An Approach to Minimal Belief via Objective Belief *

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Abstract

As a doxastic counterpart to epistemic logic based on $\mathbf{S5}$ we study the modal logic $\mathbf{KSD}$ that can be viewed as an approach to modelling a kind of objective and fair belief. We apply $\mathbf{KSD}$ to the problem of minimal belief and develop an alternative approach to nonmonotonic modal logic using a weaker concept of expansion. This corresponds to a certain minimal kind of $\mathbf{KSD}$ model and yields a new type of nonmonotonic doxastic reasoning.

1 Introduction

Modern epistemic and doxastic logic has its roots in the work of [Hintikka, 1962] which gave rise to a substantial and sustained research programme making significant contributions to philosophy, logic, AI and other fields. Though several specific features of Hintikka's approach to knowledge and belief have, over the years, been questioned and modified by different authors for different purposes, some of the core ideas have stood the test of time and are still relevant today.

One of the central, surviving ideas in this paradigm is that knowledge and belief can be understood as relations between a cognitive agent a and a proposition p, symbolised by $\mathcal{K}_a p$ – "a knows that p" and $\mathcal{B}_a p$ – "a believes that p". The semantic interpretation of these expressions proceeds via possible worlds frames equipped with an epistemic alternativeness relation, $R_{\mathcal{K}_a}$, interpreted as: $wR_{\mathcal{K}_a}w'$ if w' is compatible with everything a knows in world w. The truth condition for an epistemic operator is then given by:

$$w \models \mathcal{K}_a p \quad \text{iff} \quad w' \models p \quad \text{for all} \quad w' \quad \text{s.t.} \quad wR_{\mathcal{K}_a}w' \qquad (1)$$

So a knows p in a world w if p is true in all the epistemic alternatives to w; and analogous relations and truth conditions apply to belief sentences.

Assuming some basic, if idealising, conditions about knowledge, and simplifying by considering a single agent in each system, epistemic/doxastic logics are on this view structurally similar to normal modal logics with a necessity operator $\Box$ interpreted on Kripke frames with a single binary alternativeness or accessibility relation R on worlds. Again, despite many variations, mutations and extensions, this feature survives in many areas concerned with reasoning about knowledge and belief.

While different normal systems have been proposed to capture the logic of the knowledge operator, a special place is occupied by $\mathbf{S5}$. There are several well-known reasons for this. For one, the logic $\mathbf{S5}$ satisfies some important principles of introspection that might be considered appropriate for an ideally rational agent. For another, it is based on frames whose alternativeness relation R is in a sense maximal: R is reflexive, symmetric and transitive, ie an equivalence relation. Moreover inference in $\mathbf{S5}$ is fully characterised by the class of *universal* Kripke models, where R is a universal relation (and frames form a so-called *cluster*); so every world counts as an epistemic alternative to a given one. Third, almost all nonmonotonic systems for reasoning about knowledge make use of $\mathbf{S5}$-like frames. Furthermore, it is closely related to the important concept of *stable* (belief) set first studied by [Stalnaker, 1993] and [Moore, 1983], since these correspond to sets of formulas true on some universal $\mathbf{S5}$-frame. Stable sets again play a prominent role in nonmonotonic modal and epistemic logics in general.

1.1 Aim of the paper

The main difference between a normal *doxastic* as opposed to *epistemic* modal logic is that the $\mathbf{T}$ axiom $\Box p \to p$ that is appropriate for the latter seems unsuitable for the former. Therefore typically even strong logics of belief, such as $\mathbf{KD45}$, do not satisfy this principle. Our aim here is to investigate a doxastic modal logic that we call $\mathbf{KSD}$ as an alternative to systems like $\mathbf{KD45}$. The motivation comes from two main aspects, each of them forming one main part of our study.

The first aspect concerns monotonic doxastic logic. A leading idea is to approach doxastic reasoning by looking for a system having strong analogies to $\mathbf{S5}$, but without the $\mathbf{T}$ axiom. Let us consider then an analogy on the semantical level and take the case of universal $\mathbf{S5}$ models. An obvious way to avoid principle $\mathbf{T}$ is to consider irreflexive alternativeness relations and to interpret belief in terms of *other* possible worlds (than the one under consideration). So in place of ($\mathbf{S5}$) knowledge where we may assume that *every world is an epistemic alternative*, we would have for belief: *every* **other** *world is a doxastic alternative*. We maintain the analogy to knowledge in that we refer not just to some other worlds but

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to all of them.

There is a sense in which this interpretation can be related to an idea of objective (and fair) belief.[1] Imagine that the doxastic alternative worlds actually represent the information sets (of basic formulas) of different agents. For each world w there is a corresponding agent a_w and a set of basic (non-modal) formulas that hold there (the basic information available to a_w). Each agent is doxastically objective (or altruistic) as well as fair in the sense that it forms its (first-degree) beliefs by observing the information sets of each and every other agent (but not its own). In a model as a whole the common beliefs shared by all agents are inasmuch 'objective' as they are formed by this inter-subjective process.

How does this affect second-degree (or iterated) beliefs, eg the truth of $\Box\Box p$? To evaluate this expression a_w must inspect the first-degree belief in p held by every other agent. But any such agent in turn has $\Box p$ only if p holds at its alternatives that include w. It follows that on this account of 'objective' belief, $\Box\Box p$ depends not only on $\Box p$ but on p as well. It suggest that the positive introspection axiom $\mathbf{4}$ valid in $\mathbf{KD45}$, viz $\Box p \rightarrow \Box\Box p$, will have to be replaced by a weaker principle:

$$\Box p \wedge p \rightarrow \Box\Box p.$$

Indeed this is known as $\mathbf{w4}$ ("weak $\mathbf{4}$") and turns out to be valid in $\mathbf{KSD}$.

Returning to the models for belief, let us consider further properties. One is simple: if we interpret belief in terms of other worlds we should be sure that at least some such other worlds exist. So in addition to the set W of worlds being non-empty as usual, it should contain at least two elements. Another issue concerns the transitivity of the R relation. While symmetry seems desirable and can safely be maintained given our other concerns, in its presence transitivity may restore reflexivity. Let $R \subseteq W \times W$ be symmetric and transitive and suppose x is not a dead-end (a world seeing no other) so that xRy for some y. Then by symmetry yRx and so by transitivity xRx. To avoid this we can weaken the property of transitivity by just the right amount.

Definition 1.1 *We say that a relation $R \subseteq W \times W$ is weakly-transitive if* $(\forall x, y, z)(xRy \wedge yRz \wedge x \neq z \Rightarrow xRz)$.

So our doxastic frames should be not transitive like $\mathbf{S5}$-frames but weakly-transitive (we will also avoid dead-ends).

The second aspect of our motivation for this study concerns the KR paradigm of reasoning about knowledge with its distinctive introspective and nonmonotonic character. Here we also proceed by analogy to $\mathbf{S5}$. Nonmonotonic modal logics for reasoning about knowledge are based on a concept of (theory) expansion. In the standard cases such expansions are stable sets and can be captured by a notion of *minimal* model, again a special kind of $\mathbf{S5}$-model that can be thought to represent a concept of minimal knowledge. Evidently each world in such a model satisfies the strong $\mathbf{T}$ axiom, seemingly appropriate for knowledge but less so for belief. In fact, while $\mathbf{KD45}$ is usually considered to be a doxastic modal logic, its nonmonotonic variant is precisely autoepistemic logic, ie a system that seems closer to knowledge rather than belief.

To obtain a candidate logic for minimal belief, our strategy is to put $\mathbf{KSD}$ models in place of $\mathbf{S5}$ models and define a new concept of minimal model. Equivalently, as we show in Section 4, we can define a weaker concept of expansion and show that this corresponds to the new definition of minimal model. However, we first present the main results about $\mathbf{KSD}$ in Section 2 and then turn to the concept of minimal belief in Sections 3 and 4.

[1] We stress that this is just one possible interpretation and we do not wish to enter here into a detailed analysis of objective or fair belief, which is not the main issue.

2 The Doxastic Modal Logic KSD

The name $\mathbf{KSD}$ does not follow the typical naming convention in modal logic, ie it is not modal logic $\mathbf{K}$ plus axioms $\mathbf{S}$ and $\mathbf{D}$. It would rather be called $\mathbf{wK4BD}$ if we followed this style. $\mathbf{KSD}$ is an extension of the well known system $\mathbf{KS}$ formed by adding axiom $\mathbf{D}$. $\mathbf{KS}$ was named after Krister Segerberg who first introduced it as a modal logic of *some other time* [Segerberg, 1976]. This logic has been extensively studied by several authors, see eg [Esakia, 2001], and was rediscovered as a modal logic of *inequality* also known sometimes as $\mathbf{DL}$ [Goranko, 1990; de Rijke, 1992]. We will see that $\mathbf{KSD}$ as a simple extension of $\mathbf{KS}$ can also be viewed as a modal logic of inequality.

Definition 2.1 *The normal modal logic* $\mathbf{KSD}$ *is defined in a standard modal language with an infinite set of propositional letters* $Prop = \{p_0, p_1, p_2 \ldots\}$ *and connectives* $\wedge, \Box, \neg$. *We let* $p, q, r, \ldots$ *range over propositions, and other connectives are defined as usual. The axioms are all classical tautologies plus:*

$\mathbf{K}$: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$,
$\mathbf{D}$: $\Box p \rightarrow \Diamond p$,
$\mathbf{w4}$: $\Box p \wedge p \rightarrow \Box\Box p$,
$\mathbf{B}$: $p \rightarrow \Box\Diamond p$, *where* $\Diamond p \equiv \neg\Box\neg p$.

The rules of inference are: Modus Ponens, Substitution and Necessitation.

Let us briefly comment on differences with respect to $\mathbf{KD45}$. Obviously the first two axioms are shared by both logics. We have already remarked on how, given our semantic analogy with $\mathbf{S5}$, we expect axiom $\mathbf{4}$ to be weakened in the case of $\mathbf{KSD}$. In the case of $\mathbf{B}$ versus $\mathbf{5}$ there is no direct relation: the latter is expressed by $\Diamond p \rightarrow \Box\Diamond p$. However it is easy to see how $\mathbf{B}$ arises in virtue of our semantics of other doxastic alternative worlds. To establish $\Box\Diamond p$ at w we need to verify $\Diamond p$ at all the alternative worlds. Since each is accessible from w, truth of p in w will suffice for this, while the truth of $\Diamond p$ in w will not. So the implication $p \rightarrow \Box\Diamond p$ must be valid.

If we add the axiom $\mathbf{T}$: $\Box p \rightarrow p$ to $\mathbf{KSD}$ we obtain the classical epistemic system $\mathbf{S5}$. From these rather informal semantic considerations we pass now to the formal semantics.

2.1 Kripke semantics

We assume familiarity with the basic notions of Kripke semantics for modal logics. To fix notation and terminology we recall the main definitions found in modal logic texts.

Definition 2.2 *The pair* (W, R)*, with* W *an arbitrary set and* $R \subseteq W \times W$ *is called a Kripke frame. If we additionally have a third component* $V : Prop \times W \to \{0, 1\}$*, then we say that we have a Kripke model* $M = (W, R, V)$

The satisfaction and validity of a modal formula are defined inductively. We just state the base and modal cases here.

Definition 2.3 *For a given Kripke model* $M = (W, R, V)$ *the satisfaction of a formula at a point* $w \in W$ *is defined inductively as follows:* $w \Vdash p$ *iff* $V(p, w) = 1$*, the Boolean cases are standard, and* $w \Vdash \Box\phi$ *iff* $(\forall v)(wRv \Rightarrow v \Vdash \phi)$*. A formula* ϕ *is valid in a model* (W, R, V) *if for every point* $w \in W$ *we have* $w \Vdash \phi$*. A formula* ϕ *is valid in a frame* (W, R) *if it is valid in every model* (W, R, V) *based on a frame* (W, R)*. A formula* ϕ *is valid in a class of frames* C *if* ϕ *is valid in every frame* $(W, R) \in C$*.*

Kripke semantics for **KSD** is provided by weakly-transitive and symmetric Kripke frames without a dead-end.

Definition 2.4 *We say that a point* w *in a Kripke frame* (W, R) *is a dead-end if for all* $v \in W$*, not* wRv*.*

Observe that every dead-end is irreflexive. Weakly-transitive relations were described in Definition 1.1. Clearly, every transitive relation is weakly-transitive. Moreover it is immediate to see that weakly-transitive relations differ from transitive ones just by the occurrence of irreflexive points inside the subsets where every two distinct points are related to each other. Such sets will be called *weak-clusters*.

Definition 2.5 *A subset* $U \subseteq W$ *in a Kripke frame* (W, R) *is called a weak-cluster if* $(\forall w, v \in U)(w \neq v \Rightarrow wRv)$*. We say that a Kripke frame* (W, R) *is a weak-cluster if* W *is a weak-cluster in* (W, R) *i.e. every two distinct points of the frame are related with each other.*

As one can see the weak-cluster in Figure 1 is weakly-transitive, but not transitive.

The picture represents the diagrammatic view of a Kripke structure, where irreflexive points are coloured grey and reflexive ones are uncoloured. Arrows represent the relation between two distinct points. So, vRw and wRv, but we do not have vRv, which contradicts transitivity, but not weak transitivity as $v = v$.

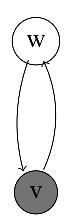

Figure 1

In the study of modal logic the class of rooted frames plays a central role. Recall that a frame (W, R) is rooted if it contains a point $w \in W$, which can see all other points in W. That is $R(w) \supseteq W - \{w\}$, where $R(w)$ is the set of all successors of w. The following proposition shows that the class of all rooted, weakly-transitive and symmetric frames without a dead-end is in one to one correspondence with the class of all weak-clusters except for the singleton irreflexive point.

Proposition 2.6 *A frame* (W, R) *without a dead-end is rooted, weakly-transitive and symmetric iff* (W, R) *is a weak-cluster, where if* W *is one-point set then* R *is reflexive.*

Proof. It follows immediately from the definition 2.5 that every weak-cluster is a rooted, weakly-transitive and symmetric frame. Besides if it does not comprise a single, irreflexive point, it does not have dead-end (see 2.4). For the other direction let (W, R) be a rooted, weakly-transitive and symmetric frame which is not a singleton irreflexive point . Let $r \in W$ be the root. If $W = \{r\}$ then by assumption we have rRr and hence (W, R) is a weak-cluster. In case W contains more than one point, take two arbitrary distinct points $w, v \in W$. As r is the root, we have: rRw and rRv. Because of symmetry we get wRr. Now as R is weakly-transitive, from $wRr \wedge rRv$ and $w \neq v$ we get wRv. Hence R is a weak-cluster.

So far we have defined the modal logic **KSD** syntactically and given a definition of weak-cluster relation. The following theorem links these two notions:

Theorem 2.7 *The modal logic* **KSD** *is sound and complete w.r.t. the class of all* ***finite, irreflexive*** *weak-clusters, excepting the single irreflexive point.*

This can be shown by adapting the completeness proof for the logic **KS** found in [Esakia, 2001], using standard techniques.

Now it is easy to see why the modal logic **KSD** is an extension of the modal logic of inequality. As the reader can easily check the interpretation of $\Box$ in irreflexive weak-clusters boils down to the following:

$$w \Vdash \Box\phi \text{ iff } (\forall v)(w \neq v \Rightarrow v \Vdash \phi).$$

Consequently our logic is indeed complete for the frames that we described in Section 1, where each world different from the world of evaluation is a doxastic alternative. Moreover it suffices that the set of worlds is finite.

We now turn to nonmonotonic logic and the idea of minimal belief.

3 Minimising Belief

The paradigm of minimal *knowledge* derives from the well-known work of [Halpern and Moses, 1985], later extended and modified by [Shoham, 1987; Lin and Shoham, 1990; Lifschitz, 1994] and others. Many approaches are based on Kripke-**S5**-models with a universal accessibility relation and the minimisation of knowledge is represented by maximising the set of possible worlds with respect to inclusion. In general, this has the effect of minimising objective knowledge, ie knowledge of basic facts and propositions. An alternative approach was developed by [Schwarz and Truszczyński, 1992] and can be seen as a special case of the general method of [Shoham, 1987] for obtaining different concepts of minimality by changing the sets of models and preference relations between them. The initial models considered by [Schwarz and Truszczyński, 1992] (see also [Schwarz, 1992]) consist of not one (**S5**) cluster but rather two clusters arranged in such a way that all worlds in one cluster are accessible from all worlds in the other (but not vice versa). In Figure 2

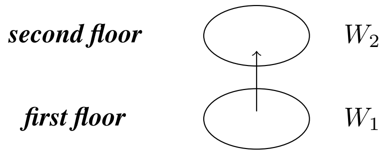

Figure 2

clusters are labelled W_1, W_2, all points are reflexive and every point in W_2 is accessible from every point in W_1. We call W_1, W_2 respectively the *first* and the *second floor* of the model. The former may be empty but the latter not.

In [Schwarz and Truszczyński, 1992], given a background theory or knowledge set I, minimal knowledge (with respect to I) is captured by an $\mathbf{S5}$-model of the theory, say $\mathcal{M}$, but now the idea of minimality is that there should be no two-floor model $\mathcal{M}'$ as in Figure 2 of the same theory I, where $\mathcal{M}$ coincides with the restriction of $\mathcal{M}'$ to the second floor W_2, and W_1 is smaller in the sense that it fails to verify some objective (non-modal) sentence true in $\mathcal{M}$. Schwarz and Truszczyński argue that this approach to minimal knowledge has some important advantages over the method of [Halpern and Moses, 1985] and they study its properties in depth, in particular showing that while the two-floor models correspond to the modal logic $\mathbf{S4F}$ first studied by [Segerberg, 1971], minimal knowledge is precisely captured by non-monotonic $\mathbf{S4F}$.

The concept of minimal *belief* was touched on by [Moore, 1983] in his analysis of autoepistemic logic. It was later shown [Truszczyński, 1991] that autoepistemic logic is in fact equivalent to non-monotonic $\mathbf{KD45}$. However, as in the case of models for minimal knowledge, autoepistemic expansions are stable sets and correspond to universal $\mathbf{S5}$-models. Our objective here is to arrive at a concept of minimal belief that is based on $\mathbf{KSD}$ (doxastic) models rather than $\mathbf{S5}$ (knowledge) models. In other respects, however, we want to follow the approach to minimality developed by Schwarz and Truszczyński. In fact the minimisation techniques proposed by Halpern are not applicable to the logic $\mathbf{KSD}$. To accomplish our aim we need a logic that, at the monotonic level, plays a role similar to the of $\mathbf{S4F}$ in the approach of Schwarz and Truszczyński, namely a logic in which we characterise the idea of minimal model and define a suitable notion of expansion. Evidently this will differ from the usual concept of expansion, since we want it to capture models that are not $\mathbf{S5}$. Our base logic will also differ from $\mathbf{KSD}$, since, like $\mathbf{S4F}$, its frames will have a two-floor structure. We call it $\mathbf{wK4Df}$.

3.1 The Logic wK4Df

In a previous paper we studied a normal system called $\mathbf{wK4f}$ in the context of minimising belief based on classical $\mathbf{KD45}$ models, where the minimisation is obtained by adding points to the $\mathbf{KD45}$ model on the first floor [Pearce and Uridia, 2010]. Here we consider an extension of $\mathbf{wK4f}$ formed by adding axiom $\mathbf{D}$. This gives us the following.

Syntax. The normal modal logic $\mathbf{wK4Df}$ is defined with the following set of axioms plus all classical tautologies. Rules of inference are: modus ponens, substitution and necessitation.

$\mathbf{K}$: $\Box(p \to q) \to (\Box p \to \Box q)$,

$\mathbf{w4}$: $\Box p \wedge p \to \Box\Box p$,

$\mathbf{D}$: $\Box p \to \Diamond p$,

$\mathbf{f}$: $p \wedge \Diamond(q \wedge \Box\neg p) \to \Box(q \vee \Diamond q)$.

Semantics. Kripke semantics for the modal logic $\mathbf{wK4Df}$ is provided by frames which have in a weak sense height at most two and which do not allow forking.

Definition 3.1 *We say that a relation $R \subseteq W \times W$ is a one-step relation if the following two conditions are satisfied:*

$1)(\forall x, y, z)((xRy \wedge yRz) \Rightarrow (yRx \vee zRy))$,

$2)(\forall x, y, z)((xRy \wedge \neg(yRx) \wedge xRz \wedge y \neq z) \Rightarrow zRy)$.

Condition 1) restricts the 'strict' height of the frame to two, where informally by 'strict' we mean that the steps are not counted *within* a cluster. The second condition is more complicated, essentially restricting the 'strict' width of the frame to one, though more points can be allowed at the bottom.

Completeness is a straightforward extension of our previous results for $\mathbf{wK4f}$ and is proved by standard techniques.

Theorem 3.2 *The modal logic* $\mathbf{wK4Df}$ *is sound and strongly complete wrt the class of all Kripke frames that do not have a dead-end, are one-step and weakly-transitive.*

Notice that rooted, one-step and weakly-transitive Kripke frames without the a dead-end are similar to the two-floor frames in the Figure 2, where the ellipses now represent not clusters but weak-clusters and the second floor is not a singleton irreflexive point. For both $\mathbf{S4F}$ and $\mathbf{wK4D}$ we refer to the rooted frames as *two-floor frames*.

4 The Logic of Minimal Belief

We now follow an approach similar to that of [Schwarz and Truszczyński, 1992] and minimise belief sets in the same way as knowledge is minimised. We thereby build a nonmonotonic logic of belief. For the underlying monotonic system we take $\mathbf{wKD4f}$. As already remarked, we cannot use the standard method to define a nonmonotonic version of $\mathbf{wKD4f}$ as this would yield minimal models that are knowledge ie $\mathbf{S5}$ models. Accordingly we need to alter the usual notions of expansion and nonmonotonic inference. The changed notion is called weak-expansion and serves as a main tool in defining the new inference relation. Let L denote some monotonic modal logic which is contained in $\mathbf{KSD}$. We first recall the definition of L-consequence relation.

Definition 4.1 *Let I be a consistent set of formulas in L. We say that the formula φ is an L-consequence of I ($I \vdash_L \varphi$) if for every L-model $\mathcal{M}$, $\mathcal{M} \Vdash \varphi$ whenever $\mathcal{M} \Vdash I$. We denote by $Cn_L(I)$ the set of all L-consequences of I.*

Definition 4.2 *Let L be a (monotonic) modal logic. We say that a set of formulas T is a* weak-expansion *(in L) for the set of formulas I if*

$$T = Cn_L(I \cup \{\neg\Box\phi \vee \neg\phi | \phi \notin T\} \cup \{\neg\Box\phi | \neg\Box\phi \in T\}).$$

If L is assumed to be given we usually drop reference to it. Observe that the weak-expansions have properties analogous to normal expansions. In fact if we think of a weak-expansion T as a belief set, we have that agent's beliefs are closed in the following way: (i) If ϕ is in an agent's belief set then he believes ϕ. (ii) If ϕ is not in the agent's belief set then the disjunction $\neg\Box\phi \vee \neg\phi$ is present. (iii) If an agent does not believe ϕ then he believes that he does not believe ϕ. In the obvious manner weak-expansions can be used to define a nonmonotonic inference relation, $\approx_L$, by setting $I \approx_L \varphi$ if φ is true in all weak-expansions of I.

Now we introduce a preference relation on rooted, one-step and weakly-transitive models in an analogous way to [Schwarz, 1992]. The preference relation is only between one floor and two floor models and only two floor models can be preferred over one floor models. Recall that by two floor we mean rooted, weakly-transitive, one-step frames, ie we allow weak-clusters on the first and the second floor.

Definition 4.3 *A two-floor model $\mathcal{M}' = (W', R', V')$ is said to be* preferred *over a one-floor model $\mathcal{M} = (W, R, V)$ (notation: $\mathcal{M}' \succ \mathcal{M}$) if: a) (W, R) is the second floor of (W', R') and V equals to the restriction of V' to the second floor; so $\mathcal{M}$ is the model obtained by deleting the first floor in $\mathcal{M}'$. b) There is a formula ψ such that $\mathcal{M} \Vdash \psi$ and $\mathcal{M}' \nVdash \psi$.*

The standard preference relation proposed by [Schwarz, 1992] is obtained by requiring the formula ψ in b) to be a propositional (modal-free) formula. Nevertheless the following theorem shows that for clusters the two defitions coincide. Let $\mathcal{M}' > \mathcal{M}$ denote the preference relation introduced by Schwarz, applied to the class of all two-floor $\mathbf{wK4Df}$-frames.

Theorem 4.4 *Let $\mathcal{M}$ be a cluster. Then for every two-floor $\mathbf{wK4Df}$-model $\mathcal{N}$ we have $\mathcal{N} > \mathcal{M}$ iff $\mathcal{N} \succ \mathcal{M}$.*

Proof. We prove the non-trivial direction. Assume $\mathcal{N} \not> \mathcal{M}$. Then for every propositional formula α, $\mathcal{M} \Vdash \alpha$ implies $\mathcal{N} \Vdash \alpha$. We show by induction on the modal depth of formula that $\mathcal{M} \Vdash \phi$ implies $\mathcal{N} \Vdash \phi$ for every formula ϕ. The base case is given by assumption. We assume the induction hypothesis holds for every formula with modal depth less than k and show the hypothesis holds for a formula ϕ with modal depth equal to k. By propositional reasoning we can rewrite ϕ equivalently by $\phi \equiv \bigwedge \delta_i$ where each δ_i is of the form $\Box\alpha_1 \vee \Box\alpha_2 \vee .. \vee \Box\alpha_n \vee \neg\Box\beta_1 \vee \neg\Box\beta_2 .. \vee \neg\Box\beta_m \vee \gamma_i$, where each α_j, β_j has modal depth less than k and γ_i is a propositional formula. $\mathcal{M} \Vdash \phi$ implies that for each i the corresponding disjunct holds in $\mathcal{M}$. So take an arbitrary i from the conjunction and take a disjunct $\Box\alpha_1 \vee \Box\alpha_2 \vee .. \vee \Box\alpha_n \vee \neg\Box\beta_1 \vee \neg\Box\beta_2 .. \vee \neg\Box\beta_m \vee \gamma_i$. If at least at one point v in $\mathcal{M}$, $\neg\Box\beta_j$ holds for some $j \in \{1, ..m\}$ then there is a point v' such that vRv' and $v' \Vdash \neg\beta_j$ hence every point on the first floor of $\mathcal{N}$ can see v' and hence $\neg\Box\beta_j$ holds on the first floor of $\mathcal{N}$. Therefore $\mathcal{N} \Vdash \phi$. If at least at one point v in $\mathcal{M}$, $\Box\beta_j$ holds for some $j \in \{1, ..n\}$ then every point $w \in \mathcal{M}$ satisfies β_j (here we use that $\mathcal{M}$ is a cluster). Hence $\mathcal{M} \Vdash \beta_j$ and by the induction hypothesis $\mathcal{N} \Vdash \beta_j$. Therefore $\mathcal{N} \Vdash \Box\beta_j$ and hence $\mathcal{N} \Vdash \phi$. If neither of the above two cases holds then $\mathcal{M} \Vdash \gamma_i$ and again by hypthesis $\mathcal{N} \Vdash \gamma_i$ hence $\mathcal{N} \Vdash \phi$. So we get $\mathcal{N} \not\succ \mathcal{M}$.$\dashv$

Our aim now is to relate weak-expansions to a notion of *minimal* model. This is exactly like the usual concept ([Schwarz, 1992]) except that it is based on the new notion of preference that is defined for weak-clusters.

Definition 4.5 *A one-floor, rooted* $\mathbf{KSD}$*-model (weak-cluster) $\mathcal{M} = (W, R, V)$ is called a* minimal *model for the set of formulas I if $\mathcal{M} \Vdash I$ and for every preferred model $\mathcal{M}'$ we have $\mathcal{M}' \nVdash I$.*

From Theorem 4.4 it follows that a *cluster* $\mathcal{M}$ is minimal for the set of formulas I in the usual sense iff $\mathcal{M}$ is minimal for the set of formulas I in a sense of Definition 4.5. The following result linking weak-expansions with minimal models is the main theorem of the paper. Throughout the proof Cn_L will stand for $Cn_{\mathbf{wK4Df}}$.

Theorem 4.6 *Let $\mathcal{M} = (W, R, V)$ be a weak-cluster and $T = \{\phi \mid \mathcal{M} \Vdash \phi\}$. T is a weak-expansion for the set of formulas I iff $\mathcal{M}$ is a minimal model for I.*

Proof. Assume the hypothesis of the theorem and suppose that $\mathcal{M} \Vdash I$. Clearly $\mathcal{M} \Vdash \{\neg\Box\phi | \neg\Box\phi \in T\}$. For any $\phi \notin T$, $\mathcal{M} \nVdash \phi$, so for some $w \in W$, $w \Vdash \neg\phi$ and since $\mathcal{M}$ is a weak-cluster, it follows that $\mathcal{M} \Vdash \neg\Box\phi \vee \neg\phi$ and so $\mathcal{M} \Vdash \{\neg\Box\phi \vee \neg\phi | \phi \notin T\}$. Now consider any two-floor model $\mathcal{N} = (N, S, U)$ of I whose second-floor coincides with $\mathcal{M}$, and choose any element x on the first-floor. For any $\neg\Box\phi \in T$, ϕ is false at some point v in $\mathcal{M}$ and since xSv, $x \nVdash \Box\phi$ and so $\mathcal{N} \Vdash \neg\Box\phi$. For any $\phi \notin T$, $\neg\phi$ holds at some point u in $\mathcal{M}$ and, since xSu, $x \Vdash \neg\Box\phi$. Therefore both models $\mathcal{M}$ and $\mathcal{N}$ satisfy the set of formulas $X = I \cup \{\neg\Box\phi \vee \neg\phi | \phi \notin T\} \cup \{\neg\Box\phi | \neg\Box\phi \in T\}$ and therefore also $Cn_L(X)$, and so $Cn_L(X) \subseteq T$. Now if T is a weak-expansion for I, then $T = Cn_L(X)$, consequently $\mathcal{N}$ is not preferred to $\mathcal{M}$ and so $\mathcal{M}$ is a minimal model for I.

For the other direction, let $\mathcal{M}$ be a minimal model for I. With X as above we have already shown that $Cn_L(X) \subseteq T$. We now show that every $\mathbf{wK4Df}$-model which validates $Cn_L(X)$ also validates T. This by Theorem 3.2 will imply that $Cn_L(X) \vdash T$ in $\mathbf{wK4Df}$ so $T \subseteq Cn_L(X)$. Now let us prove the assumption. Suppose for some $\mathbf{wK4Df}$-model $\mathcal{N} = (N, S, U)$ we have $\mathcal{N} \Vdash Cn_L(X)$. We have two subcases:

Case 1 $\mathcal{N}$ is a one-floor frame. Assume $\phi \in T$. Then $\Box\phi \in T$. This implies that $\mathcal{M} \Vdash \Box\phi$ and since $\mathcal{M}$ is a weak-cluster and $\mathcal{M}$ is not one irreflexive point we have $\mathcal{M} \Vdash \Diamond\Box\phi$. This implies that $\Diamond\Box\phi \in T$. Now as $\Diamond\Box\phi \equiv \neg\Box\neg\Box\phi$, by the definition of weak-expansions we conclude that $\Diamond\Box\phi \in Cn_L(X)$. This implies that $\mathcal{N} \Vdash \Diamond\Box\phi$. Now because $\mathcal{N}$ is a one-floor frame it is symmetric, yielding $N \Vdash \phi$.

Case 2 $\mathcal{N}$ is a two-floor $\mathbf{wK4Df}$-model. Let us denote the floors by N_1 and N_2 respectively. Let $\mathcal{N}_\mathcal{M}$ be the model obtained by substituting $\mathcal{M}$ in $\mathcal{N}$ instead of the second floor N_2. Let us show by induction on the complexity of formulas that for every formula ϕ and for every point $v \in N_1$ we have $\mathcal{N}_\mathcal{M}, v \Vdash \phi$ iff $\mathcal{N}, v \Vdash \phi$. The only non-trivial case is when the formula is of the form $\Box\phi$. Suppose for some $v \in N_1$, $\mathcal{N}, v \Vdash \Box\phi$. This means that for every $v' \in \mathcal{N}$ if $v \neq v'$ then $v' \Vdash \phi$. Besides by the induction hypothesis it means that for every $v' \in N_1$ such that $v \neq v'$, $\mathcal{N}_\mathcal{M}, v' \Vdash \phi$. Assume for the contradiction that $\mathcal{N}_\mathcal{M}, v \nVdash \Box\phi$. This means there is a point $w \in \mathcal{M}$ with $w \nVdash \phi$. That $\mathcal{M} \nVdash \phi$ means that $\phi \notin T$ and hence $\Diamond\neg\phi \vee \neg\phi \in Cn_L(X)$. So we obtain $\mathcal{N} \Vdash \Diamond\neg\phi \vee \neg\phi$. But this yields a contradiction as for every $u \in N_2$ we have that $u \Vdash \phi$ so no point in N_2 satisfies the formula $\Diamond\neg\phi \vee \neg\phi$.

Conversely assume for every $v \in N_1, \mathcal{N}_\mathcal{M}, v \Vdash \Box\phi$. This implies that $\mathcal{M} \Vdash \phi$ and $\mathcal{M} \Vdash \Box\phi$ as $\mathcal{M}$ is a final weak-cluster. Therefore $\mathcal{M} \Vdash \Diamond\Box\phi$ and hence $\Diamond\Box\phi \in T$. So $\Diamond\Box\phi \in Cn_L(X)$. Hence $\mathcal{N} \Vdash \Diamond\Box\phi$. This implies that for every $u \in N_2$, $u \Vdash \Diamond\Box\phi$ hence $u \Vdash \phi$. As for the points in N_1, by the induction hypothesis $v' \neq v$ implies that $v' \Vdash \phi$. So we get that $\mathcal{N}, v \Vdash \Box\phi$.

From what we have already proved we see that $\mathcal{N}_\mathcal{M} \Vdash I$ and as $\mathcal{M}$ is I-minimal we have that $\mathcal{N}_\mathcal{M}$ is not preferred over $\mathcal{M}$ which by Definition 4.3 implies that $\mathcal{N}_\mathcal{M} \Vdash T$. Now applying again the fact that the two models validate the same formulas on the first floors, we get that $\mathcal{N}, v \Vdash T$ for every $v \in N_1$. Now let us take an arbitrary $\phi \in T$. As T is closed under consequence, $\Box\phi \in T$ so for every point $v \in N_1$ we have $\mathcal{N}, v \Vdash \Box\phi$, hence for every $u \in N_2, \mathcal{N}, u \Vdash \phi$. This means that $\mathcal{N} \Vdash \phi$. Hence $\mathcal{N} \Vdash T$. $\dashv$

Weak-expansions may exist where no corresponding expansion does, as the following example shows.

Example 4.7 *Let $I = \{\Box p \vee \Box q\}$. It is easy to check that I has no autoepistemic expansions nor any* **wK4Df** *expansions. However let $\mathcal{M} = (W, R, V)$ be the* **KSD**-*model where $W = \{w, v\}$, $R = \{(w, v), (v, w)\}$ and $w \Vdash p \wedge \neg q, v \Vdash q \wedge \neg p$. Then $\mathcal{M}$ is minimal for I since $\mathcal{M} \Vdash I$ and for every preferred model at any point on the first floor both $\Box p$ and $\Box q$ (hence $\Box p \vee \Box q$) are falsified.*

This illustrates how weak-expansions need not contain instances of the **T** axiom $\Box p \rightarrow p$. Note that I is an example of what [Halpern and Moses, 1985] calls a dishonest formula.

4.1 Relations to other logics

Though weak-expansions may exist where expansions do not, Theorem 4.4 implies the following simple relationship. Let $\mid\!\sim_L$ be the usual (skeptical) nonmonotonic inference relation for a modal logic L based on expansions, ie $\mid\!\sim_{wK4Df}$ corresponds to nonmonotonic **wK4Df**. Let $\mid\!\approx_{wK4Df}$ be the inference relation based on weak-expansions defined earlier.

Corollary 4.8 *For any formula ϕ and set of formulas I such that $I \mid\!\sim_{wK4Df} \phi$ is defined, $I \mid\!\approx_{wK4Df} \phi \Rightarrow I \mid\!\sim_{wK4Df} \phi$.*

This relation is the expected one for cautious inference, since it states that reasoning with doxastic (**KSD**) models is more skeptical than reasoning with epistemic (**S5**) models. The $\mid\!\sim_{wK4Df}$ relation coincides with $\mid\!\sim_{wK4f}$ examined in [Pearce and Uridia, 2010]. From the study of minimal models carried out there it is easy to see that **wK4f**-expansions are also **KD45**-expansions and **S4F**-expansions too. It follows that in general skeptical inference in **wK4Df** is at least as strong as in autoepistemic logic or the logic of minimal knowledge of [Schwarz and Truszczyński, 1992]. A comparison of these logics with $\mid\!\approx_{wK4Df}$ is a topic for future work.

Lastly, let us mention that by the so-called splitting translation **S5** can be embedded in **KSD**. This translation, call it t, replaces each formula of the form $\Box p$ by $(p \wedge \Box p)$ and yields: $I \vdash_{S5} \phi \Leftrightarrow t(I) \vdash_{KSD} t(\phi)$. This is close to the idea of knowledge as a form of justified true belief.

5 Conclusion

We have presented a new approach to minimal belief and reasoning in nonmonotonic doxastic logic in which **S5** epistemic models are replaced by **KSD** models and a weaker notion of expansion is employed. **KSD** does not validate the **T** axiom and at least on one reading can be seen as a logic of objective and fair belief. We hope to provide a more detailed analysis of the $\mid\!\approx_L$ relation in future work.

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