{TPP}(y,x) \\
\mathrm{NTPPi}(x,y) & \text{iff } \mathrm{NTPP}(y,x)
\end{array}$$

where $\mathrm{P}(x,y)$ is defined as $x \neq 0 \wedge x \leq y$ and $\mathrm{PP}(x,y)$ as $\mathrm{P}(x,y) \wedge y \not\leq x$. Note that in RCC models as defined above the relation EQ entails the identity relation (they are *strict* models in the sense of [Stell, 2000]). From the definitions it follows that for each RCC model, the set of RCC8 relations induces a relational partition scheme on the set of its non-null regions.

Definition 1 (cp. [Ligozat and Renz, 2004]). A *relational partition scheme* on D is a finite set, $\mathcal{B}$, of (possibly empty) binary relations on D that forms a partition of $D \times D$, contains the diagonal (or identity) relation $\{(x,x) : x \in D\}$, and is closed under converses (i.e., $B^{-1} := \{(y,x) : (x,y) \in B\} \in \mathcal{B}$ for $B \in \mathcal{B}$). The elements of $\mathcal{B}$ are referred to as *base relations*, and unions of base relations are referred to as *general relations* of the partition scheme.

In general, the set of general relations is not closed under composition, i.e., for general relations R and S, the composition $R \circ S := \{(x,z) \in D^2 : \exists y((x,y) \in R \wedge (y,z) \in S)\}$ need not be a general relation of the partition scheme. This is specifically the case for RCC8 partition schemes induced by RCC models (see, e.g., [Bennett and Düntsch, 2007]).

To enable reasoning on the symbolic level (see section 2) when satisfiability of formulae needs to be checked with respect to a class of models, one identifies a composition function (usually recorded as a table) that provides the strongest upper approximation for the composition of concrete relations in all models of that class. The RCC8 composition function $\circ^{\mathrm{RCC8}}$ is defined in Table 1 (see e.g. [Randell *et al.*, 1992a]). The entry in row R and column S shows the base relations $T_1, \ldots, T_k$ such that $\{T_1, \ldots, T_k\} = R \circ^{\mathrm{RCC8}} S$. It can be shown that each RCC model satisfies the RCC8 composition table in the sense that it satisfies all formulae

$$\forall xyz(R(x,y) \wedge S(y,z) \Rightarrow T_1(x,z) \vee \cdots \vee T_k(x,z)) \qquad (*)$$

where R and S are symbols in the relational signature of RCC8 base relations, τ, and $R \circ^{\mathrm{RCC8}} S = \{T_1, \ldots, T_k\}$. If we write general relations R representing a union of base relations $B_1 \cup \cdots \cup B_m$ as sets $\{B_1, \ldots, B_m\}$, then converse and composition of general relations can be defined as follows: $R^{-1} := \{B^{-1} : B \in R\}$ and $R \circ^{\mathrm{RCC8}} R' := \bigcup_{B \in R, B' \in R'} B \circ^{\mathrm{RCC8}} B'$. By these settings the set of general relations becomes a set algebra that is closed under this approximative composition.

Definition 2. Let τ be the relational signature of RCC8 relations. An *RCC8 model* is a τ-structure, $\mathfrak{A}$, with non-empty domain A such that (a) $\{\mathrm{EQ}^{\mathfrak{A}}, \mathrm{DC}^{\mathfrak{A}}, \ldots, \mathrm{NTPPi}^{\mathfrak{A}}\}$ defines a relational partition scheme on A, (b) $\mathrm{EQ}^{\mathfrak{A}}$ is the identity relation, (c) $\mathrm{DC}^{\mathfrak{A}}$, $\mathrm{EC}^{\mathfrak{A}}$ and $\mathrm{PO}^{\mathfrak{A}}$ are symmetric, (d) $\mathrm{TPPi}^{\mathfrak{A}}$ is the converse of $\mathrm{TPP}^{\mathfrak{A}}$ and $\mathrm{NTPPi}^{\mathfrak{A}}$ the converse of $\mathrm{NTPP}^{\mathfrak{A}}$, and (e) when $R \circ^{\mathrm{RCC8}} S = \{T_1, \ldots, T_k\}$, then $\mathfrak{A}$ satisfies $(*)$.

An RCC8 model $\mathfrak{A}$ is called *extensional* if composition in $\mathfrak{A}$ coincides with the RCC8 composition table. Examples of RCC8 models include the closed disks model (regions are closed disks in the Euclidean plane) and the model by Egenhofer [1991] (regions are Jordan curve bounded regions of the Euclidean plane, i.e., regions that are homeomorphic to the unit disk). Both models are known to be extensional [Bennett and Düntsch, 2007; Li and Ying, 2003].

2 Constraint-based Reasoning with RCC8

Given the relational signature of RCC8 relations, τ, we now introduce basic concepts of constraint-based reasoning with RCC8. An *RCC8 constraint network* is defined by a finite set of variables V and a finite set of constraints between pairs of variables. Each constraint specifies a general relation that needs to hold between these variables. Since the set of RCC8 relations is closed under intersections, we may cast a constraint network as a directed and simple labeled graph $\langle V,E,l \rangle$. Its arcs E are simply the scopes of the constraints in the network, and to each such arc, the labeling function l assigns the constraint relation R (i.e., some subset of the signature τ). A *solution* of a constraint network $\langle V,E,l \rangle$ with respect to an RCC8 model $\mathfrak{A}$ with domain A is a function $*^I : V \to A$ such that for each arc (v_1, v_2) in E, $(v_1^I, v_2^I) \in R^{\mathfrak{A}}$ for some $R \in l(x,y)$. A constraint network is said to be *satisfiable in* $\mathfrak{A}$ if it has a solution in $\mathfrak{A}$ and it is said to be *satisfiable* if it has a solution in some RCC8 model.

.

Table 1: The composition table of RCC8. “1” indicates the set of all base relations of RCC8; compositions with EQ are omitted.

$\circ^{\text{RCC8}}$	**DC**	**EC**	**PO**	**TPP**	**NTPP**	**TPPi**	**NTPPi**
DC	1	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC	DC
EC	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPP, TPPi, EQ	DC, EC, PO, TPP, NTPP	EC, PO, TPP, NTPP	PO, TPP, NTPP	DC, EC	DC
PO	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPPi, NTPPi	1	PO, TPP, NTPP	PO, TPP, NTPP	DC, EC, PO, TPPi, NTPPi	DC, EC, PO, TPPi, NTPPi
TPP	DC	DC, EC	DC, EC, PO, TPP, NTPP	TPP, NTPP	NTPP	DC, EC, PO, TPP, TPPi, EQ	DC, EC, PO, TPPi, NTPPi
NTPP	DC	DC	DC, EC, PO, TPP, NTPP	NTPP	NTPP	DC, EC, PO, TPP, NTPP	1
TPPi	DC, EC, PO, TPPi, NTPPi	EC, PO, TPPi, NTPPi	PO, TPPi, NTPPi	PO, TPP, TPPi, EQ	PO, TPP, NTPP	TPPi, NTPPi	NTPPi
NTPPi	DC, EC, PO, TPPi, NTPPi	PO, TPPi, NTPPi	PO, TPPi, NTPPi	PO, TPPi, NTPPi	PO, TPP, TPPi, NTPP, NTPPi, EQ	NTPPi	NTPPi

The network consistency problem for RCC8 is the following computational problem: given as input an RCC8 constraint network, decide whether the network is satisfiable in some RCC8 model. This problem is NP-complete in general (see, e.g., [Renz and Nebel, 2007]). The network consistency problem is the central problem for reasoning with RCC8, since many other reasoning tasks, such as entailment checking, can be reduced to it.

A constraint network $\langle V,E,l\rangle$ is called *atomic* if for each $(x,y) \in E$, $l(x,y)$ is a singleton set, *complete* if $E = V \times V$, and *normalized* if it is complete and it holds $l(x,x) = \{\text{EQ}\}$ as well as $l(x,y) = l(y,x)^{-1}$. A constraint network $\langle V',E',l'\rangle$ is called a *refinement* of $\langle V,E,l\rangle$ if $V = V'$, $E \subseteq E'$, and $l'(x,y) \subseteq l(x,y)$ for each arc $(x,y) \in E$, and called a *subnetwork* if $V' \subseteq V$, $E' \subseteq E$, and $l(x,y) \subseteq l'(x,y)$ for each arc $(x,y) \in E'$. A normalized RCC8 constraint network $\langle V,E,l\rangle$ is said to be *path-consistent* if $l(x,y) \subseteq l(x,z) \circ^{\text{RCC8}} l(z,y)$ for each triple of variables x,y,z in V. Note that each RCC8 constraint network can be transformed into an equivalent normalized network. Moreover, each constraint network can be refined into an equivalent path-consistent constraint network. This can be achieved if each of the labels $l(x,y)$ is refined by applying the operation

$$l(x,y) \leftarrow l(x,y) \cap (l(x,z) \circ^{\text{RCC8}} l(z,y)),$$

where z is any third variable occurring in the network, until a fixpoint is reached. For implementation details of this path-consistency procedure see, for example, [Mackworth, 1977; Renz and Nebel, 2007]). If the resulting constraint network has empty labels, the original network is unsatisfiable.

3 Homomorphisms, Embeddings, Amalgamation

In the proof of our tractability result we use a result from [Bodirsky and Dalmau, 2008], which is formulated in a slightly different formalism, namely the homomorphism formulation of constraint satisfaction problems. Let $\mathfrak{A}$ be a relational structure with a finite relational signature τ. Then the *constraint satisfaction problem* of $\mathfrak{B}$, CSP($\mathfrak{B}$), is the computational problem to decide for a finite τ-structure $\mathfrak{A}$ whether $\mathfrak{A}$ homomorphically maps to $\mathfrak{B}$. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a map $f : A \to B$ such that for all $a,b \in A$ and each binary $R \in \tau$ with $a\, R^{\mathfrak{A}}\, b$, it holds $f(a)\, R^{\mathfrak{B}}\, f(b)$; for higher-ary relations, the definition is analogous, but not needed here. We will see later that the network consistency problem for RCC8 can be viewed as a constraint satisfaction problem for a certain infinite structure $\mathfrak{S}$; more importantly, we show that $\mathfrak{S}$ has certain pleasant model-theoretic properties that allow us to derive our tractability result. We make the convention that the domains of structures $\mathfrak{A},\mathfrak{B},\mathfrak{C},\dots$ will be denoted by $A,B,C,\dots$, respectively. For a subset S of the domain A of $\mathfrak{A}$, we write $\mathfrak{A}[S]$ for the substructure of $\mathfrak{A}$ induced by S. In this paper, substructure always means *induced substructure*, as in [Hodges, 1997]. An *embedding* of a τ-structure $\mathfrak{A}$ in a τ-structure $\mathfrak{B}$ is a mapping $f : A \to B$ that is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}[f(A)]$. The *age* of a relational structure $\mathfrak{A}$ is the set of finite structures that embed into $\mathfrak{A}$. A class of finite structures $\mathcal{C}$ with relational signature τ is an *amalgamation class* if $\mathcal{C}$ is nonempty, closed under isomorphisms and taking substructures, and has the *amalgamation property*, which says that for all $\mathfrak{A},\mathfrak{B}_1,\mathfrak{B}_2 \in \mathcal{C}$ and embeddings $e_1 : A \to B_1$ and $e_2 : A \to B_2$ there exists $\mathfrak{C} \in \mathcal{C}$ and embeddings $f_1 : B_1 \to C$ and $f_2 : B_2 \to C$ such that $f_1 e_1 = f_2 e_2$. We call $\mathfrak{C}$ the *amalgam of* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *over* $\mathfrak{A}$. When we can always amalgamate such that $f_1(B_1) \cap f_2(B_2) = f_1(e_1(A))$, we say that $\mathcal{C}$ is a *strong* amalgamation class. A structure $\mathfrak{A}$ is called *homogeneous* (sometimes *ultra-homogeneous* [Hodges, 1997]) if every isomorphism between finite substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. The most famous example of a homogeneous structure is given by the rationals (as an ordered set). In fact, each order isomorphism between finite subsets can be extended to an automorphism on the rationals. Moreover, the set of all finite linear orders forms an amalgamation class.

Theorem 1 (Fraïssé 1954; see also [Hodges, 1997]). *A countable class $\mathcal{C}$ of finite relational structures with countable signature is the age of a countable homogeneous structure $\mathfrak{B}$ if and only if $\mathcal{C}$ is an amalgamation class. In this case $\mathfrak{B}$ is up to isomorphism unique and called the* Fraïssé limit *of $\mathcal{C}$.*

A relational structure $\mathfrak{B}$ (i.e., a structure for a purely relational signature) is called *ω-categorical* if all countable models of the first-order theory[1] of $\mathfrak{B}$ are isomorphic to each other. Homogeneous structures provide a rich source of ω-categorical structures.

Proposition 1 (see [Hodges, 1997]). *A countable homogeneous $\mathfrak{B}$ over a finite relational signature is ω-categorical.*

4 A Homogeneous Representation for RCC8

In this section we construct a homogeneous representation for RCC8. In this context, a *representation for RCC8* is an RCC8 model that is extensional, i.e., it has the RCC8 composition table as its composition table. The relevance of homogeneous representations in the context of relation algebras and constraint satisfaction has first been recognized in [Hirsch, 1997], where such representations are described for several fundamental relation algebras such as Allen's interval algebra and the containment algebra.

We will use $\mathrm{PP}(x,y)$ as a shortcut for the relation defined by $\mathrm{NTPP}(x,y) \vee \mathrm{TPP}(x,y)$, $\mathrm{P}(x,y)$ as a shortcut for the relation defined by $\mathrm{PP}(x,y) \vee \mathrm{EQ}(x,y)$ (which is equivalent to our previous settings), and Pi and PPi to denote the converses of these relations. In this section, τ denotes the signature of RCC8 base relations. The structure $\mathfrak{R}$ will be defined as the Fraïssé limit of an appropriate class of finite τ-structures $\mathcal{R}$.

Definition 3. The class $\mathcal{R}$ is the class of all finite τ-structures $\mathfrak{A}$ that satisfy conditions (a) – (d) of Definition 2 as well as the following universally quantified axioms:

$$\mathrm{DC}(x,y) \wedge \mathrm{Pi}(y,z) \Rightarrow \mathrm{DC}(x,z) \quad (1)$$

$$\mathrm{EC}(x,y) \wedge \mathrm{P}(y,z) \Rightarrow \mathrm{EC}(x,z) \vee \mathrm{PO}(x,z) \vee \mathrm{PP}(x,z) \quad (2)$$

$$\mathrm{EC}(x,y) \wedge \mathrm{Pi}(y,z) \Rightarrow \mathrm{DC}(x,z) \vee \mathrm{EC}(x,z) \quad (3)$$

$$\mathrm{EC}(x,y) \wedge \mathrm{NTPPi}(y,z) \Rightarrow \mathrm{DC}(x,z) \quad (4)$$

$$\mathrm{PO}(x,y) \wedge \mathrm{P}(y,z) \Rightarrow \mathrm{PO}(x,z) \vee \mathrm{PP}(x,z) \quad (5)$$

$$\mathrm{NTPP}(x,y) \wedge \mathrm{P}(y,z) \Rightarrow \mathrm{NTPP}(x,z) \quad (6)$$

$$\mathrm{P}(x,y) \wedge \mathrm{NTPP}(y,z) \Rightarrow \mathrm{NTPP}(x,z) \quad (7)$$

$$\mathrm{PP}(x,y) \wedge \mathrm{PP}(y,z) \Rightarrow \mathrm{PP}(x,z) \quad (8)$$

$$\mathrm{Pi}(x,y) \wedge \mathrm{P}(y,z) \Rightarrow \neg\mathrm{DC}(x,z) \quad (9)$$

Lemma 1. *The τ-structures in $\mathcal{R}$ with non-empty domain are exactly the finite RCC8 models.*

Proof. We have to show that each (finite) RCC8 model implies (1) – (9). But this can easily from be seen from the RCC8 composition table (see Table 1). For example, $\mathrm{DC}(x,y) \wedge \mathrm{Pi}(y,z) \rightarrow \mathrm{DC}(x,z)$ is implied by the entries (1,6) and (1,7) in the composition table, $\mathrm{EC}(x,y) \wedge \mathrm{P}(y,z) \rightarrow \mathrm{EC}(x,z) \vee \mathrm{PO}(x,z) \vee \mathrm{PP}(x,z)$ is implied by the entries (2,4) and (2,5), etc.

For the other direction, we show that any τ-structure that satisfies (1) – (9) is an RCC8 model, i.e., a model of the RCC8 composition table. There are 8^2 table entries. Three table entries contain all 8 base relations, in which case there is nothing to show. Since the relation EQ is the same as the equality relation, this leaves only $7^2 - 3 = 46$ table entries to verify. We identify those entries by pairs (row, column).

(1) covers the table entries (1,6), (1,7), (4,1), (5,1), and by contraposition and symmetry of DC, it also covers (1,2), (1,3), (1,4), (1,5), (2,1), (3,1), (6,1), and (7,1). (2) covers the table entries (2,4), (6,2). (3) covers the table entries (2,6), (4,2). (4) covers the table entries (2,7), (5,2), and by contraposition and double application, it also covers (2,2). (5) covers the table entries (3,4), (3,5), (6,3), (7,3), and by contraposition and symmetry of PO, also (2,3), (3,2), (3,6), (3,7), (4,3), and (5,3). (6) covers the table entries (5,4), (5,5), (6,7), and (7,7). And, by contraposition, also (4,7) and (5,6). (7) covers the table entries (4,5) and (7,6). (5) covers the table entries (4,4) and (6,6). Moreover, rules (9) and (3) jointly cover the table entry (7,5) and rules (2) and (4) jointly cover (2,5), (7,2). The fields (7,4), (6,5) are covered by (9) in combination with a contraposition of (6). The field (6,4) is covered by (9) in combination with a contraposition of (7). The field (4,6) is covered by a contraposition of (7). □

Proposition 2. *The class $\mathcal{R}$ has the amalgamation property.*

Proof. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be structures from $\mathcal{R}$ such that $B_1 = A \cup \{e_1, \ldots, e_k\}$ and $B_2 = A \cup \{f_1, \ldots, f_l\}$, where $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_l\}$ are disjoint, and $\mathfrak{A}$ is the substructure that is induced by A both in $\mathfrak{B}_1$ and in $\mathfrak{B}_2$. It suffices to show that there exists a τ-structure $\mathfrak{C}$ in $\mathcal{R}$ with domain $C = A \cup \{e_1, \ldots, e_k, f_1, \ldots, f_l\}$ such that both B_1 and B_2 are induced substructures of $\mathfrak{C}$ (this will in fact show that $\mathcal{R}$ has strong amalgamation).

Let G be the directed graph with vertex set $A \cup \{e_1, \ldots, e_k, f_1, \ldots, f_l\}$ and edge set

$$\mathrm{NTPP}^{\mathfrak{B}_1} \cup \mathrm{TPP}^{\mathfrak{B}_1} \cup \mathrm{NTPP}^{\mathfrak{B}_2} \cup \mathrm{TPP}^{\mathfrak{B}_2} .$$

Observe that, if there is a directed path in G from x to y that lies fully within B_i, for $i \in \{1,2\}$, then $(x,y) \in \mathrm{EQ}^{\mathfrak{B}_i} \cup \mathrm{NTPP}^{\mathfrak{B}_i} \cup \mathrm{TPP}^{\mathfrak{B}_i}$; this is straightforward to verify since $\mathfrak{B}_i$ satisfies (1) – (9).

Moreover, by a straightforward induction one can show that if there is any directed path from x to y in G, for $x, y \in B_i$, then $(x,y) \in \mathrm{EQ}^{\mathfrak{B}_i} \cup \mathrm{NTPP}^{\mathfrak{B}_i} \cup \mathrm{TPP}^{\mathfrak{B}_i}$.

We now describe our amalgamation procedure. For each $(x,z) \in C^2$,

- we add (x,z) to $R^{\mathfrak{C}}$ if $(x,z) \in R^{\mathfrak{B}_1} \cup R^{\mathfrak{B}_2}$ for some $R \in \tau$;
- otherwise, we add (x,z) to $\mathrm{NTPP}^{\mathfrak{C}}$ if there exists a directed path from x to z in G;
- otherwise, we add (x,z) to $\mathrm{NTPPi}^{\mathfrak{C}}$ if there exists a directed path from z to x in G;

[1]The first-order theory of a τ-structure $\mathfrak{B}$ is the set of all τ-sentences that are true in $\mathfrak{B}$.

- otherwise, we add (x,z) to $\mathrm{PO}^{\mathfrak{C}}$ if there exists a y and directed paths in G from y to x and from y to z, but not from x to z or from z to x;
- otherwise, we add (x,z) to $\mathrm{PO}^{\mathfrak{C}}$ if there exists $(x,y) \in \mathrm{PO}^{\mathfrak{B}_1} \cup \mathrm{PO}^{\mathfrak{B}_2} \cup \mathrm{EC}^{\mathfrak{B}_1} \cup \mathrm{EC}^{\mathfrak{B}_2}$ and a directed path from y to z in G;
- otherwise, add (x,z) to $\mathrm{DC}^{\mathfrak{C}}$.

It is clear that each pair $(x,z) \in C^2$ is by termination of this procedure contained in exactly one of the relations of $\mathfrak{C}$. We claim that $\mathfrak{C}$ satisfies (1) – (9), and verify item by item. In each of the following cases, we can assume without loss of generality that $x \in \{e_1,\dots,e_k\}$, $y \in A$, and $z \in \{f_1,\dots,f_l\}$.

(1) Assume that $(x,y) \in \mathrm{DC}^{\mathfrak{B}_1}$, $(y,z) \in \mathrm{Pi}^{\mathfrak{B}_2}$; we have to show that $(x,z) \in \mathrm{DC}^{\mathfrak{C}}$. But this follows from: (a) The pair (x,z) cannot be added to $\mathrm{NTPP}^{\mathfrak{C}}$: any path $p_1,\dots,p_s$ from x to z must pass through A, say in p_i. Then we must have $(p_i,z) \in \mathrm{NTPP}^{\mathfrak{B}_2} \cup \mathrm{TPP}^{\mathfrak{B}_2}$ and hence also $(p_i,y) \in \mathrm{NTPP}^{\mathfrak{B}_2} \cup \mathrm{TPP}^{\mathfrak{B}_2}$. Consequently, $(p_i,y) \in \mathrm{NTPP}^{\mathfrak{B}_1} \cup \mathrm{TPP}^{\mathfrak{B}_1}$, which leads to a contradiction to the assumption that $\mathfrak{B}_1$ satisfies (1). (b) The pair (x,z) cannot be added to $\mathrm{NTPPi}^{\mathfrak{C}}$: any path $p_1,\dots,p_s$ from z to x must pass through A, say in p_i. Then $(z,p_i) \in \mathrm{NTPP}^{\mathfrak{B}_2} \cup \mathrm{TPP}^{\mathfrak{B}_2}$ and hence $(p_i,y) \notin \mathrm{DC}^{\mathfrak{B}_2}$ by (9). Considering now the three vertices $\{p_i,y,x\}$ of $\mathfrak{B}_1$, we reach a contradiction to the assumption that $\mathfrak{B}_1$ satisfies (1). (c) The pair (x,z) cannot be added to $\mathrm{PO}^{\mathfrak{C}}$: if there is a vertex y' with directed paths from y' to x and from y' to z, then we must have $(y',y) \in \mathrm{NTPP}^{\mathfrak{B}_1} \cup \mathrm{TPP}^{\mathfrak{B}_1}$; so we obtain a contradiction to the assumption that $\mathfrak{B}_1$ satisfies (9). If there is $(x,y') \in \mathrm{PO}^{\mathfrak{B}_1}$ and a path from y' to z, then we also have $(y',y) \in \mathrm{NTPP}^{\mathfrak{B}_1} \cup \mathrm{TPP}^{\mathfrak{B}_1}$, in contradiction to the assumption that $\mathfrak{B}_1$ satisfies (5).

(2) Arguing similarly as above, the pair (x,z) cannot be added to $\mathrm{NTPP}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satisfies (9), and not to $\mathrm{NTPPi}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satisfies (8). The pair (x,z) will then be added to $\mathrm{PO}^{\mathfrak{C}}$ and therefore (2) is satisfied by x,y,z.

(3) The pair (x,z) cannot be added to $\mathrm{NTPP}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satisfies (1), not to $\mathrm{NTPPi}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satsfies (8), and not to $\mathrm{PO}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satsfies (8). The pair (x,z) will then be added to $\mathrm{DC}^{\mathfrak{C}}$ and therefore (3) is satisfied by x,y,z.

(4) The argument here is the same as the argument in the previous item.

(5) If the pair (x,z) has been added to $\mathrm{NTPP}^{\mathfrak{C}}$ then x,y,z satisfy (5). It cannot be added to $\mathrm{NTPPi}^{\mathfrak{C}}$ since $\mathfrak{B}_1$ satisfies (8). So, if (x,y) has not been added to $\mathrm{NTPP}^{\mathfrak{C}}$, then it must have been added to $\mathrm{PO}^{\mathfrak{C}}$ and (5) is again satisfied by x,y,z.

(6) (x,z) has been added to $\mathrm{NTPP}^{\mathfrak{C}}$, hence (6) is satisfied.

(7) (x,z) has been added to $\mathrm{NTPP}^{\mathfrak{C}}$, hence (7) is satisfied.

(8) (x,z) has been added to $\mathrm{NTPP}^{\mathfrak{C}}$, hence (8) is satisfied.

(9) If (x,z) is added to $\mathrm{NTPP}^{\mathfrak{C}}$ or $\mathrm{NTPPi}^{\mathfrak{C}}$, then (9) is satisfied. Otherwise, it will be added to $\mathrm{PO}^{\mathfrak{C}}$ and again (9) is satisfied by x,y,z in $\mathfrak{C}$. □

In terms of constraint networks, the amalgamation property can be restated as follows: A *scenario* of a constraint network C is complete, atomic, and satisfiable refinement of C. If C has a scenario, then it is obviously satisfiable. Note that for many classes of RCC8 models, satisfiability for atomic networks can be decided by the path consistency procedure. It is clear that each finite RCC8 model of a network C defines a scenario of C, and conversely, that each scenario of C defines an RCC8 model of C. Let S_1 and S_2 be scenarios of constraint networks $C_1 = \langle V_1, A_1, l_1 \rangle$ and $C_2 = \langle V_2, A_2, l_2 \rangle$, respectively, that coincide on $V_1 \cap V_2$. The RCC8 model induced by S_1 on the variable set $V_1 \cap V_2$, then, can be embedded into both the RCC8 models induced by S_1 and S_2. Hence the amalgam of these models is an RCC8 structure that satisfies all constraints in S_1 and S_2, and hence it induces a scenario of the constraint network $C_1 \cup C_2$.

Theorem 2. *RCC8 has a representation by a countably infinite homogeneous structure $\mathfrak{R}$. All finite (in fact, all countable) RCC8 models embed into $\mathfrak{R}$.*

Proof. Let $\mathfrak{R}$ be the Fraïssé-limit of $\mathcal{R}$ (see e.g. [Hodges, 1997]). The Fraïssé-limit is always homogeneous; since $\mathfrak{R}$ has a finite relational signature, it is also ω-categorical (see e.g. [Hodges, 1997]). Since all structures $\mathfrak{A}$ in $\mathcal{R}$ have pairwise disjoint relations that cover all of A^2, the same is true for $\mathfrak{R}$. Similarly, it is clear that $\mathfrak{R}$ satisfies (1) – (9). Note that all the formulas to be verified for $\mathfrak{R}$ are universal: a structure satisfies a universal formula if and only if the formula is satisfied in all finite substructures. Lemma 1 implies that $\mathfrak{R}$ is an RCC8 model. It is also well-known that the Fraïssé-limit of a class embeds all countable structures that have the same age as the Fraïssé-limit (Lemma 6.1.3 and 6.1.4 in [Hodges, 1997]). Homogeneity of $\mathfrak{R}$ implies that $\mathfrak{R}$ is extensional. □

5 Networks of Bounded Treewidth

Let $G = \langle V, E \rangle$ be a graph. A *tree decomposition* of G is a pair $\langle X, T \rangle$, where $X = \{X_1,\dots,X_n\}$ is a family of subsets of V (called *bags*), and T is a tree with node set X such that (a) for each edge (v,w) in G, $\{v,w\} \subseteq X_i$ for some bag X_i, and (b) for each path $X_i\ T\ \dots\ T\ X_j\ T\ \dots\ T\ X_k$ in X, it holds $X_i \cap X_k \subseteq X_j$. The *width* of a tree decomposition is defined as $\max_i(|X_i|) - 1$. The *treewidth* of a graph G is the minimum width possible for arbitrary tree decompositions of G. Note that trees have treewidth one. The treewidth of a constraint network N is simply the treewidth of the underlying graph of N, i.e., the graph on the nodes of the network where two nodes are joined if and only if there is a constraint on those two nodes.

Let $\mathfrak{B}$ be a structure with a finite relational signature τ, and let $\mathfrak{A}$ be an instance of the CSP for $\mathfrak{B}$. Then the *constraint graph* of $\mathfrak{A}$ is the graph with vertices A where two vertices are joined if there exists a tuple in a relation in $\mathfrak{A}$ that contains both vertices.

Theorem 3 ([Bodirsky and Dalmau, 2008], Cor. 1). *Let $\mathfrak{B}$ be an ω-categorical structure. Then for any k, the CSP of $\mathfrak{B}$ is polynomial-time tractable when restricted to instances whose constraint graphs have treewidth at most k.*

We want to remark that for $k = 2$ and when $\mathfrak{B}$ has only binary relation symbols, the algorithm applied in [Bodirsky and Dalmau, 2008] equals the path consistency procedure.

Theorem 4. *For any k, the network consistency problem for RCC8 restricted to networks of treewidth at most k can be solved in polynomial time.*

Proof Sketch. Consider the expansion $\mathfrak{S}$ of $\mathfrak{R}$ by a relation for each union of the relations from $\mathfrak{R}$. It is well-known that first-order expansions preserve ω-categoricity [Hodges, 1997], and so $\mathfrak{S}$ is ω-categorical. An RCC8 constraint network N can be translated into an instance $\mathfrak{A}$ of CSP($\mathfrak{S}$) in the obvious way. If N has a solution, then it clearly also has a solution over a *finite* RCC8 model, and $\mathfrak{A}$ homomorphically maps to $\mathfrak{S}$ by Theorem 2. Conversely, if $\mathfrak{A}$ homomorphically maps to $\mathfrak{S}$, then N is satisfiable. The result follows from Theorem 3. □

Remark. In order to apply the algorithm used in Theorem 4, it is not necessary to determine the treewidth of the input network, which is in general an NP-hard problem; rather, it is sufficient to know that its treewidth is bounded.

Acknowledgments

M. B. acknowledges support from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement no. 257039). S. W. acknowledges support by Deutsche Forschungsgemeinschaft (SFB/TR 8 Spatial Cognition, project R4-[LogoSpace]). We thank the reviewers for their valuable comments. We also thank Till Mossakowski for discussions on an early version of the paper.

References

[Atserias *et al.*, 2007] Albert Atserias, Andrei A. Bulatov, and Víctor Dalmau. On the power of *k*-consistency. In *ICALP*, LNCS 4596, pages 279–290. Springer, 2007.

[Bennett and Düntsch, 2007] Brandon Bennett and Ivo Düntsch. Axioms, algebras and topology. In M. Aiello, I. Pratt-Hartmann, and J. v. Benthem, editors, *Handbook of Spatial Logics*, pages 99–159. Springer, 2007.

[Bennett, 1998] Brandon Bennett. Determining consistency of topological relations. *Constraints*, 3(2/3):213–225, 1998.

[Bodirsky and Dalmau, 2008] Manuel Bodirsky and Víctor Dalmau. Datalog and constraint satisfaction with infinite templates. *An extended abstract appeared in the proceedings of STACS'06. The full version is available online at arXiv:0809.2386 [cs.LO]*, 2008.

[Bodlaender, 1993] Hans L. Bodlaender. A tourist guide through treewidth. *Acta Cybern.*, 11(1-2):1–22, 1993.

[Cohn and Hazarika, 2001] Anthony G. Cohn and Shyamanta M. Hazarika. Qualitative spatial representation and reasoning: An overview. *Fundam. Inform.*, 46(1-2):1–29, 2001.

[Egenhofer, 1991] Max J. Egenhofer. Reasoning about binary topological relations. In *SSD*, LNCS 525, pages 143–160. Springer, 1991.

[Fraïssé, 1954] R. Fraïssé. Ann. sci. école norm. sup. *Sur l'extension aux relations de quelques propriétés des ordres*, 71:363–388, 1954.

[Hirsch, 1997] R. Hirsch. Expressive power and complexity in algebraic logic. *Journal of Logic and Computation*, 7(3):309 – 351, 1997.

[Hodges, 1997] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[Li and Ying, 2003] Sanjiang Li and Mingsheng Ying. Extensionality of the RCC8 composition table. *Fundam. Inform.*, 55(3-4):363–385, 2003.

[Li *et al.*, 2008] Jason Jingshi Li, Tomasz Kowalski, Jochen Renz, and Sanjiang Li. Combining binary constraint networks in qualitative reasoning. In *ECAI*, FAIA 178, pages 515–519. IOS Press, 2008.

[Li *et al.*, 2009] Jason Jingshi Li, Jinbo Huang, and Jochen Renz. A divide-and-conquer approach for solving interval algebra networks. In *IJCAI*, pages 572–577, 2009.

[Ligozat and Renz, 2004] Gérard Ligozat and Jochen Renz. What is a qualitative calculus? A general framework. In *PRICAI*, LNCS 3157, pages 53–64. Springer, 2004.

[Mackworth, 1977] Alan K. Mackworth. Consistency in networks of relations. *Artif. Intell.*, 8(1):99–118, 1977.

[Ragni *et al.*, 2007] Marco Ragni, Bolormaa Tseden, and Markus Knauff. Cross-cultural similarities in topological reasoning. In *COSIT*, pages 32–46. Springer, 2007.

[Randell *et al.*, 1992a] David A. Randell, Anthony G. Cohn, and Zhan Cui. Computing transivity tables: A challenge for automated theorem provers. In *CADE*, LNCS 607, pages 786–790. Springer, 1992.

[Randell *et al.*, 1992b] David A. Randell, Zhan Cui, and Anthony G. Cohn. A spatial logic based on regions and connection. In *KR*, pages 165–176, 1992.

[Renz and Ligozat, 2005] Jochen Renz and Gérard Ligozat. Weak composition for qualitative spatial and temporal reasoning. In *CP*, LNCS 3709, pages 534–548. Springer, 2005.

[Renz and Nebel, 2007] Jochen Renz and Bernhard Nebel. Qualitative spatial reasoning using constraint calculi. In *Handbook of Spatial Logics*, pages 161–215. Springer, 2007.

[Renz *et al.*, 2000] Jochen Renz, Reinhold Rauh, and Markus Knauff. Towards cognitive adequacy of topological spatial relations. In *Spatial Cognition*, LNCS 1849, pages 184–197. Springer, 2000.

[Renz, 2002] Jochen Renz. *Qualitative Spatial Reasoning with Topological Information*. LNCS 2293. Springer, 2002.

[Samer and Szeider, 2010] Marko Samer and Stefan Szeider. Constraint satisfaction with bounded treewidth revisited. *J. Comput. Syst. Sci.*, 76(2):103–114, 2010.

[Stell, 2000] John G. Stell. Boolean connection algebras: A new approach to the region-connection calculus. *Artif. Intell.*, 122(1-2):111–136, 2000.