Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence

Reasoning about Fuzzy Belief and Common Belief: With Emphasis on Incomparable Beliefs

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Abstract

We formalize reasoning about fuzzy belief and fuzzy common belief, especially incomparable beliefs, in multi-agent systems by using a logical system based on Fitting's many-valued modal logic, where incomparable beliefs mean beliefs whose degrees are not totally ordered. Completeness and decidability results for the logic of fuzzy belief and common belief are established while implicitly exploiting the duality-theoretic perspective on Fitting's logic that builds upon the author's previous work. A conceptually novel feature is that incomparable beliefs and qualitative fuzziness can be formalized in the developed system, whereas they cannot be formalized in previously proposed systems for reasoning about fuzzy belief. We believe that belief degrees can ultimately be reduced to truth degrees, and we call this "the reduction thesis about belief degrees", which is assumed in the present paper and motivates an axiom of our system. We finally argue that fuzzy reasoning sheds new light on old epistemic issues such as coordinated attack problem.

1 Introduction

Epistemic logic has been studied in order to formalize reasoning about knowledge and belief (see [13; 7]) with widespread applications to many research areas, including computer science and artificial intelligence ([7; 16]), economics and game theory ([1]), and philosophy ([2; 13]). The logic of common knowledge and belief is one of the central concerns of epistemic logic (see [7; 16]).

In this paper we formalize reasoning about fuzzy belief and fuzzy common belief, especially incomparable beliefs, in multi-agent systems by using a logical system based on Fitting's many-valued modal logic (for this logic, see [8; 9; 10]), where incomparable beliefs are defined as beliefs whose degrees are not totally ordered. We remark that many-valued modal logics have already been studied from various perspectives (see [4; 14; 15]). Results in the paper are established while implicitly exploiting the duality-theoretic perspective on Fitting's logic that builds upon the author's previous study (see [14; 15]).

Let us explain our motivations for studying the logic of fuzzy belief and common belief. It is not so unusual that one believes something to some degree, or the degree of one's belief may be neither 0 nor 1. The notion of fuzzy belief is appropriate in such a case. Moreover, the notion of fuzzy common belief can be appropriate even in a case where any agent of a group does not have a fuzzy belief. To see this, consider the following question. Is there anything that all the people in the world believe? Strictly speaking, there may be no such thing as a common belief among all the people in the world. Even if so, there may be something that most of the people in the world believe. For instance, most but not all of the people in the world probably believe that any human being is mortal or that the law of identity (i.e., $\varphi \to \varphi$) is valid (note that some logicians do not believe it). The notion of fuzzy common belief is appropriate in such a case as well as in a case where an agent of a group has a fuzzy belief.

Here, we would like to clarify our philosophical standpoint. We consider that the degree of a belief φ by an agent i is equivalent to the truth degree of the proposition that i believes φ (in fact, this is imprecise; to be precise, see **B2** in Definition 11; not T_a but U_a is appropriate also here), that is, degrees of belief can ultimately be reduced to degrees of truth in this way (in the sense of U_a as in **B2**), which we call "the reduction thesis about belief degrees" (this has no relation with Peirce's reduction thesis). We may identify the reduction thesis with the axiom **B2** in Definition 11. Although the thesis may be contested, we work under the assumption of it in this paper. We believe that the reduction thesis is philosophically justifiable to some degrees, but anyway it is certainly beneficial (and would thus be justifiable) from a technical point of view as is shown in the results of the paper.

Epistemic logic based on classical logic is inadequate to formalize reasoning about fuzzy (common) belief, which is due to the fact that either 0 or 1 is assigned to every formula in classical epistemic logic. We are thus led to consider epistemic logic based on many-valued logic, since the truth value of a proposition may be neither 0 nor 1 in many-valued logic. Among many existing many-valued logics, we employ a modified version of Fitting's lattice-valued logic, the reasons of which are explained later, and we add to the lattice-valued logic epistemic operators including a common belief operator, thus developing a logical system for reasoning about fuzzy belief and common belief.

Several authors have already developed logical systems to formalize reasoning about fuzzy belief (see, e.g., [3; 6; 11]), for example, by combining probabilistic logic and epistemic logic. However, there seems to have been no study of reasoning about fuzzy common belief via the combination of many-valued logic and epistemic logic. Moreover, the degrees of beliefs are supposed to be totally ordered in the previously proposed systems. For this reason, the notion of incomparable beliefs cannot be formalized in them.

There are indeed many incomparable beliefs in ordinary life. For instance, consider the following situations: (1) Suppose that there are two collections X, Y of grains here, that X is taller and thinner than Y and that it is not obvious whether or not each collection of grains makes a heap, as shown in the following figure (of course, this is based on the well-known sorites paradox, which motivates many-valued logic).

Let a (resp. b) be the degree of one's belief that X (resp. Y) makes a heap. Then, a and b may be incomparable, since their magnitudes are incomparable. (2) Suppose that a child believes that she loves her mother and that she loves her father. Then, the degrees of her two beliefs can be incomparable. Thus, one's beliefs are sometimes incomparable.

Hence, it would be significant to be able to formalize the notion of incomparable beliefs in a logical system. In our system, a degree of a belief is expressed as an element of a lattice which is not necessarily totally ordered. Therefore, the notion of incomparable beliefs can be formalized in our system, which is impossible in previously developed systems for reasoning about fuzzy belief such as those in [3; 6; 11]. This is one of the reasons why we employ a version of Fitting's lattice-valued logic as the underlying logic of our system for reasoning about fuzzy belief and common belief.

We remark that Fitting's lattice-valued logic (for different lattice-valued logics, see [17]) may be considered as a kind of fuzzy logic, but the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ is not necessarily valid in Fitting's logic, while it is valid in fuzzy logics such as Łukasiewicz logic and Gödel logic (for these logics, see [12]). We also note that the lattice of truth values is finite in Fitting's logic (see also [4]), since L-valued modal logic may not be recursively axiomatizable for an infinite lattice L. In practice or in the real world, a sufficiently large finite lattice and an infinite lattice would not make a significant difference (we could not distinguish between them).

This paper is organized as follows. In Section 2, lattice-valued logic L-**VL** is discussed. In Section 3, a logic of fuzzy belief in an n-agent system, L-$\mathbf{K}_n$, is discussed. In the two sections, we mainly aim to reformulate algebraic axiomatizations in [14] in terms of Hilbert-style deductive systems. In Section 4, the usual Kripke semantics for a common belief operator is naturally extended to the L-valued case and then a logic of fuzzy belief and common belief, L-$\mathbf{K}_n^C$, is discussed. Especially, we develop a Hilbert-style deductive system for L-$\mathbf{K}_n^C$ and show that it is sound and complete with respect to the extended Kripke semantics and that L-$\mathbf{K}_n^C$ is decidable and enjoys the finite model property. We remark that we can also obtain other versions of these results such as **KD45**-style, **S5**-style, **K45**-style, and **S4**-style ones.

2 Lattice-Valued Logic: L-VL

Throughout this paper, let L denote a finite distributive lattice with the top element 1 and the bottom element 0. Then, as is well known, L forms a finite Heyting algebra. For $a, b \in L$, let $a \to b$ denote the pseudo complement of a relative to b. Let **2** denote the two-element Boolean algebra.

Definition 1. *We augment L with unary operations $\mathrm{T}_a(\text{-})$'s for all $a \in L$ defined as follows: $\mathrm{T}_a(x) = 1$ if $x = a$ and $\mathrm{T}_a(x) = 0$ if $x \neq a$. We also augment L with unary operations $\mathrm{U}_a(\text{-})$'s for all $a \in L$ defined by: $\mathrm{U}_a(x) = 1$ if $x \geq a$ and $\mathrm{U}_a(x) = 0$ if $x \ngeq a$*

We define L-valued logic L-**VL** as follows. The connectives of L-**VL** are $\wedge$, $\vee$, $\to$, 0, 1, T_a and U_a for each $a \in L$, where T_a and U_a are unary connectives, 0 and 1 are nullary connectives, and the others are binary connectives. Let **PV** denote the set of propositional variables. Then, the set of formulas of L-**VL**, which is denoted by **Form**, are recursively defined in the usual way. Let $\varphi \leftrightarrow \psi$ be the abbreviation of $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg\varphi$ the abbreviation of $\varphi \to 0$. The intended meaning of $\mathrm{T}_a(\varphi)$ is that the truth value of a proposition φ is an element a of L. The intended meaning of $\mathrm{U}_a(\varphi)$ is that the truth value of φ is more than or equal to a. L-valued semantics is then introduced as follows.

Definition 2. *A function v : **Form** $\to L$ is an L-valuation on **Form** iff it satisfies the following:*

1. $v(\mathrm{T}_a(\varphi)) = \mathrm{T}_a(v(\varphi))$ *for each* $a \in L$;
2. $v(\mathrm{U}_a(\varphi)) = \mathrm{U}_a(v(\varphi))$ *for each* $a \in L$;
3. $v(\varphi \wedge \psi) = \inf(v(\varphi), v(\psi))$;
4. $v(\varphi \vee \psi) = \sup(v(\varphi), v(\psi))$;
5. $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$;
6. $v(a) = a$ *for* $a = 0, 1$.

*Then, $\varphi \in$ **Form** is called valid in L-**VL** iff $v(\varphi) = 1$ for any L-valuation v on **Form**.*

If L is the two-element Boolean algebra **2**, then the above semantics coincides with the ordinary two-valued semantics for classical logic, where note that $\mathrm{T}_1(\varphi) \leftrightarrow \varphi$ and $\mathrm{T}_0(\varphi) \leftrightarrow \neg\varphi$ are valid in **2-VL**, whence all T_a's and U_a's are actually redundant in **2-VL**. We then give a Hilbert-style axiomatization of L-**VL**.

Definition 3. *$\varphi \in$ **Form** is provable in L-**VL** iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are*

A1. *all instances of tautologies of intuitionistic logic;*

A2. $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\psi)) \to \mathrm{T}_{a@b}(\varphi@\psi)$ *for* $@ = \to, \wedge, \vee$; $\mathrm{T}_b(\varphi) \to (\mathrm{T}_{@(b)}(@(\varphi))$ *for* $@ = \mathrm{T}_a, \mathrm{U}_a$;

A3. $\mathrm{T}_0(0)$; $\mathrm{T}_a(0) \leftrightarrow 0$ *for* $a \neq 0$; $\mathrm{T}_1(1)$; $\mathrm{T}_a(1) \leftrightarrow 0$ *for* $a \neq 1$;

A4. $\bigvee_{a \in L} \mathrm{T}_a(\varphi)$*;* $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\varphi)) \leftrightarrow 0$ *for* $a \neq b$*;* $\mathrm{T}_a(\varphi) \vee \neg \mathrm{T}_a(\varphi)$*;*

A5. $\mathrm{T}_1(\mathrm{T}_a(\varphi)) \leftrightarrow \mathrm{T}_a(\varphi)$*;* $\mathrm{T}_0(\mathrm{T}_a(\varphi)) \leftrightarrow (\mathrm{T}_a(\varphi) \to 0)$*;* $\mathrm{T}_b(\mathrm{T}_a(\varphi)) \leftrightarrow 0$ *for* $b \neq 0, 1$*;*

A6. $\mathrm{T}_1(\varphi) \to \varphi$*;* $\mathrm{T}_1(\varphi \wedge \psi) \leftrightarrow \mathrm{T}_1(\varphi) \wedge \mathrm{T}_1(\psi)$*;*

A7. $\mathrm{U}_a(\varphi) \leftrightarrow \bigvee\{\mathrm{T}_x(\varphi)\ ;\ a \leq x \text{ and } x \in L\}$*;*

A8. $(\bigwedge_{a \in L}(\mathrm{T}_a(\varphi) \leftrightarrow \mathrm{T}_a(\psi))) \to (\varphi \leftrightarrow \psi)$,

where $a, b \in L$ *and* $\varphi, \psi \in$ **Form**. *The rules of inference are*

R1. *From* φ *and* $\varphi \to \psi$ *infer* ψ*;*

R2. *From* $\varphi \leftrightarrow \psi$ *infer* $\chi \leftrightarrow \chi'$*, where* χ' *is the formula obtained from* χ *by replacing an occurrence of* φ *with* ψ*;*

R3. *From* $\varphi \to \psi$ *infer* $\mathrm{T}_1(\varphi) \to \mathrm{T}_1(\psi)$,

where $\varphi, \psi, \chi \in$ **Form**.

$(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\psi)) \to \mathrm{T}_{a \to b}(\varphi \to \psi)$ intuitively means that if the truth value of φ is a and the truth value of ψ is b then the truth value of $\varphi \to \psi$ is $a \to b$. The intuitive meanings of the axioms in **A2** can be explained in similar ways. Note that $\mathrm{T}_a(b)$ and $\mathrm{U}_a(b)$ in **A2** are either 0 or 1. An axiom $\bigvee_{a \in L} \mathrm{T}_a(\varphi)$ in **A4** is called the L-valued excluded middle, since the **2**-valued excluded middle coincides with the ordinary excluded middle.

The notion of deducibility for L-**VL** is defined in the usual way: For $\varphi \in$ **Form** and $X \subset$ **Form**, φ is deducible from X in L-**VL** iff φ can be deduced from X and the axioms of L-**VL** by the inference rules of L-**VL**. We then have the following deduction theorem for L-**VL**, which can be shown in almost the same way as delta deduction theorems for fuzzy logics with Baaz delta (see [5, Theorem 6]).

Proposition 4. *Let* $\varphi, \psi_1, ..., \psi_k \in$ **Form** *where* $k \in \omega \setminus \{0\}$*. If* φ *is deducible from* $\{\psi_1, ..., \psi_k\}$ *in* L-**VL***, then* $\mathrm{T}_1(\psi_1 \wedge ... \wedge \psi_k) \to \varphi$ *is provable in* L-**VL**.

In the following, we show that the above axiomatization of L-**VL** is sound and complete with respect to the L-valued semantics. We first define the notion of L-**VL** consistency as follows: For $X \subset$ **Form**, X is L-**VL** consistent iff 0 is not deducible from X in L-**VL**. Note that a maximal L-**VL** consistent subset of **Form** is closed under the inference rules of L-**VL** by the maximality of it.

Lemma 5. *Let* X *be a maximal* L-**VL** *consistent subset of* **Form***. Then, for any* $\varphi \in$ **Form***, there is a unique* $a \in L$ *such that* $\mathrm{T}_a(\varphi) \in X$*.*

Proof. Let $\varphi \in$ **Form**. Assume that there is no $a \in L$ such that $\mathrm{T}_a(\varphi) \in X$. Since X is closed under $\wedge$ and T_1, it follows from the maximality of X and Proposition 4 that for each $a \in L$, there is $\psi_a \in X$ such that $(\psi_a \wedge \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is provable in L-**VL**. Let $\psi = \bigwedge_{a \in L} \psi_a$. Note that $\psi \in X$. Now, $(\psi \wedge \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is provable in L-**VL**. By **A1**, $(\psi \wedge \bigvee_{a \in L} \mathrm{T}_a(\varphi)) \leftrightarrow 0$ is also provable in L-**VL**. Thus it follows from **A1**, **A4**, and **R2** that $\psi \leftrightarrow 0$ is provable in L-**VL**. Then, since X is closed under modus ponens, we have $0 \in X$ by $\psi \in X$, which is a contradiction. Hence there is $a \in L$ such that $\mathrm{T}_a(\varphi) \in X$. The uniqueness of such $a \in L$ is shown by using the L-**VL** consistency of X and the following axiom in **A4**: $(\mathrm{T}_a(\varphi) \wedge \mathrm{T}_b(\varphi)) \leftrightarrow 0$ for $a \neq b$. □

Theorem 6. *For* $\varphi \in$ **Form***,* φ *is provable in* L-**VL** *iff* φ *is valid in* L-**VL**.

Proof. It is straightforward to show the soundness. We show the completeness by proving the contrapositive. Assume that φ is not provable in L-**VL**. Then, $\mathrm{T}_1(\varphi)$ is not provable in L-**VL** by **A6**. If $\{\neg \mathrm{T}_1(\varphi)\}$ is not L-**VL** consistent, then it follows from **A1** and an axiom $\mathrm{T}_a(\varphi) \vee \neg \mathrm{T}_a(\varphi)$ in **A4** that $\mathrm{T}_1(\varphi)$ is provable in L-**VL**, which is a contradiction. Thus, $\{\neg \mathrm{T}_1(\varphi)\}$ is L-**VL** consistent. By a standard argument using Zorn's lemma we have a maximal L-**VL** consistent subset X of **Form** containing $\neg \mathrm{T}_1(\varphi)$. Then we define a function v_X from **Form** to L as follows: For $\psi \in$ **Form**, $v_X(\psi) = a \Leftrightarrow \mathrm{T}_a(\psi) \in X$. Then it follows from Lemma 5 that v_X is well defined. Since $\neg \mathrm{T}_1(\varphi) \in X$, we have $\mathrm{T}_1(\varphi) \notin X$ by **A1**, which implies that $v_X(\varphi) \neq 1$. Now it remains to show that v_X is an L-valuation. We first verify that $v_X(\psi \to \chi) = v_X(\psi) \to v_X(\chi)$. Let $a = v_X(\psi)$ and $b = v_X(\chi)$. Then we have $\mathrm{T}_a(\psi), \mathrm{T}_b(\chi) \in X$. Thus, since X is closed under modus ponens, it follows from **A2** that $\mathrm{T}_{a \to b}(\psi \to \chi) \in X$. Therefore, by the definition of v_X, we have $v_X(\psi \to \chi) = a \to b = v_X(\psi) \to v_X(\chi)$. The other cases are similarly verified. □

By using the above theorem, it is straightforward to show the following three propositions.

Proposition 7. *Let* $a \in L$ *and* $\varphi, \psi \in$ **Form**. *(i)* $\mathrm{U}_a(\varphi \wedge \psi) \leftrightarrow \mathrm{U}_a(\varphi) \wedge \mathrm{U}_a(\psi)$ *is provable in* L-**VL**. *(ii)* $(\varphi \to \psi) \to (\mathrm{U}_a(\varphi) \to \mathrm{U}_a(\psi))$ *is provable in* L-**VL**.

Proposition 8. *Let* $a, b \in L$ *with* $a \neq 0$ *and* $\varphi \in$ **Form**. *(i)* $\mathrm{U}_a(\mathrm{U}_b(\varphi)) \leftrightarrow \mathrm{U}_b(\varphi)$ *is provable in* L-**VL**. *(ii)* $\mathrm{U}_a(\mathrm{T}_b(\varphi)) \leftrightarrow \mathrm{T}_b(\varphi)$ *is provable in* L-**VL**.

Although the law of excluded middle does not necessarily hold in L-**VL**, it holds for a special kind of formulas. The same thing holds also for De Morgan's law and for the commutativity of U_a and $\vee$.

Proposition 9. *Let* $a, b \in L$ *with* $a \neq 0$ *and* $\varphi, \psi \in$ **Form**. *Assume that* $\mathrm{U}_a(\varphi) \leftrightarrow \varphi$ *and* $\mathrm{U}_a(\psi) \leftrightarrow \psi$ *are provable in* L-**VL**. *(i)* $\varphi \vee \neg\varphi$ *is provable in* L-**VL**. *(ii)* $\neg(\varphi \wedge \chi) \leftrightarrow (\neg\varphi \vee \neg\chi)$ *and* $\neg(\varphi \vee \chi) \leftrightarrow (\neg\varphi \wedge \neg\chi)$ *are provable in* L-**VL**. *(iii)* $\mathrm{U}_b(\varphi \vee \psi) \leftrightarrow \mathrm{U}_b(\varphi) \vee \mathrm{U}_b(\psi)$ *is provable in* L-**VL**.

3 Logic of Fuzzy Belief: L-$\mathbf{K}_n$

In this section, we introduce a logical system for reasoning about fuzzy belief in an n-agent system for a non-negative integer n, which is denoted by L-$\mathbf{K}_n$. The connectives of L-$\mathbf{K}_n$ are unary connectives B_i for $i = 1, ..., n$ and the connectives of L-**VL**. Then, let $\mathbf{Form}_n$ denote the set of formulas of L-$\mathbf{K}_n$. The intended meaning of $B_i(\varphi)$ is that the i-th agent believes that φ. Thus, the intended meaning of $B_i\mathrm{U}_a(\varphi)$ is that the i-th agent believes φ at least to the degree of a or the degree of the i-th agent's belief φ is more than or equal to a.

An example of reasoning about fuzzy belief is: If the degree of the i-th agent's belief φ is a and if the degree of the i-th agent's belief ψ is more than or equal to a, then the degree of the i-th agent's belief $\varphi \to \psi$ is 1. This reasoning is expressed in L-$\mathbf{K}_n$ as $(B_i\mathrm{T}_a(\varphi) \wedge B_i\mathrm{U}_a(\psi)) \to B_i\mathrm{T}_1(\varphi \to$

ψ), which is both valid and provable in the following semantics and proof system for L-$\mathbf{K}_n$.

An example of reasoning about incomparable beliefs is: If the degree of the i-th agent's belief φ and the degree of the i-th agent's belief ψ are incomparable (examples of incomparable beliefs φ, ψ are in Section 1), then it does not hold that either if the i-th agent believes φ then the i-th agent believes ψ or if the i-th agent believes ψ then the i-th agent believes φ. This is expressed in L-$\mathbf{K}_n$ as follows (let a and b be incomparable in L with $a \vee b \neq 1$): $(B_i\mathrm{T}_a(\varphi) \wedge B_i\mathrm{T}_b(\psi)) \to \neg\mathrm{T}_1((B_i\varphi \to B_i\psi) \vee (B_i\psi \to B_i\varphi))$, which is both valid and provable in the following semantics and proof system for a lattice L in which there are such a and b (there are indeed many such lattices L). Recall that $(B_i\varphi \to B_i\psi) \vee (B_i\psi \to B_i\varphi)$ is valid in classical epistemic logic, which is a so-called paradox of material implication. The paradox is avoided in our logical system.

L-valued Kripke semantics for L-$\mathbf{K}_n$ is defined as follows.

Definition 10. *Let $(M, R_1, ..., R_n)$ be a Kripke n-frame, i.e., R_i is a binary relation on a set M for each $i = 1, ..., n$. Then, a function $e : M \times \mathbf{Form}_n \to L$ is an L-$\mathbf{K}_n$ valuation on $(M, R_1, ..., R_n)$ iff it satisfies the following for each $w \in M$:*

1. $e(w, B_i(\varphi)) = \bigwedge\{e(w', \varphi)\ ;\ wR_iw'\}$ *for* $i = 1, ..., n$;
2. $e(w, \mathrm{T}_a(\varphi)) = \mathrm{T}_a(e(w, \varphi))$ *for each* $a \in L$;
3. $e(w, \mathrm{U}_a(\varphi)) = \mathrm{U}_a(e(w, \varphi))$ *for each* $a \in L$;
4. $e(w, \varphi @ \psi) = e(w, \varphi) @ e(w, \psi)$ *for* $@ = \wedge, \vee, \to$;
5. $e(w, a) = a$ *for* $a = 0, 1$.

We call $(M, R_1, ..., R_n, e)$ an L-$\mathbf{K}_n$ Kripke model. Then, $\varphi \in \mathbf{Form}_n$ is said to be valid in L-$\mathbf{K}_n$ iff $e(w, \varphi) = 1$ for any L-$\mathbf{K}_n$ Kripke model $(M, R_1, ..., R_n, e)$ and any $w \in M$.

If L is the two-element Boolean algebra, then the above Kripke semantics coincides with the usual Kripke semantics for the $\mathbf{K}$-style logic of belief in an n-agent system. A Hilbert-style axiomatization of L-$\mathbf{K}_n$ is given as follows.

Definition 11. *$\varphi \in \mathbf{Form}_n$ is provable in L-$\mathbf{K}_n$ iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are $\mathbf{A1}, ..., \mathbf{A8}$ in Definition 3 and*

B1. $B_i(\varphi \wedge \psi) \leftrightarrow B_i(\varphi) \wedge B_i(\psi)$ *for each* $i = 1, ..., n$;

B2. $B_i\mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_aB_i(\varphi)$ *for each* $i = 1, ..., n$,

where $a \in L$ and $\varphi, \psi \in \mathbf{Form}_n$. The rules of inference are $\mathbf{R1}, \mathbf{R2}, \mathbf{R3}$ in Definition 3 and

R4. *From* $\varphi \to \psi$ *infer* $B_i(\varphi) \to B_i(\psi)$ *for* $i = 1, ..., n$.

We may call the axiom **B2** the reduction thesis about belief degrees (see Section 1). If **B2** is contested, it is possible to develop another deductive system without **B2** that corresponds to Kripke semantics with L-valued accessibility relations. L-$\mathbf{K}_n$ consistency (and L-$\mathbf{K}_n^C$ consistency in the next section) are defined in the same way as L-**VL** consistency. The axiomatic system above is sound and complete.

Theorem 12. *For $\varphi \in \mathbf{Form}_n$, φ is provable in L-$\mathbf{K}_n$ iff φ is valid in L-$\mathbf{K}_n$.*

Proof. It is straightforward to show the soundness. We show the completeness by proving the contrapositive. Assume that φ is not provable in L-$\mathbf{K}_n$. Let Con be the set of all maximal L-$\mathbf{K}_n$ consistent subsets of $\mathbf{Form}_n$. We can consider the L-$\mathbf{K}_n$ Kripke model $(\mathrm{Con}, R_1, ..., R_n, e)$ such that for each $i = 1, ..., n$ and $V, W \in \mathrm{Con}$, VR_iW iff, for any $a \in L$ and $\psi \in \mathbf{Form}_n$, $\mathrm{U}_a(B_i\psi) \in V$ implies $\mathrm{U}_a(\psi) \in W$ and that for each propositional variable p, $e(W, p) = a$ iff $\mathrm{T}_a(p) \in W$.

We claim that, for any $\psi \in \mathbf{Form}_n$ and $W \in \mathrm{Con}$,

$$e(W, \psi) = a \text{ iff } \mathrm{T}_a(\psi) \in W.$$

We show the claim by induction on the structure of formulas. We consider only the case that ψ is of the form $B_i(\chi)$ for $i \in \{1, ..., n\}$, since arguments in the other cases are similar to those in the proof of Theorem 6. In order to show that $e(W, B_i\chi) = a$ iff $\mathrm{T}_a(B_i\chi) \in W$, it suffices to show that $e(W, B_i\chi) \geq a$ iff $\mathrm{U}_a(B_i\chi) \in W$, since by **A7** we have: $\mathrm{T}_a(B_i\chi) \in W$ iff $\mathrm{U}_x(B_i\chi) \in W$ for any $x \in L$ with $x \leq a$ and $\mathrm{U}_x(B_i\chi) \notin W$ for any $x \in L$ with $x \not\leq a$. If $\mathrm{U}_a(B_i\chi) \in W$, then $\mathrm{U}_a(\chi) \in V$ for any $V \in \mathrm{Con}$ with WR_iV, whence by the induction hypothesis, we have $e(W, B_i\chi) = \bigwedge\{e(V, \chi)\ ;\ WR_iV\} \geq a$. We next show the converse by proving the contrapositive. Assume $\mathrm{U}_a(B_i\chi) \notin W$. Let

$$G = \{\mathrm{U}_b(\eta)\ ;\ \eta \in \mathbf{Form}_n,\ b \in L \text{ and } B_i\mathrm{U}_b(\eta) \in W\}.$$

We first verify that $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is L-$\mathbf{K}_n$ consistent. Suppose for contradiction that $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is not L-$\mathbf{K}_n$ consistent. Then, there is $\zeta \in G$ such that $\zeta \to \mathrm{U}_a(\chi)$ is provable in L-$\mathbf{K}_n$. Thus, $B_i\zeta \to B_i\mathrm{U}_a(\chi)$ is provable in L-$\mathbf{K}_n$. Since $B_i\zeta \in W$ by the definition of G, we have $B_i\mathrm{U}_a(\chi) \in W$ and so $\mathrm{U}_a(B_i\chi) \in W$ by **B2**, which contradicts $\mathrm{U}_a(B_i\chi) \notin W$. Thus, $G \cup \{\neg\mathrm{U}_a(\chi)\}$ is L-$\mathbf{K}_n$ consistent. By a standard argument using Zorn's lemma, we have a maximal L-$\mathbf{K}_n$ consistent subset H of $\mathbf{Form}_n$ containing $G \cup \{\neg\mathrm{U}_a(\chi)\}$. Since H contains $\neg\mathrm{U}_a(\chi)$, it follows from the induction hypothesis that $e(H, \chi) \not\geq a$. Since H contains G, it follows from **B2** that WR_iH. Thus we have $e(W, B_i\chi) \not\geq a$. This completes the proof of the above claim. Now it is straightforward to verify that $(\mathrm{Con}, R_1, ..., R_n, e)$ is a counter-model for φ. □

4 Logic of Fuzzy Common Belief: L-$\mathbf{K}_n^C$

In this section, we introduce a logical system for reasoning about fuzzy belief and common belief in an n-agent system, which is denoted by L-$\mathbf{K}_n^C$. The connectives of L-$\mathbf{K}_n^C$ are unary connectives E and C, and the connectives of L-$\mathbf{K}_n$. Let $\mathbf{Form}_n^C$ denote the set of formulas of L-$\mathbf{K}_n^C$. The intended meaning of $E(\varphi)$ is that every agent in the system believes that φ. The intended meaning of $C(\varphi)$ is that it is a common belief among all the agents in the system that φ (for the difference between $E(\varphi)$ and $C(\varphi)$, see, e.g., [7]). Thus, the intended meaning of $\mathrm{U}_aC(\varphi)$ is that φ is a common belief at least to the degree of a or the degree of a common belief φ is more than or equal to a.

An example of reasoning about fuzzy common belief is: If the degree of a common belief φ in the n-agent system is a, then the degree of any agent's belief φ is more than or equal to a. This reasoning is expressed in L-$\mathbf{K}_n^C$ as $\mathrm{T}_aC(\varphi) \to$

$(\mathrm{U}_a B_1(\varphi) \wedge ... \wedge \mathrm{U}_a B_n(\varphi))$, which is both valid and provable in the following semantics and proof system for L-$\mathbf{K}_n^C$.

L-valued Kripke semantics for L-$\mathbf{K}_n^C$ is defined as follows. For a non-negative integer k, $E^k(\varphi)$ is defined by $E^1(\varphi) = E(\varphi)$ and $E^{k+1}(\varphi) = E(E^k(\varphi))$.

Definition 13. *Let $(M, R_1, ..., R_n)$ be a Kripke n-frame. Then, a function $e : M \times \mathbf{Form}_n^C \to L$ is an L-$\mathbf{K}_n^C$ valuation on $(M, R_1, ..., R_n)$ iff it satisfies the following for each $w \in M$:*

1. *$e(w, E(\varphi)) = \bigwedge\{e(w, B_i(\varphi))\ ;\ i = 1, ..., n\}$;*
2. *$e(w, C(\varphi)) = \bigwedge\{e(w, E^k(\varphi))\ ;\ k \in \omega \setminus \{0\}\}$;*
3. *The other conditions are as in Definition 10.*

We call $(M, R_1, ..., R_n, e)$ an L-$\mathbf{K}_n^C$ Kripke model. Then, $\varphi \in \mathbf{Form}_n^C$ is said to be valid in L-$\mathbf{K}_n^C$ iff $e(w, \varphi) = 1$ for any L-$\mathbf{K}_n^C$ Kripke model $(M, R_1, ..., R_n, e)$ and any $w \in M$.

If L is the two-element Boolean algebra, then the above Kripke semantics coincides with the usual Kripke semantics for the (**K**-style) logic of common belief in an n-agent system (for logics of common knowledge and belief, see [7]). The following notion of reachability is useful for understanding the common belief operator C.

Definition 14. *Let $(M, R_1, ..., R_n)$ be a Kripke n-frame and $w, w' \in M$. For a non-negative integer k, w is reachable from w' in k steps iff there are $w_0, ..., w_k \in M$ such that $w_0 = w'$, $w_k = w$, and for any $l \in \{0, ..., k-1\}$, there is $i \in \{1, ..., n\}$ with $w_l R_i w_{l+1}$.*

Proposition 15. *Let $(M, R_1, ..., R_n, e)$ be an L-$\mathbf{K}_n^C$ Kripke model. For $w \in M$, $a \in L$ and $\varphi \in \mathbf{Form}_n^C$, $e(w, C(\varphi)) \geq a$ iff, for any $k \in \omega \setminus \{0\}$, if w' is reachable from w in k steps, then $e(w', \varphi) \geq a$.*

We now give a Hilbert-style axiomatization of L-$\mathbf{K}_n^C$.

Definition 16. *$\varphi \in \mathbf{Form}_n^C$ is provable in L-$\mathbf{K}_n^C$ iff it is either an instance of one of the following axioms or deduced from provable formulas by one of the following rules of inference: The axioms are $\mathbf{A1}, ..., \mathbf{A8}, \mathbf{B1}, \mathbf{B2}$ in Definition 3 and Definition 11, and*

E1. *$E(\varphi) \leftrightarrow (B_1(\varphi) \wedge ... \wedge B_n(\varphi))$;*

C1. *$C\mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_a C(\varphi)$;*

C2. *$\mathrm{U}_a C(\varphi) \to \mathrm{U}_a E(\varphi \wedge C(\varphi))$,*

where $a \in L$ and $\varphi \in \mathbf{Form}_n^C$. The rules of inference are $\mathbf{R1}, \mathbf{R2}, \mathbf{R3}, \mathbf{R4}$ in Definition 3 and Definition 11, and

R5. *From $\varphi \to E(\varphi \wedge \psi)$ infer $\varphi \to C(\psi)$.*

Using the axioms **B1** and **B2**, we can show the following.

Proposition 17. *Let $a \in L$ and $\varphi \in \mathbf{Form}_n^C$. (i) $E\mathrm{U}_a(\varphi) \leftrightarrow \mathrm{U}_a E(\varphi)$ is provable in L-$\mathbf{K}_n^C$. (ii) $E(\varphi \wedge \psi) \leftrightarrow E(\varphi) \wedge E(\psi)$ is provable in L-$\mathbf{K}_n^C$.*

The following theorem states that the above axiomatization of L-$\mathbf{K}_n^C$ is sound and complete with respect to L-valued Kripke semantics for L-$\mathbf{K}_n^C$.

Theorem 18. *For $\varphi \in \mathbf{Form}_n^C$, φ is provable in L-$\mathbf{K}_n^C$ iff φ is valid in L-$\mathbf{K}_n^C$.*

Our proof of the above theorem is rather long and so we put it in the last part of this section. L-$\mathbf{K}_n^C$ enjoys the finite model property:

Theorem 19. *For $\varphi \in \mathbf{Form}_n^C$, φ is provable in L-$\mathbf{K}_n^C$ iff $e(w, \varphi) = 1$ for any finite L-$\mathbf{K}_n^C$ Kripke model $(M, R_1, ..., R_n, e)$ and any $w \in M$.*

Proof. The proof of Theorem 18 given below contains the proof of this theorem, since a counter-model $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ in the proof of Theorem 18 is actually a finite L-$\mathbf{K}_n^C$ Kripke model. □

By Theorem 18 and Theorem 19, we obtain the decidability of L-$\mathbf{K}_n^C$.

Theorem 20. *For $\varphi \in \mathbf{Form}_n^C$, it is effectively decidable whether or not φ is valid in L-$\mathbf{K}_n^C$.*

Finally, we give the proof of Theorem 18 (though we omit some details) by generalizing the proof of [7, Theorem 4.3]

Proof of Theorem 18. It is straightforward to verify the soundness. The completeness is proved as follows. To show the contrapositive, assume that φ is not provable in L-$\mathbf{K}_n^C$. Let $\mathrm{Sub}_C(\varphi)$ be the set of the following formulas: (i) all subformulas of φ; (ii) $B_1\psi, ..., B_n\psi$ for each subformula $E\psi$ of φ; (iii) $\psi \wedge C\psi, B_1(\psi \wedge C\psi), ..., B_n(\psi \wedge C\psi), E(\psi \wedge C\psi)$ for each subformula $C\psi$ of φ. Let

$$\mathrm{Sub}_C^+(\varphi) = \{\mathrm{T}_a(\psi), \mathrm{U}_a(\psi)\ ;\ \psi \in \mathrm{Sub}_C(\varphi) \text{ and } a \in L\}.$$

Define $\mathrm{Con}_C(\varphi)$ as the set of all maximal L-$\mathbf{K}_n^C$ consistent subsets of $\mathrm{Sub}_C^+(\varphi)$. Then, we can consider the L-$\mathbf{K}_n^C$ Kripke model $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ such that for each $i = 1, ..., n$ and $V, W \in \mathrm{Con}_C(\varphi)$, VR_iW iff, for any $a \in L$ and $\psi \in \mathrm{Sub}_C^+(\varphi)$, $\mathrm{U}_a(B_i\psi) \in V$ implies $\mathrm{U}_a(\psi) \in W$ and that for each propositional variable $p \in \mathrm{Sub}_C(\varphi)$, $e(W, p) = a$ iff $\mathrm{T}_a(p) \in W$. Then we claim that, for $a \in L$, $\psi \in \mathrm{Sub}_C(\varphi)$ and $W \in \mathrm{Con}_C(\varphi)$,

$$e(W, \psi) = a \text{ iff } \mathrm{T}_a(\psi) \in W.$$

If the claim holds, then $(\mathrm{Con}_C(\varphi), R_1, ..., R_n, e)$ is a counter-model for φ, since there is $W \in \mathrm{Con}_C(\varphi)$ with $\mathrm{T}_1(\varphi) \notin W$ (i.e., $e(W, \varphi) \neq 1$) by the assumption that φ is not provable in L-$\mathbf{K}_n^C$ (such W is obtained as follows: Construct a maximal L-$\mathbf{K}_n^C$ consistent subset X of $\mathbf{Form}_n^C$ containing $\neg\mathrm{T}_1(\varphi)$ and then let $W = X \cap \mathrm{Sub}_C^+(\varphi)$). Thus, to show the completeness, it suffices to verify the claim. Note that the above claim is equivalent to the following: For $a \in L$ with $a \neq 0$, $\psi \in \mathrm{Sub}_C(\varphi)$ and $W \in \mathrm{Con}_C(\varphi)$, $e(W, \psi) \geq a$ iff $\mathrm{U}_a(\psi) \in W$, since we have: $\mathrm{T}_a(\psi) \in W$ iff $\mathrm{U}_x(\psi) \in W$ for any $x \in L$ with $x \leq a$ and $\mathrm{U}_x(\psi) \notin W$ for any $x \in L$ with $x \not\leq a$. We show the claim by induction on the structure of formulas. We consider only the case that ψ is of the form $C(\chi)$, since arguments in the other cases are similar to those in the proofs of Theorem 6 and Theorem 12.

Suppose that ψ is of the form $C(\chi)$. It suffices to show that, for any $W \in \mathrm{Con}_C(\varphi)$ and $a \in L$ with $a \neq 0$, $e(W, C\chi) \geq a$ iff $\mathrm{U}_a C(\chi) \in W$. We first show that $\mathrm{U}_a C(\chi) \in W$ implies $e(W, C\chi) \geq a$. Assume $\mathrm{U}_a C(\chi) \in W$. We claim that, for any $k \in \omega \setminus \{0\}$, if $V \in \mathrm{Con}_C(\varphi)$ is reachable from W in k

steps, then both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_aC(\chi)$ are in V. This is shown by induction on $k \in \omega \setminus \{0\}$. We first consider the case $k = 1$. By **C2**, $\mathrm{U}_aC(\chi) \to \mathrm{U}_aE(\chi \wedge C(\chi))$ is provable in L-$\mathbf{K}_n^C$. Then, by $W \in \mathrm{Con}_C(\varphi)$, we have $\mathrm{U}_aE(\chi \wedge C(\chi)) \in W$, whence $\mathrm{U}_aE\chi \in W$ and $\mathrm{U}_aE(C\chi) \in W$ by (ii) in Proposition 17 and (i) in Proposition 7. If V is reachable from W in 1 step, then it follows from **E1** and the definition of R_i that $\mathrm{U}_a(\chi)$ and $\mathrm{U}_aC(\chi)$ are in V. We next consider the case $k = k' + 1$ for $k' \in \omega \setminus \{0\}$. If V is reachable from W in $k'+1$ steps, then there is $V' \in \mathrm{Con}_C(\varphi)$ such that V is reachable from V' in 1 step and that V' is reachable from W in k' steps. By the induction hypothesis, both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_aC(\chi)$ are in V'. By arguing as in the case $k = 1$, it is verified that both $\mathrm{U}_a(\chi)$ and $\mathrm{U}_aC(\chi)$ are in V. Thus, the claim has been proved. Hence, for any $k \in \omega \setminus \{0\}$, if V is reachable from W in k steps, then $\mathrm{U}_a(\chi) \in V$ and so $e(V, \chi) \geq a$ by the induction hypothesis for the first claim. By Proposition 15, we have $e(W, C\chi) \geq a$.

In order to complete the proof, we show that $e(W, C\chi) \geq a$ implies $\mathrm{U}_aC(\chi) \in W$. Assume $e(W, C\chi) \geq a$. Define

$$\Gamma = \{V \in \mathrm{Con}_C(\varphi) \,;\, e(V, C\chi) \geq a\}$$
$$\zeta = \bigvee\{\bigwedge V \,;\, V \in \Gamma\}.$$

where note that both V and $\{\bigwedge V \,;\, V \in \Gamma\}$ are finite sets. We first show that if $\mathrm{U}_a(\zeta) \to \mathrm{U}_aE(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$, then $\mathrm{U}_aC(\chi) \in W$. Assume that $\mathrm{U}_a(\zeta) \to \mathrm{U}_aE(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$. Then, by (i) in Proposition 17 and (i) in Proposition 7, $\mathrm{U}_a(\zeta) \to E(\mathrm{U}_a(\chi) \wedge \mathrm{U}_a(\zeta))$ is provable in L-$\mathbf{K}_n^C$. Then it follows from **R5** and **C1** that $\mathrm{U}_a(\zeta) \to \mathrm{U}_aC(\chi)$ is provable in L-$\mathbf{K}_n^C$. Since $(\bigwedge W) \to \zeta$ is provable in L-$\mathbf{K}_n^C$ by $W \in \Gamma$, $\mathrm{U}_a(\bigwedge W) \to \mathrm{U}_a(\zeta)$ is provable in L-$\mathbf{K}_n^C$ by (ii) in Proposition 7. By these facts, $\mathrm{U}_a(\bigwedge W) \to \mathrm{U}_aC(\chi)$ is provable in L-$\mathbf{K}_n^C$. Since $\mathrm{U}_a(\bigwedge W) \leftrightarrow (\bigwedge W)$ is provable in L-$\mathbf{K}_n^C$ by Proposition 8, $(\bigwedge W) \to \mathrm{U}_aC(\chi)$ is provable in L-$\mathbf{K}_n^C$. Hence, we have $\mathrm{U}_aC(\chi) \in W$. Now it only remains to show that the assumption of this argument holds, i.e., $\mathrm{U}_a(\zeta) \to \mathrm{U}_aE(\chi \wedge \zeta)$ is provable in L-$\mathbf{K}_n^C$, which we omit because of the limitation of space. □

5 Conclusions

We have studied the logic of fuzzy belief and common belief with emphasis on incomparable beliefs (which cannot be formalized in previously known systems in [3; 6; 11]), establishing completeness and decidability results (implicitly based on duality-theoretic intuitions). We can also obtain many other versions of the results such as **KD45**-style and **S5**-style ones (and many more). We will study their complexity issues in future work.

We emphasize that our system based on L-valued logic can treat both qualitative fuzziness and quantitative fuzziness in the context of epistemic reasoning, while systems based on $[0, 1]$-valued logic or probabilistic logic (such as those in [3; 6; 11]) can only encompass the latter.

In the previous work such as [14; 15], we studied the mathematically profound aspects of Fitting's many-valued modal logic. The results in the paper suggest that Fitting's logic be beneficial also in the context of artificial intelligence.

We consider that fuzzy reasoning is useful to understand some epistemic problems such as two generals' problem or coordinated attack problem (for this problem, see, e.g., [7]). In theory, the two generals cannot attack at any time, while, in practice, they will attack after they have sent messages a few times. We can understand this as follows. The more times they send messages, the higher the degrees of their beliefs become. Even if the degrees cannot reach 1, the generals will attack when they become sufficiently high. This seems to be a very natural understanding of coordinated attack problem, which is impossible if we stick to classical logic and becomes possible only if we allow fuzzy reasoning.

In the paper we considered the reduction thesis about belief degrees as being provisionally true. Although the reduction thesis would be justified to some degrees by its theoretical merit, in future work, we will discuss in more detail to what extent the reduction thesis is justifiable, since it would be of philosophical interest.

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