Reasoning About Typicality in Low Complexity DLs: The Logics $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$*

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Abstract

We propose a nonmonotonic extension of low complexity Description Logics $\mathcal{EL}^{\perp}$ and *DL-Lite*$_{core}$ for reasoning about typicality and defeasible properties. The resulting logics are called $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$. Concerning *DL-Lite*$_c\mathbf{T}_{min}$, we prove that entailment is in Π^p_2. With regard to $\mathcal{EL}^{\perp}\mathbf{T}_{min}$, we first show that entailment remains EXPTIME-hard. Next we consider the known fragment of Left Local $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and we prove that the complexity of entailment drops to Π^p_2.

1 Introduction

Nonmonotonic extensions of Description Logics (DLs) have been actively investigated since the early 90s, [Straccia, 1993; Bonatti *et al.*, 2006; Baader and Hollunder, 1995; Bonatti *et al.*, 2009; Donini *et al.*, 2002; Giordano *et al.*, 2009b; 2009a; 2008; Casini and Straccia, 2010]. The reason is that DLs are used to represent classes and their properties, so that a nonmonotonic mechanism is wished to express defeasible inheritance of prototypical properties. A simple but powerful nonmonotonic extension of DL is proposed in [Giordano *et al.*, 2009b; 2009a; 2008]: in this approach "typical" or "normal" properties can be directly specified by means of a "typicality" operator $\mathbf{T}$ enriching the underlying DL; the tipicality operator $\mathbf{T}$ is essentially characterised by the core properties of nonmonotonic reasoning axiomatized by *preferential logic* [Kraus *et al.*, 1990]. In $\mathcal{ALC} + \mathbf{T}$ [Giordano *et al.*, 2009b] and $\mathcal{EL}^{\perp}\mathbf{T}$ [Giordano *et al.*, 2009a], one can consistently express defeasible inclusions and exceptions such as: typical students do not pay taxes, but working students do typically pay taxes, but working student having children normally do not: $\mathbf{T}(Student) \sqsubseteq \neg TaxPayer$; $\mathbf{T}(Student \sqcap Worker) \sqsubseteq TaxPayer$; $\mathbf{T}(Student \sqcap Worker \sqcap \exists HasChild.\top) \sqsubseteq \neg TaxPayer$ (for the formalization in $\mathcal{EL}^{\perp}\mathbf{T}$ see Example 2.6). Although the operator $\mathbf{T}$ is nonmonotonic in itself, the logics $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{EL}^{\perp}\mathbf{T}$ are monotonic. As a consequence, unless a KB contains explicit assumptions about typicality of individuals (e.g. that john is a typical student), there is no way of inferring defeasible properties of them (e.g. that john does not pay taxes). In [Giordano *et al.*, 2008], a non monotonic extension of $\mathcal{ALC} + \mathbf{T}$ based on a minimal models semantics is proposed. The resulting logic, called $\mathcal{ALC} + \mathbf{T}_{min}$, supports typicality assumptions, so that if one knows that john is a student, one can nonmonotonically assume that he is also a *typical* student and therefore that he does not pay taxes. As an example, for a TBox specified by the inclusions above, in $\mathcal{ALC} + \mathbf{T}_{min}$ the following inference holds: TBox $\cup \{Student(john)\} \models_{\mathcal{ALC}+\mathbf{T}_{min}} \neg TaxPayer(john)$.

Similarly to other nonmonotonic DLs, adding the typicality operator with its minimal-model semantics to a standard DL, such as $\mathcal{ALC}$, leads to a very high complexity (namely query entailment in the resulting logic is in CO-NEXP$^{\text{NP}}$ [Giordano *et al.*, 2008]). This fact may motivate the study of nonmonotonic extensions of low complexity DLs such as *DL-Lite*$_{core}$ [Calvanese *et al.*, 2007] and $\mathcal{EL}^{\perp}$ of the $\mathcal{EL}$ family [Baader *et al.*, 2005] which are nonetheless well-suited for encoding large knowledge bases (KBs). In this paper, we hence consider the extensions of the low complexity logics *DL-Lite*$_{core}$ and $\mathcal{EL}^{\perp}$ with the typicality operator based on the minimal model semantics introduced in [Giordano *et al.*, 2008]. We study the complexity of the resulting logics $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$. For $\mathcal{EL}^{\perp}$, it turns out that its extension $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ is unfortunately EXPTIME-hard. This result is analogous to the one for *circumscribed* $\mathcal{EL}^{\perp}$ KBs [Bonatti *et al.*, 2009]. However, the complexity decreases to Π^p_2 for the fragment of *Left Local* $\mathcal{EL}^{\perp}$ KBs, corresponding to the homonymous fragment in [Bonatti *et al.*, 2009]. We obtain the same complexity upper bound for *DL-Lite*$_c\mathbf{T}_{min}$.

2 The typicality operator $\mathbf{T}$ and the Logic $\mathcal{EL}^{\perp}\mathbf{T}_{min}$

Before describing $\mathcal{EL}^{\perp}\mathbf{T}_{min}$, let us briefly recall the underlying monotonic logic $\mathcal{EL}^{\perp}\mathbf{T}$ [Giordano *et al.*, 2009a], obtained by adding to $\mathcal{EL}^{\perp}$ the typicality operator $\mathbf{T}$. The intuitive idea is that $\mathbf{T}(C)$ selects the *typical* instances of a concept C. In $\mathcal{EL}^{\perp}\mathbf{T}$ we can therefore distinguish between the properties that hold for all instances of concept C ($C \sqsubseteq D$), and those that only hold for the normal or typical instances of C ($\mathbf{T}(C) \sqsubseteq D$).

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Formally, the $\mathcal{EL}^{\perp}\mathbf{T}$ language is defined as follows.

Definition 2.1 *We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individuals $\mathcal{O}$. Given $A \in \mathcal{C}$ and $r \in \mathcal{R}$, we define*

$$C := A \mid \top \mid \bot \mid C \sqcap C$$
$$C_R := C \mid C_R \sqcap C_R \mid \exists r.C$$
$$C_L := C_R \mid \mathbf{T}(C)$$

A KB is a pair (TBox, ABox). TBox contains a finite set of concept inclusions $C_L \sqsubseteq C_R$. ABox contains assertions of the form $C_L(a)$ and $r(a,b)$, where $a, b \in \mathcal{O}$.

The semantics of $\mathcal{EL}^{\perp}\mathbf{T}$ [Giordano *et al.*, 2009a] is defined by enriching ordinary models of $\mathcal{EL}^{\perp}$ by a *preference relation* $<$ on the domain, whose intuitive meaning is to compare the "typicality" of individuals: $x < y$, means that x is more typical than y. Typical members of a concept C, that is members of $\mathbf{T}(C)$, are the members x of C that are minimal with respect to this preference relation.

Definition 2.2 (Semantics of $\mathbf{T}$) *A model $\mathcal{M}$ is any structure $\langle \Delta, <, I \rangle$ where Δ is the domain; $<$ is an irreflexive and transitive relation over Δ that satisfies the following* Smoothness Condition*: for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$, where $Min_<(S) = \{u : u \in S$ and $\nexists z \in S$ s.t. $z < u\}$. Furthermore, $<$ is multilinear: if $u < z$ and $v < z$, then either $u = v$ or $u < v$ or $v < u$. I is the extension function that maps each concept C to $C^I \subseteq \Delta$, and each role r to $r^I \subseteq \Delta^I \times \Delta^I$. For concepts of $\mathcal{EL}^{\perp}$, C^I is defined in the usual way. For the $\mathbf{T}$ operator: $(\mathbf{T}(C))^I = Min_<(C^I)$.*

Definition 2.3 (Model satisfying a Knowledge Base) *Given a model $\mathcal{M}$, I can be extended so that it assigns to each individual a of $\mathcal{O}$ a distinct element a^I of the domain Δ. $\mathcal{M}$ satisfies a KB (TBox,ABox), if it satisfies both its TBox and its ABox, where:*

- *$\mathcal{M}$ satisfies TBox if for all inclusions $C \sqsubseteq D$ in TBox, for all elements $x \in \Delta$, if $x \in C^I$, then $x \in D^I$.*
- *$\mathcal{M}$ satisfies ABox if: (i) for all $C(a)$ in ABox, $a^I \in C^I$, (ii) for all aRb in ABox, $(a^I, b^I) \in r^I$.*

The operator $\mathbf{T}$ [Giordano *et al.*, 2009b] is characterized by a set of postulates that are essentially a reformulation of KLM [Kraus *et al.*, 1990] axioms of *preferential logic* $\mathbf{P}$. Roughly speaking, the assertion $\mathbf{T}(C) \sqsubseteq D$ corresponds to the conditional $C \mid\!\sim D$ of $\mathbf{P}$. $\mathbf{T}$ has therefore all the "core" properties of nonmonotonic reasoning as it is axiomatised by $\mathbf{P}$.

The semantics of the typicality operator can be specified by modal logic. The interpretation of $\mathbf{T}$ can be split into two parts: for any x of the domain Δ, $x \in (\mathbf{T}(C))^I$ just in case (i) $x \in C^I$, and (ii) there is no $y \in C^I$ such that $y < x$. Condition (ii) can be represented by means of an additional modality $\Box$, whose semantics is given by the preference relation $<$ interpreted as an accessibility relation. Observe that by the Smoothness Condition, $\Box$ has the properties of Gödel-Löb modal logic of provability G.

The interpretation of $\Box$ in $\mathcal{M}$ is as follows:

$$(\Box C)^I = \{x \in \Delta \mid \text{for every } y \in \Delta, \text{ if } y < x \text{ then } y \in C^I\}.$$

We immediatly get that $x \in (\mathbf{T}(C))^I$ iff $x \in (C \sqcap \Box\neg C)^I$. From now on, we consider $\mathbf{T}(C)$ as an abbreviation for $C \sqcap \Box\neg C$. Finally, we recall that the problem of entailment in $\mathcal{EL}^{\perp}\mathbf{T}$ is in CO-NP (as shown in [Giordano *et al.*, 2009a]).

The main limit of $\mathcal{EL}^{\perp}\mathbf{T}$ is that it is *monotonic*. Even if the typicality operator $\mathbf{T}$ itself is nonmonotonic (i.e. $\mathbf{T}(C) \sqsubseteq E$ does not imply $\mathbf{T}(C \sqcap D) \sqsubseteq E$), the logic $\mathcal{EL}^{\perp}\mathbf{T}$ is monotonic: what is inferred from KB can still be inferred from any KB' with KB $\subseteq$ KB'. In order to perform nonmonotonic inferences, as done in [Giordano *et al.*, 2008], we strengthen the semantics of $\mathcal{EL}^{\perp}\mathbf{T}$ by restricting entailment to a class of minimal (or preferred) models. We call the new logic $\mathcal{EL}^{\perp}\mathbf{T}_{min}$. Intuitively, the idea is to restrict our consideration to models that *minimize the non typical instances of a concept*. The preference relation on models (with the same domain) is defined by comparing individuals in the interpretation of modal (or more precisely $\Box$-ed) formulas.

Given a KB, we consider a finite set $\mathcal{L}_{\Box}$ of concepts: these are the concepts whose non typical instances we want to minimize. We assume that the set $\mathcal{L}_{\Box}$ contains at least all concepts C such that $\mathbf{T}(C)$ occurs in the KB or in the query α. A *query* α is either an assertion $C(a)$ or an inclusion relation $C \sqsubseteq D$. We have that x is a typical instance of C if $x \in (C \sqcap \Box\neg C)^I$. Since we do not want to change the extension of concept C by minimization, we minimize only objects not satisfying $\Box\neg C$ for $C \in \mathcal{L}_{\Box}$. Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, we first define: $\mathcal{M}^{\Box^-}_{\mathcal{L}_{\Box}} = \{(x, \neg\Box\neg C) \mid x \notin (\Box\neg C)^I, \text{ with } x \in \Delta, C \in \mathcal{L}_{\Box}\}$.

Definition 2.4 (Preferred and minimal models) *Given a model $\mathcal{M} = \langle \Delta <, I \rangle$ of a knowledge base KB, and a model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ of KB, we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to $\mathcal{L}_{\Box}$, and we write $\mathcal{M} <_{\mathcal{L}_{\Box}} \mathcal{M}'$, if (i) $\Delta = \Delta'$ and (ii) $\mathcal{M}^{\Box^-}_{\mathcal{L}_{\Box}} \subset \mathcal{M}'^{\Box^-}_{\mathcal{L}_{\Box}}$. A model $\mathcal{M}$ is a* minimal model *for KB (with respect to $\mathcal{L}_{\Box}$) if it is a model of KB and there is no other model $\mathcal{M}'$ of KB such that $\mathcal{M}' <_{\mathcal{L}_{\Box}} \mathcal{M}$.*

We can now define entailment in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$. Let us first say that a query α holds in a model $\mathcal{M} = \langle \Delta <, I \rangle$ of a KB (Definition 2.3) whenever: (i) $\alpha = C \sqsubseteq D$, and for all $x \in \Delta$, if $x \in C^I$, then $x \in D^I$, or (ii) $\alpha = C(a)$, and $a^I \in C^I$.

Definition 2.5 (Minimal Entailment in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$) *A query α is minimally entailed in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ by a knowledge base KB with respect to $\mathcal{L}_{\Box}$ if it holds in all models of KB that are minimal with respect to $\mathcal{L}_{\Box}$. We write KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$.*

Example 2.6 The KB of the Introduction can be reformulated as follows in $\mathcal{EL}^{\perp}\mathbf{T}$: $TaxPayer \sqcap NotTaxPayer \sqsubseteq \bot$; $Parent \sqsubseteq \exists HasChild.\top$; $\exists HasChild.\top \sqsubseteq Parent$; $\mathbf{T}(Student) \sqsubseteq NotTaxPayer$; $\mathbf{T}(Student \sqcap Worker) \sqsubseteq TaxPayer$; $\mathbf{T}(Student \sqcap Worker \sqcap Parent) \sqsubseteq NotTaxPayer$. Let $\mathcal{L}_{\Box} = \{Student, Student \sqcap Worker, Student \sqcap Worker \sqcap Parent\}$. Then TBox $\cup$ $\{Student(john)\}$ $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}}$ $NotTaxPayer(john)$, since $john^I \in (Student \sqcap \Box\neg Student)^I$ for all minimal models $\mathcal{M} = \langle \Delta <, I \rangle$ of the KB. Similarly we obtain TBox $\cup$ $\{\exists HasChild.(Student \sqcap Worker)(jack)\}$ $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}}$ $\exists HasChild.TaxPayer(jack)$. The latter shows that minimal entailment applies to *implict individuals* as well, without any ad-hoc mechanism.

Observe that minimal entailment is nonmonotonic, since, for KB $\subset$ KB', the minimal models of KB' are not necessarily *minimal* models of KB.

3 Complexity of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$

In this section we analyze the complexity of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$. We consider the complexity of minimal entailment of a query as defined above. First we show that, contrary to the mere addition of the operator $\mathbf{T}$ to $\mathcal{EL}^{\perp}$, the adoption of the nonmonotonic semantics drastically worsens the complexity of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ making it EXPTIME hard. Next we identify a syntactic restriction that allows one to obtain a reasonable complexity (laying in the second level of the polynomial hierarchy), comparable to the one of other nonmonotonic mechanisms for $\mathcal{EL}^{\perp}$, such as circumscription.

3.1 An EXPTIME lower bound for $\mathcal{EL}^{\perp}\mathbf{T}_{min}$

We prove here that $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ is EXPTIME hard. The proof is based on the reduction of the problem of TBox satisfiability in $\mathcal{EL}^{\neg A}$ to the complement of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ subsumption. The logic $\mathcal{EL}^{\neg A}$ is just $\mathcal{EL}^{\perp}$ extended by the complement of concept names and the satisfiability problem for it is known to be EXPTIME-hard [Baader *et al.*, 2005].

Theorem 3.1 *Entailment in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ is* EXPTIME*-hard.*

Proof. Let T be a TBox in $\mathcal{EL}^{\neg A}$. We define a KB=(T',A') in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ by introducing, for each concept name A in T, the fresh names: $\overline{A}$ (for representing the negation of A, $\neg A$), U_A (for representing the fact that A has an undefined truth value), N_A (meaning normal A), $N_{\neg A}$ (meaning normal $\neg A$), N_{U_A} (meaning normal U_A). Also, we introduce the new concept names D (for defining a subset of the domain), E and U and the new concept role R.

We translate each concept C into C^* as follows: (i) $C^* = C$, if C is a concept name; (ii) $C^* = \overline{A}$, if $C = \neg A$; (iii) $C^* = D \sqcap \exists r.(C_1^* \sqcap D)$, if $C = \exists r.C_1$ (iv) $C^* = C_1^* \sqcap C_2^*$ if $C = C_1 \sqcap C_2$. For all the inclusions $C_1 \sqsubseteq C_2 \in T$ the translated TBox T' contains the inclusion $C_1^* \sqsubseteq C_2^*$. The translated TBox T', contains, in addition, the following inclusions (for all concept names A):

$$\begin{array}{lll} D \sqsubseteq N_A, & D \sqsubseteq N_{\overline{A}}, & D \sqsubseteq N_{U_A} \\ \mathbf{T}(N_A) \sqsubseteq A & \mathbf{T}(N_{\overline{A}}) \sqsubseteq \overline{A} & \mathbf{T}(N_{U_A}) \sqsubseteq U_A \\ A \sqcap \overline{A} \sqsubseteq \bot & A \sqcap U_A \sqsubseteq \bot & U_A \sqcap \overline{A} \sqsubseteq \bot \\ D \sqsubseteq E \quad & D \sqcap \mathbf{T}(E) \sqsubseteq \exists R.(D \sqcap U_A) & D \sqcap U_A \sqsubseteq U \end{array}$$

A' contains the assertion $D(a)$, for a new individual name a.

A D element of the domain is a normal A, a normal $\overline{A}$ and a normal U_A element (first line) and it can be (in mutual exclusion) either a typical normal A element or a typical normal $\overline{A}$ element or a typical normal U_A element (second and third line). Thus any D element, provides a truth value (true, false or undefined) to each concept name A. The last line says that each D is an E and the typical E are in the relation R with D element for which U_A is true. Observe that, if there is a $D \sqcap U_A$ element in the domain, there is no reason why a D element should not be assumed to be a typical E element (and in the preferred models, that D element would be in the relation R with a $D \sqcap U_A$ element, unless there are no $D \sqcap U_A$ elements in the model). Last subsumption says that U must hold for those D elements which make some concept name A undefined. Finally, the assertion $D(a)$ guarantees the existence of at least a D element in the models of KB. Any preferred model of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$, in which there are no D elements with undefined concept names ($(D \sqcap U_A)^I = \emptyset$), can be mapped to a model in $\mathcal{EL}^{\neg A}$, by removing all the relation $<$ and by restricting the domain of the model to the set of elements in which D holds. If the resulting model does not contain any undefined element (i.e. any element in which U_A holds for some A), then that model is a (two valued) model of T. Hence, we can verify that T is satisfiable in $\mathcal{EL}^{\neg A}$, by verifying that KB has a minimal model $\mathcal{M}$ in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ that does not contain $D \sqcap U_A$ elements. This is true if there is a minimal model that does not contain U elements, which can be tested by verifying that KB $\not\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} D \sqsubseteq \exists R.U$, with $\mathcal{L}_{\square} = \{N_A, N_{\overline{A}}, N_{U_A}, E\}$. This condition holds whenever there is a model $\mathcal{M} = \langle \Delta, <, I \rangle$ of KB in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and an element y in its domain such that: $y \in D^I$ and $y \notin (\exists R.U)^I$. This means that there is no $D \sqcap U_A$ element in the model $\mathcal{M}$. Then, we can conclude that entailment in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ is EXPTIME-hard. ■

The previous result corresponds to an analogous result for circumscribed KBs [Bonatti *et al.*, 2009]. In order to lower the complexity of minimal entailment in $\mathcal{EL}^{\perp}\mathbf{T}_{min}$, we put the following syntactic restrictions on the KB.

3.2 Left Local Knowledge Bases

In this section, we consider a syntactical restriction on knowledge bases called Left Local KBs. This restriction is similar to the one introduced in [Bonatti *et al.*, 2009].

Definition 3.2 (Left Local knowledge base) *A Left Local KB only contains subsumptions $C_L^{LL} \sqsubseteq C_R$, where C and C_R are as in Definition 2.1 and:*

$$C_L^{LL} := C \mid C_L^{LL} \sqcap C_L^{LL} \mid \exists r.\top \mid \mathbf{T}(C)$$

There is no restriction on the ABox.

Observe that the KB in the Example 2.6 is Left Local.

We provide an upper bound for the complexity of Left Local KBs by a small model theorem. Intuitively, what allows us to keep the size of the small model polynomial is that we reuse the same world to verify the same existential concept throughout the model.

We write KB $\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ to say that α holds in all minimal models of the KB whose size is polynomially bounded by the size of KB.

Let KB=(TBox,ABox) be a Left Local KB, and suppose KB $\not\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. We want to show that KB $\not\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ (this is one half of Theorem 3.11).

Let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal model of KB in which α does not hold. Let $x \in \Delta$ be an element falsifying α. Since α can be an existential formula, x is the only world for which we have to preserve the truth value of all existential formulas (also negative) in the small model. For this reason we consider x first. The small model $\mathcal{M}'$ is built as follows.

1. $\Delta_0 := Unres := \{x\} \cup \{a^I \in \Delta \mid a$ occurs in ABox $\}$
2. **for each** $\exists r.C$ occurring in KB s.t. $x \in (\exists r.C)^I$ **do**

3. choose $w \in \Delta$ s.t. $(x, w) \in r^I$ and $w \in C^I$
4. $\Delta_0 := \Delta_0 \cup \{w\}$
5. add (x, w) to $r^{I'}$
6. **while** *Unres* $\neq \emptyset$ **do**
7. extract one y from *Unres*
8. **for each** $\exists r.C$ in KB s.t. $y \in (\exists r.C)^I$, $y \neq x$ **do**
9. **if** $\exists w \in \Delta_0$ s.t. $w \in C^{I'}$ **then**
10. add (y, w) to $r^{I'}$
11. **else**
12. choose $w \in \Delta$ s.t. $(y, w) \in r^I$ and $w \in C^I$
13. $\Delta_0 := \Delta_0 \cup \{w\}$
14 add (y, w) to $r^{I'}$
15. *Unres*:=*Unres* $\cup \{w\}$
16. **for each** $y_i \in \Delta$ such that $y_i < y$ **do**
17. $\Delta_0 := \Delta_0 \cup \{y_i\}$
18. *Unres*:=*Unres* $\cup \{y_i\}$

We then define $\mathcal{M}'$ in the following way.

Definition 3.3 *The model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ is as follows: (i) $\Delta' = \Delta_0$; (ii) we define $<'$ by adding $u <' v$ if $u < v$, for each $u, v \in \Delta'$; (iii) the extension function I' is defined as follows: - for all atomic concepts $C \in \mathcal{C}$, for all elements in Δ', we define $u \in C^{I'}$ if $u \in C^I$; - for all roles $r \in \mathcal{R}$, $r^{I'}$ is as computed by the above algorithm; (iv) I' is extended so that it assigns a^I to each individual a in the ABox .*

Lemma 3.4 *The relation $<'$ is irreflexive, transitive, multilinear and satisfies the Smoothness Condition.*

Proof. This is so because these properties hold in $\mathcal{M}$ for all descending chains of $<$. Here, we have imported some of the chains and we have added no new relations to $<'$ not belonging to $<$. ■

Lemma 3.5 *Given a concept C with form C_L^{LL} of Definition 3.2, for all $y \in \Delta'$, $y \in C^{I'}$ iff $y \in C^I$.*

Proof. By induction on the complexity of C. To save space, we only present the two most interesting cases. Let C be $\exists r.\top$. If $y \in (\exists r.\top)^I$, then at step 5 or 10 or 14 (y, z) has been inserted to $r^{I'}$, and $y \in (\exists r.\top)^{I'}$. For the other direction, if $(y \in \exists r.\top)^{I'}$, then there must be z such that (y, z) has been added to $r^{I'}$ in step 5 or 10 or 14 for a given $y \in (\exists r.C)^I$. Hence $y \in (\exists r.\top)^I$ also in $\mathcal{M}$.
Let C be $\mathbf{T}(C_1)$, and $y \in (\mathbf{T}(C_1))^I$. Then $y \in C_1^I$ and for all $z < y$, $z \notin C_1^I$, i.e. $y \in (\Box\neg C_1)^I$. By inductive hypothesis, $y \in C_1^{I'}$. Furthermore, by definition of $<'$, for all $z \in \Delta'$, if $z <' y$, then $z < y$, and by inductive hypothesis, $z \notin C_1^{I'}$, hence $y \in (\Box\neg C_1)^{I'}$. Hence, $y \in (\mathbf{T}(C_1))^{I'}$. On the other hand, if $y \in (\mathbf{T}(C))^{I'}$, we can reason analogously: from $y \in C_1^{I'}$ we conclude that $y \in C_1^I$, and from $y \in (\Box\neg C_1)^{I'}$, we conclude that $y \in (\Box\neg C_1)^I$, i.e., for all $z < y$, $z \notin C_1^I$ (since for all $z < y$, $z \in \Delta_0$ and $z <' y$). Hence, $y \in (\mathbf{T}(C_1))^I$. ■

Lemma 3.6 *Given a concept C with form C_R, for all $y \in \Delta'$, if $y \in C^I$ then $y \in C^{I'}$.*

Proof. Let C be $\exists r.C_1$. If $y \in (\exists r.C_1)^I$, by construction, $(y, w) \in r^{I'}$ for some $w \in C_1^I$. By Lemma 3.5, $w \in C_1^{I'}$. It follows that $y \in (\exists r.C_1)^{I'}$. ■

Lemma 3.7 *Given any concept C, $x \in C^{I'}$ iff $x \in C^I$.*

Proof. With respect to Lemma 3.5 we also consider here existential concepts, for which we can prove the converse of Lemma 3.6, namely that if $x \in (\exists r.C_1)^{I'}$, then $x \in (\exists r.C_1)^I$. This follows from the fact that for the starting individual x, for all w such that $(x, w) \in r^{I'}$, $(x, w) \in r^I$, and by Lemma 3.5, if $w \in {C_1}^{I'}$, then $w \in {C_1}^I$. ■

Lemma 3.8 *$\mathcal{M}'$ is a model of KB. Furthermore, α does not hold in $\mathcal{M}'$.*

Proof. Assertions in the ABox hold in $\mathcal{M}'$ by Lemma 3.5 or Lemma 3.6. Concerning the TBox, consider $C \sqsubseteq D$. By definition of Left Local, for C Lemma 3.5 holds. Hence, if $y \in C^{I'}$, also $y \in C^I$, hence $y \in D^I$ (since $\mathcal{M}$ is a model of KB) and, by Lemma 3.6, $y \in D^{I'}$. Last, by hypothesis x falsifies α in $\mathcal{M}$, i.e., if $\alpha = C \sqsubseteq D$, then $x \in C^I$ and $x \notin D^I$; if $\alpha = C(a)$, $x = a^I$ and $x \notin C^I$. From Lemma 3.7 in $\mathcal{M}'$, $x \in C^{I'}$ but $x \notin D^{I'}$, in the first case, and $x \notin C^{I'}$, in the second case. In both cases, α does not hold in $\mathcal{M}'$. ■

Lemma 3.9 *$\mathcal{M}'$ has a polynomial size in the size of KB.*

Proof. First of all, since $\mathcal{M}$ is minimal, each descending $<$-chain must be polynomial in size. Indeed, for each $w' < w$ in the chain there must be a concept C (occurring in KB) such that $w \notin (\Box\neg C)^I$ and $w' \in (\Box\neg C)^I$ (it can be shown that if it was not so, we could build a model preferred to $\mathcal{M}$, which, hence, would not be minimal; this model would be obtained by eliminating w from the $<$-chain and by putting w in the extension of the same concepts to which belongs a given individual with no preferred element, w.r.t. $<$). Since the number of concepts occurring in KB is polynomial, each chain is polynomial in size. By construction, this also holds in $\mathcal{M}'$. Furthermore, in $\mathcal{M}'$ the number of chains inserted corresponds to the number of distinct existential concepts occurring in KB, whose number is polynomial. Hence the overall size of $\mathcal{M}'$ is polynomial in the size of KB. ■

Lemma 3.10 *$\mathcal{M}'$ is a minimal model for KB.*

Proof. We have already shown in Lemma 3.5, when considering the case of $\mathbf{T}(C_1)$, that for all w in Δ', $w \notin (\Box\neg C_1)^{I'}$ iff $w \notin (\Box\neg C_1)^I$. Hence, $\mathcal{M}'^{\Box-}_{\mathcal{L}_\Box} \subseteq \mathcal{M}^{\Box-}_{\mathcal{L}_\Box}$ (we use $\subseteq$ instead of $=$ because the domain of $\mathcal{M}'$ is smaller than the domain of $\mathcal{M}$). Suppose now $\mathcal{M}'$ was not minimal. There would be an $\mathcal{M}''$ whose domain is the same than $\mathcal{M}'$'s and $\mathcal{M}''^{\Box-}_{\mathcal{L}_\Box} \subset \mathcal{M}'^{\Box-}_{\mathcal{L}_\Box}$. However, in this case we can build an $\mathcal{M}'''$ with the same domain as $\mathcal{M}$ such that $\mathcal{M}'''^{\Box-}_{\mathcal{L}_\Box} \subset \mathcal{M}^{\Box-}_{\mathcal{L}_\Box}$, thus proving that $\mathcal{M}$ is not minimal, against the hypothesis. $\mathcal{M}'''$ can be built from $\mathcal{M}''$ by reintroducing the domain elements of $\mathcal{M}$ that do not belong to $\mathcal{M}''$. We chose an element y in $\mathcal{M}''$ with no $y_1 <'' y$ (by the Smoothness Condition this element exists) and we let I''' be the extension of I'' that handles the added elements by putting them in the extension of exactly the same concepts to which y belongs in $\mathcal{M}''$. Since y satisfies all positive boxed formulas in $\mathcal{M}''$, $\mathcal{M}'''^{\Box-}_{\mathcal{L}_\Box} = \mathcal{M}''^{\Box-}_{\mathcal{L}_\Box} \subset \mathcal{M}'^{\Box-}_{\mathcal{L}_\Box}$. However, as stated above, $\mathcal{M}'^{\Box-}_{\mathcal{L}_\Box} \subseteq \mathcal{M}^{\Box-}_{\mathcal{L}_\Box}$, hence we conclude that $\mathcal{M}'''^{\Box-}_{\mathcal{L}_\Box} \subset \mathcal{M}^{\Box-}_{\mathcal{L}_\Box}$, and $\mathcal{M}$ would not be minimal, contradiction. ■

Theorem 3.11 (Small model theorem) *KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ if and only if KB $\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$.*

Proof. If KB $\not\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$, by the lemmas above we can build a small minimal model of KB that does not satisfy α, hence KB $\not\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. On the other hand, if KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ in all minimal models of KB, this also holds for *small* minimal models, hence KB $\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. ■

Given Theorem 3.11 above, we can conclude that, when evaluating entailment, we can restrict our consideration to small models, namely, to polynomial multilinear models of the KB. This provides an upper bound on the complexity of determining if KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. Indeed, by Theorem 3.11, this amounts to considering whether KB $\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. We consider the complementary problem of KB $\not\models^{small}_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$. This problem can be solved by nondeterministically generating a model $\mathcal{M}$ whose size is polynomial in the size of KB (NP), and then by calling an NP oracle which verifies that $\mathcal{M}$ is a minimal model of KB that does not satisfy α. To see this, the verification that $\mathcal{M}$ is not a minimal model of the KB can be done by an NP algorithm which nondeterministically generates a new model $\mathcal{M}'$ of KB (whose size is polynomial in the size of $\mathcal{M}$) and checks whether it is preferred to $\mathcal{M}$ (this can be checked in polynomial time in the size of $\mathcal{M}$). Hence, the problem of verifying that KB $\not\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ is in Σ^p_2, and that of verifying KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ is in Π^p_2. We can therefore conclude that:

Theorem 3.12 (Complexity for $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ Left Local KBs) *If KB is Left Local, the problem of deciding whether KB $\models_{\mathcal{EL}^{\perp}\mathbf{T}_{min}} \alpha$ is in Π^p_2.*

In [Giordano *et al.*, 2011] we have introduced a tableau calculus for deciding whether a query α is minimally entailed from a Left Local KB in the logic $\mathcal{EL}^{\perp}\mathbf{T}_{min}$. The calculus is a variant of the calculus $\mathcal{TAB}^{\mathcal{ALC}+\mathbf{T}}_{min}$ [Giordano *et al.*, 2008] for $\mathcal{ALC}+\mathbf{T}_{min}$, and it is essentially obtained by: (i) modifying the application of the rule $(\exists^+)$ in a way such that, when applied to a formula $u : \exists R.C$, it introduces a new label only when a label x_C for C does not belong to the current branch. Otherwise, if x_C has been already introduced, then $u \xrightarrow{R} x_C$ is added to the conclusion of the rule. In this way, in a given branch, the rule $(\exists^+)$ only introduces a new label x_C for each concept C occurring in the inital KB in some $\exists R.C$; (ii) modifying the rule for negated boxed formulas $\neg\Box\neg C$ according to the multilinear models semantics of $\mathcal{EL}^{\perp}\mathbf{T}_{min}$.

The calculus performs a two-phase computation in order to build an open branch representing a minimal model satisfying KB $\cup \{\neg\alpha\}$. In the first phase, the tableau calculus simply verifies whether KB $\cup \{\neg\alpha\}$ is satisfiable by an $\mathcal{EL}^{\perp}\mathbf{T}$ model, building candidate models. In the second phase the calculus checks whether the candidate models found in the first phase are *minimal* models of KB, i.e. for each open branch of the first phase, the second phase tries to build a "smaller" model, i.e. a model whose individuals satisfy less formulas $\neg\Box\neg C$ than the corresponding candidate model.

The resulting calculus is sound, complete, terminating, and its complexity matches the results of Theorem 3.12.

4 The Logic *DL-Lite*$_c\mathbf{T}_{min}$

In this section we present the extension of the low complexity description logic *DL-Lite*$_{core}$ [Calvanese *et al.*, 2007] with the $\mathbf{T}$ operator. We call *DL-Lite*$_c\mathbf{T}_{min}$ the resulting logic.

The language of *DL-Lite*$_c\mathbf{T}_{min}$ is defined as follows.

Definition 4.1 *We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individuals $\mathcal{O}$. Given $A \in \mathcal{C}$ and $r \in \mathcal{R}$, we define*

$$C_L := A \mid \exists R.\top \mid \mathbf{T}(A) \qquad R := r \mid r^-$$
$$C_R := A \mid \neg A \mid \exists R.\top \mid \neg\exists R.\top$$

A DL-Lite$_c\mathbf{T}_{min}$ KB is a pair (TBox, ABox). TBox contains a finite set of concept inclusions of the form $C_L \sqsubseteq C_R$. ABox contains assertions of the form $C(a)$ and $r(a, b)$, where C is a concept C_L or C_R, $r \in \mathcal{R}$, and $a, b \in \mathcal{O}$.

As for $\mathcal{EL}^{\perp}\mathbf{T}_{min}$, a model $\mathcal{M}$ for *DL-Lite*$_c\mathbf{T}_{min}$ is any structure $\langle\Delta, <, I\rangle$, defined as in Definition 2.2, where I is extended to take care of inverse roles: given $r \in \mathcal{R}$, $(r^-)^I = \{(a, b) \mid (b, a) \in r^I\}$.

For *DL-Lite*$_c\mathbf{T}_{min}$ KBs we can make a small model construction that is similar to that one for Left Local KBs. As a difference, in this case, we exploit the fact the, for each atomic role r, the same element of the domain can be used to satisfy all occurrences of the existential $\exists r.\top$. Also, the same element of the domain can be used to satisfy all occurrences of the existential $\exists r^-.\top$. This will guarantee that existential concepts (as well as all other concepts) have the same valuation in $\mathcal{M}'$ and in $\mathcal{M}$ on the common part of the domain. In the construction, we consider the two cases for $\exists r.\top$ and for $\exists r^-.\top$ separately.

Let $\mathcal{M} = \langle\Delta, <, I\rangle$ be a minimal model of KB in which α does not hold, and let x be the element of the domain falsifying α. The small model $\mathcal{M}'$ is built as follows:

$\Delta_0 := Unres := \{x\} \cup \{a^I \in \Delta \mid a$ occurs in ABox $\}$
while $Unres \neq \emptyset$ **do**
  extract one y from $Unres$
  for each $\exists r.\top$ occurring in KB s.t. $y \in (\exists r.\top)^I$ **do**
    if $\exists z, w \in \Delta_0$ s.t. $(z, w) \in r^{I'}$ **then**
      add (y, w) to $r^{I'}$
    else
      choose $w \in \Delta$ s.t. $(y, w) \in r^I$
      $\Delta_0 := \Delta_0 \cup \{w\}$
      add (y, w) to $r^{I'}$
      $Unres := Unres \cup \{w\}$
  for each $\exists r^-.\top$ in KB s.t. $y \in (\exists r^-.\top)^I$ **do**
    if $\exists z, w \in \Delta_0$ s.t. $(w, z) \in r^{I'}$ **then**
      add (w, y) to $r^{I'}$
    else
      choose $w \in \Delta$ s.t. $(w, y) \in r^I$
      $\Delta_0 := \Delta_0 \cup \{w\}$
      add (w, y) to $r^{I'}$
      $Unres := Unres \cup \{w\}$
  for each $y_i \in \Delta$ such that $y_i < y$ **do**
    $\Delta_0 := \Delta_0 \cup \{y_i\}$
    $Unres := Unres \cup \{y_i\}$

We define $\mathcal{M}'$ in the same way as in Definition 3.3. By reasoning similarly to what done for Left Local KBs, we can now show the following lemmas.

Lemma 4.2 *The relation $<'$ is irreflexive, transitive, multilinear and satisfies the Smoothness Condition.*

Lemma 4.3 *For all DL-Lite$_c\mathbf{T}$ concepts C occurring in the KB (or in the query), for all $y \in \Delta'$, $y \in C^{I'}$ iff $y \in C^I$.*

Proof. By induction on the structure of C. If C is a concept name or a boolean combination of concept names, the proof follows straightforwardly from the definition of I'.

Let us consider the case for C is $\exists R.\top$. We use the following facts concerning the construction of model $\mathcal{M}'$: (1) when (y, w) is added to $r^{I'}$ in steps 6 and 10, it must be the case that $(z, w) \in r^I$, for some $z \in \Delta$, i.e. that $w \in (\exists r^-.\top)^I$. (2) when (w, y) is added to $r^{I'}$ in steps 14 and 18, it must be the case that $(w, z) \in r^I$, for some $z \in \Delta$, i.e. that $w \in (\exists r.\top)^I$.

Let $y \in (\exists r.\top)^I$. As $y \in \Delta'$, y must have been selected at step 3. Then, in steps 6 or 10, (y, w) is added to $r^{I'}$ for some w in Δ_0. Hence, $y \in (\exists r.\top)^{I'}$. Let $y \in (\exists r.\top)^{I'}$. Then there is a $w \in \Delta'$ such that $(y, w) \in r^{I'}$. There are two cases. In the first one, (y, w) has been added to $r^{I'}$ in steps 6 or 10, since the condition of the for loop in step 4 holds. In this case, $y \in (\exists r.\top)^I$. In the second case, (y, w) has been added to $r^{I'}$ in steps 14 or 18. By fact (2) above, there must be some $z \in \Delta$ such that $(y, z) \in r^I$. Hence, $y \in (\exists r.\top)^I$.

Let $y \in (\exists r^-.\top)^I$. As $y \in \Delta'$, y must have been selected at step 3. Then, in steps 14 or 18, (w, y) is added to $r^{I'}$ for some w in Δ_0. Hence, $y \in (\exists r^-.\top)^{I'}$. Let $y \in (\exists r^-.\top)^{I'}$. Then there is a $w \in \Delta'$ such that $(w, y) \in r^{I'}$. There are two cases. In the first one, (w, y) has been added to $r^{I'}$ in steps 14 or 18, since the condition of the for loop in step 12 holds. In this case, $y \in (\exists r^-.\top)^I$. In the second case, (w, y) has been added to $r^{I'}$ in steps 6 or 10. By fact (1) above, there must be some $z \in \Delta$ such that $(z, y) \in r^I$. Hence, $y \in (\exists r^-.\top)^I$. ■

Notice that, due to the differences in the treatment of existential concepts here and in the case of Left Local KBs, Lemma 4.3 is stronger than Lemmas 3.5, 3.6, and 3.7.

By Lemma 4.3, and by reasoning as for Left Local KBs, we can prove the following lemma and theorem:

Lemma 4.4 *$\mathcal{M}'$ is a minimal model of KB whose size is polynomial in the size of KB.*

Theorem 4.5 (Small model theorem) *KB $\models_{DL\text{-}Lite_c\mathbf{T}_{min}}$ α if and only if KB $\models^{small}_{DL\text{-}Lite_c\mathbf{T}_{min}}$ α.*

By the same argument used for Left Local KBs, we can prove:

Theorem 4.6 (Complexity for *DL-Lite*$_c\mathbf{T}_{min}$) *The problem of deciding whether KB $\models_{DL\text{-}Lite_c\mathbf{T}_{min}}$ α is in Π^p_2.*

5 Conclusions

We have proposed a nonmonotonic extension of low complexity DLs $\mathcal{EL}^\perp$ and *DL-Lite*$_{core}$ for reasoning about typicality and defeasible properties. We have shown that entailment is EXPTIME-hard for $\mathcal{EL}^\perp\mathbf{T}_{min}$, whereas it drops to Π^p_2 when considering the Left Local Fragment of $\mathcal{EL}^\perp\mathbf{T}_{min}$. The same complexity has been found for *DL-Lite*$_c\mathbf{T}_{min}$. These results match the complexity upper bounds of the same fragments in circumscribed KBs [Bonatti *et al.*, 2009]. In [Bonatti *et al.*, 2010] a fragment of $\mathcal{EL}^\perp$ has been identified for which the complexity of circumscribed KBs is polynomial. In future work we shall investigate the complexity of minimal entailment for such fragment extended with $\mathbf{T}$.

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