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Syntactic Computation of Hybrid Possibilistic Conditioning under Uncertain Inputs

Salem Benferhat
CRIL, Université d'Artois
Faculté Jean Perrin, France
benferhat@cril.univ-artois.fr

Célia da Costa Pereira
Université de Nice Sophia
Antipolis, Lab. I3S, France
celia.pereira@unice.fr

Andrea G. B. Tettamanzi
Université de Nice Sophia
Antipolis, Lab. I3S, France
andrea.tettamanzi@unice.fr

Abstract

We extend hybrid possibilistic conditioning to deal with inputs consisting of a set of triplets composed of propositional formulas, the level at which the formulas should be accepted, and the way in which their models should be revised. We characterize such conditioning using elementary operations on possibility distributions. We then solve a difficult issue that concerns the syntactic computation of the revision of possibilistic knowledge bases, made of weighted formulas, using hybrid conditioning. An important result is that there is no extra computational cost in using hybrid possibilistic conditioning and in particular the size of the revised possibilistic base is polynomial with respect to the size of the initial base and the input.

1 Introduction

In possibility theory, in presence of a new sure piece of information, there are two different ways to define the conditioning depending on how possibility degrees are interpreted: either we fully use the numerical unit interval [0,1] or we only use a relative total preordering induced by the unit interval [0,1]. This distinction leads to two different frameworks called quantitative (or product-based) framework and qualitative (or min-based) framework.

A possibilistic counterpart of Jeffrey's rule for cases where new information is uncertain has been proposed and used in [Dubois and Prade, 1997; 1993]. In [Benferhat *et al.*, 2011], an analysis of the uniqueness and the existence of the solution has been performed.

At the syntactic level, when the input is a single weighted formula (ϕ, a), such a form of revision comes down to adding ϕ to a knowledge base Σ at the prescribed level a. What makes the problem difficult is that the knowledge base must be modified so that the added formula, ϕ, maintains its prescribed priority, a, that is, it is neither implicitly inhibited by higher-priority formulas that contradict it, nor pushed to higher priority levels by formulas that imply it. A first step towards an efficient way for doing this when the input is a single weighted formula has been proposed in [Benferhat *et al.*, 2002].

Most possibilistic-based revision approaches are based on homogeneous operators for conditioning (the same operator is used to revise the possibility distribution on the models of all elements of the input). However, since the conditioning of the models of each component of the inputs proceeds independently of and in parallel with the conditioning of the models of the other components, one might as well decide to use different operators on different components, as proposed in [Benferhat *et al.*, 2012] in a belief-revision setting, thus obtaining a family of several hybrid operators.

The ability to perform hybrid revision might prove useful, for example, in fault diagnosis, where possibly conflicting information from different sensors, requiring different treatments, must be combined simultaneously.

Despite the important role played by possibilistic Jeffrey's rule in belief revision (e.g. [Dubois *et al.*, 1998; Benferhat *et al.*, 2010a]), no syntactic equivalent to the possibilistic counterpart of Jeffrey's rule has been given yet in the literature, where the input consists of more than a single weighted formula. Here, we bridge this gap, by proposing a syntactic characterization of a set of basic operations that may be applied on possibility distributions. These operations will be used to define a syntactic computation of hybrid conditioning that revises possibilistic knowledge bases. Our approach generalizes possibilistic counterparts of Jeffrey's rule of conditioning in the sense that different forms of conditioning may be applied on different components of the input.

Performing conditioning at the syntactic level has clear advantages, since performing revision at the semantic level has a higher computational cost, as the cardinality of the set of interpretations grows exponentially with the number of propositional atoms, and even impossible if the language adopted has an infinite number of interpretations (consider, e.g., First-Order Logic or expressive Description Logics).

The paper is organized as follows: after recalling the main notions of possibilistic logic in Section 2 and presenting the two different types of conditioning in possibility theory in Section 3, Section 4 introduces the concept of generalized conditioning operators. Section 5 presents a set of basic elementary operations on possibility distributions which are then used in Section 6 to define syntactic computations of hybrid conditioning. Section 7 concludes the paper.

2 Possibilistic Logic: A refresher

Let us consider a finite set of propositional variables V and a propositional language PL_V built from $V \cup \{\top, \bot\}$ and the connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ in the usual way. Formulas, i.e., elements of PL_V are denoted by Greek letters.

Let Ω be the finite set of interpretations on V. We denote by $[\phi]$ the set of models of ϕ.

2.1 Possibility Distributions and Possibility Measures

A *possibility distribution* is a mapping from the set of interpretations Ω to the interval [0,1]. A possibility distribution π represents the available knowledge about what the real world is. By convention, $\pi(\omega) = 1$ means that it is totally possible for ω to be the real world, $1 > \pi(\omega) > 0$ means that ω is only somehow possible, while $\pi(\omega) = 0$ means that ω is certainly not the real world. When $\pi(\omega_1) > \pi(\omega_2)$, ω_1 is a preferred candidate to ω_2 for being the real state of the world. π is said to be *normalized* if $\exists \omega \in \Omega$, such that $\pi(\omega) = 1$.

A possibility distribution π induces two mappings grading respectively the possibility and the certainty of a formula:

- The possibility measure $\Pi(\phi) = \max\{\pi(\omega) : \omega \models \phi\}$ which evaluates to what extent ϕ is consistent with the available knowledge expressed by π.
- The necessity measure: $N(\phi) = 1 - \Pi(\neg\phi)$ which evaluates to what extent ϕ is entailed by the knowledge expressed by π.

2.2 Possibilistic Knowledge Bases

A possibilistic knowledge base is a finite set of weighted formulas $\Sigma = \{(\phi_i, a_i), i = 1, \ldots, n\}$, where a_i is understood as a lower bound of the degree of necessity $N(\phi_i)$ (i.e., $N(\phi_i) \geq a_i$). Formulas with $a_i = 0$ are not explicitly represented in the knowledge base, i.e., only knowledge which is somewhat accepted by the agent is explicitly represented. The higher the weight, the more certain the formula.

Definition 1 *Let Σ be a possibilistic base, and a∈[0,1]. We call the a-cut of Σ (resp. strict a-cut), denoted by $\Sigma_{\geq a}$ (resp. $\Sigma_{>a}$), the set of classical formulas in Σ having a certainty degree at least equal to (resp. strictly greater than) a.*

A possibilistic knowledge base Σ is said to be consistent if the classical knowledge base, obtained by forgetting the weights, is classically consistent. Each inconsistent possibilistic base is associated with a level of inconsistency in the following way:

Definition 2 *Let Σ be a possibilistic knowledge base. The inconsistency degree of Σ is:* $\mathrm{Inc}(\Sigma) = \max\{a : \Sigma_{\geq a} \equiv \bot\}$, *with* $\max(\emptyset) = 0$.

2.3 From Syntactic to Semantic Representation

Possibilistic knowledge bases are compact representations of possibility distributions. Given a possibilistic knowledge base Σ, we can generate a possibility distribution from Σ by associating to each interpretation, its level of compatibility with the agent's knowledge, namely with Σ, as explained in [Dubois *et al.*, 1994].

Definition 3 *: The least specific possibility distribution associated to knowledge base Σ is defined, for all $\omega \in \Omega$, by:*

$$\pi_\Sigma(\omega) = 1 - \max\{a_i : (\phi_i, a_i) \in \Sigma \text{ and } \omega \notin [\phi_i]\},$$

with $\max(\emptyset) = 0$.

Example 1 *Let $\Sigma = \{(q, 0.3), (q \vee r, 0.5)\}$. Then:*

$\omega \longrightarrow$	qr	$q\neg r$	$\neg qr$	$\neg q \neg r$
$\pi_\Sigma(\omega)$	*1*	*1*	*0.7*	*0.5*

The two interpretations qr and $q\neg r$ are the preferred ones since they are the only ones which are consistent with Σ, and $\neg qr$ is preferred to $\neg q\neg r$, since the highest knowledge falsified by $\neg qr$ (i.e. $(q, 0.3)$) is less certain than the highest knowledge falsified by $\neg q\neg r$ (i.e., $(q \vee r, 0.5)$).

The possibility distribution π_Σ is not necessarily normal; it is normal iff Σ is consistent.

3 Conditioning in Possibility Theory

3.1 Conditioning under Certain Input

Two different types of conditioning [Dubois and Prade, 1998] have been defined in possibility theory for revising a possibility distribution π by a new and certain formula ϕ (when $\Pi(\phi) > 0$):

- In an ordinal setting, we have:

$$\pi(\omega \mid_m \phi) = \begin{cases} 1, & \text{if } \pi(\omega) = \Pi(\phi) \text{ and } \omega \models \phi, \\ \pi(\omega), & \text{if } \pi(\omega) < \Pi(\phi) \text{ and } \omega \models \phi, \\ 0, & \text{if } \omega \notin [\phi]. \end{cases}$$

  This is the definition of *minimum-based conditioning.*

- In a numerical setting, we get:

$$\pi(\omega \mid_. \phi) = \begin{cases} \frac{\pi(\omega)}{\Pi(\phi)}, & \text{if } \omega \models \phi, \\ 0, & \text{otherwise.} \end{cases}$$

  This is the definition of *product-based conditioning*, which is also known as Dempster rule of conditioning [Shafer, 1976].

These two definitions of conditioning satisfy an equation of the form $\forall \omega, \pi(\omega) = \pi(\omega \mid \phi) \diamond \Pi(\phi)$, which is similar to Bayesian conditioning, where $\diamond$ is the minimum or the product, respectively. The rule based on the product is much closer to genuine Bayesian conditioning than the qualitative conditioning defined from the minimum which is purely based on the comparison of levels. Besides, when $\Pi(\phi) = 0, \pi(\omega \mid_m \phi) = \pi(\omega \mid_. \phi) = 1, \forall \omega$, by convention (see also [Coletti and Vantaggi, 2009] for an analysis of this particular case in a coherent possibility theory setting).

3.2 Conditioning under Uncertain Input

Jeffrey's rule [Jeffrey, 1965] allows revising a probability distribution p into p' given the uncertainty bearing on a set of mutually exclusive and exhaustive events $\mu = \{(\lambda_i, a_i), i = 1, \ldots, n\}$. In [Dubois and Prade, 1997] a possibilistic counterpart of Jeffrey's rule has been defined. Given the initial beliefs encoded by a possibility distribution π and the uncertainty bearing on an exhaustive and mutually exclusive set of events λ_i in the form of (λ_i, a_i) such that $\Pi'(\lambda_i) = a_i$, then the revised possibility distribution π' according to Jeffrey's rule must satisfy the following conditions:

Property 1 (Input preservation condition) $\forall \lambda_i, \Pi'(\lambda_i) = a_i$.

Property 2 (Conditioning preservation condition) $\forall \lambda_i \subset \Omega$, $\forall \phi \subseteq \Omega, \Pi'(\phi|\lambda_i) = \Pi(\phi|\lambda_i)$.

Property 1 states that, after the revision operation, the a posteriori possibility of each event λ_i must be equal to a_i (namely, $\Pi'(\lambda_i) = a_i$). Hence, this way for specifying the uncertainty relative to the uncertain evidence expresses the fact that the uncertain evidence is seen as a constraint or an effect once the new evidence is accepted [Chan and Darwiche, 2005]. Property 2 states that the conditional possibility of any event ϕ given any uncertain event λ_i remains the same in the original and the revised distributions.

In [Dubois and Prade, 1997], two possibilistic counterparts of Jeffrey's rules have been defined. More precisely, revising with μ can be achieved using the following definition:

$$\forall (\lambda_i, a_i) \in \mu, \forall \omega \models \lambda_i, \pi(\omega \mid \mu) = a_i \diamond \pi(\omega \mid_\diamond \lambda_i)$$

where $\diamond$ is either the minimum or the product, depending on whether conditioning is based on the product or the minimum operator. From the above definitions, it is clear that the new ranking on models of ϕ is simply obtained using conditioning with a sure input. Note that, if $\diamond$ = product, both Properties 1 and 2 are satisfied. When $\diamond = \min$, instead, only Property 1 is satisfied [Benferhat *et al.*, 2011].

4 Generalized Conditioning Operators

We now define a family of generalized conditioning operators that take an uncertain input of the form $\mu = \{(\lambda_i, a_i, \diamond_i), i = 1, \dots, n\}$, where the propositional formulas λ_i induce a partition[1] of the set of interpretations, a_i is the possibility of formula λ_i, and $\diamond_i$ indicates the conditioning operator to be applied to the models of λ_i.

Hybrid conditioning with uncertain input $\mu = \{(\lambda_i, a_i, \diamond_i), i = 1, \dots, n\}$ proceeds to parallel changes on each models of λ_i. Here, the $(\lambda_i, a_i, \diamond_i)$'s are interpreted as constraints to be satisfied by the revised possibility distribution, namely, the revised possibility distribution denoted $\pi(\cdot \mid_h \mu)$ should be such that, for all i, $\Pi(\lambda_i \mid_h \mu) = a_i$.

Definition 4 *Let π be a possibility distribution. Let $\mu = \{(\lambda_i, a_i, \diamond_i), i = 1, \dots, n\}$ be an uncertain input, with $\max(a_i) = 1$. We assume that $\forall \lambda_i, \Pi(\lambda_i) > 0$. We define hybrid conditioning of π by μ, denoted by $\pi(\cdot \mid_h \mu)$ as follows: $\forall \lambda_i \in \mu$ and $\forall \omega \models \lambda_i$,*

- *if $\diamond_i = \min$,*

$$\pi(\omega \mid_h \mu) = \begin{cases} a_i, & \pi(\omega) = \Pi(\lambda_i) \text{ or } \pi(\omega) \geq a_i; \\ \pi(\omega) & \text{otherwise}; \end{cases}$$

- *if $\diamond_i = $ product, $\pi(\omega \mid_h \mu) = \frac{\pi(\omega)}{\Pi(\lambda_i)} \cdot a_i$.*

The symbol $\diamond_i$ indicates the way hybrid conditioning should be defined. Hybrid here is given the sense that models of λ_i and λ_j, with $i \neq j$, may use different forms of conditioning. For instance, models of λ_i may be revised with min-based conditioning, whereas models of λ_j are revised with product-based conditioning.

One can check that Property 1 (Section 3.2) is satisfied whatever the $\diamond_i$ used. Property 2 (Section 3.2), however, is only satisfied when $\forall i$, $\diamond_i$ = product-based conditioning. When at least one of the $\diamond_i$ is min-based conditioning, one may reuse the same counterexamples provided in [Benferhat *et al.*, 2011].

Clearly, hybrid conditioning generalizes possibilistic Jeffrey's rule of conditioning where *all* λ_i are revised using the same form of conditioning; with respect to this, the symbols $\diamond_i$ may be regarded as parameters that indicate the way the models of λ_i should be revised. Standard homogeneous conditioning (recalled in Section 3) is obtained as a special case when $\diamond_i = \diamond_j$ for all i, j. Likewise, the hybrid belief change operators studied in [Benferhat *et al.*, 2012] are obtained as special cases when $\mu = \{(\lambda, 1, \diamond_1), (\neg\lambda, a, \diamond_2)\}$, where $\diamond_1 \neq \diamond_2$.

Example 2 *Let us consider the possibility distribution π_Σ of Example 1 and let us revise it for the uncertain input $\mu = \{(q \wedge r, 0.8, \min), (q \wedge \neg r, 0.4, \min), (\neg q, 1, \cdot)\}$.*

Definition 4 yields the new possibility degree associated with each interpretation $\omega \in \Omega$:

- *Interpretation qr is the only model of $q \wedge r$, hence we get $\pi(qr \mid_h \mu) = 0.8$ (first item of Definition 4).*
- *Interpretation $q\neg r$ is the only model of $q \wedge \neg r$, hence we get $\pi(q\neg r \mid_h \mu) = 0.4$ (again using the first item of Definition 4).*
- *Interpretation $\neg qr$ is one of the models of $\neg q$. We have from π_Σ of Example 1 $\Pi(\neg q) = 0.7$. Using item 2 of Definition 4, we get $\pi(\neg qr \mid_h \mu) = \frac{0.7}{0.7} \cdot 1 = 1$.*
- *Interpretation $\neg q\neg r$ is one of the models of $\neg q$. We have from π_Σ of Example 1 $\Pi(\neg q) = 0.7$. Using item 2 of Definition 4, we get $\pi(\neg q\neg r \mid_h \mu) = \frac{0.5}{0.7} \cdot 1 = 0.7143$.*

Hence, computing $\pi(\omega \mid_h \mu)$ according to Definition 4 yields:

$\omega \longrightarrow$	qr	$q\neg r$	$\neg qr$	$\neg q\neg r$
$\pi(\omega \mid_h \mu)$	*0.8*	*0.4*	*1*	*0.7143*

One might wonder whether the same effect could not be obtained by first applying the min-based conditioning on an input $\mu_{\min}$ consisting of only the triples with $\diamond_i = \min$ and then applying the product-based conditioning on an input μ_{product} consisting of only the triples with $\diamond_i = \text{product}$. The problem is that neither $\mu_{\min}$ nor μ_{product} would be a partition; they would have to be completed by arbitrarily setting the possibility degree of the remaining models, thus yielding, in general, a different result.

5 Elementary Operations on Possibility Distributions

We now introduce a set of elementary operations on possibility distributions, summarized in Table 1, which will be used to characterize hybrid conditioning under uncertain input. For

[1] In fact, this restriction may be relaxed by weakening Property 1 so that $\forall \lambda_i$, $\Pi'(\lambda_i) \leq a_i$ (a_i's may reflect a reliability of the sources). This corresponds to a different view of conditioning than the one we adopt here.

each elementary operation given in Table 1 we will later provide its syntactic counterpart when possibility distributions are compactly represented by possibilistic knowledge bases. This will help us define the syntactic computation of hybrid possibilistic conditioning.

Unnormalized conditioning simply declares countermodels of ϕ as impossible while preserving initial possibility degrees of models of ϕ. It is called *unnormalized* because $\max\{\pi_\phi(\omega) : \omega \in \Omega\}$ may be different of 1.

a**-Discounting** decreases the possibility degrees of some (or all) interpretations. The degree $a \in [0, 1]$ can be viewed as a reliability degree of the source that provides the possibility distribution. There are two different ways to discount a possibility distribution:

- (Proportional a-discounting) either we proportionally shift down all interpretations, namely we replacie a possibility degree $\pi(\omega)$ of each interpretation ω by $a \cdot \pi(\omega)$, or
- (Threshold a-discounting) we shift down only interpretations that are greater than a. Namely, Threshold a-discounting consists of viewing the degree a as threshold, where any interpretation having a possibility degree greater than a should be discounted to a.

a**-Enhancing** is the dual operation to a-discounting. It increases the possibility degrees of some or all interpretations. The degree a here means that all possible events have at least a degree equal to a. Again, there are two different ways to enhance a possibility distribution:

- Proportional a-enhancing, which replaces a possibility degree $\pi(\omega)$ of each interpretation ω by $\max\{1, \pi(\omega)/a\}$ with the constraint that $\pi(\omega)/a \leq 1$ (here, it is assumed that $a > 0$).
- Threshold a-enhancing, which leaves π unchanged if there exists at least an interpretation having a degree greater than a. Otherwise, it only enhances the best interpretations to a.

5.1 Characterizing Hybrid Conditioning

We begin by observing that Definition 4 may be viewed as a set of parallel changes to an initial possibility distribution induced by each element of the input:

Definition 5 *Let π be a possibility distribution. Let $\mu = \{(\lambda_i, a_i, \diamond_i), i = 1, \dots, n\}$ be an uncertain input. Let us define, for each $(\lambda_i, a_i, \diamond_i)$, a possibility distribution π_{λ_i} as follows: $\forall \omega \in \Omega$,*

$$\pi_{\lambda_i}(\omega) = \begin{cases} \pi(\omega|_{\diamond_i}\mu), & \text{if } \omega \models \lambda_i, \\ 0, & \text{otherwise,} \end{cases}$$

where $\diamond_i$ is either a minimum or a product operator.

Example 3 *The following table gives the possibility distributions associated with the input considered in Example 2.*

	qr	$q\neg r$	$\neg qr$	$\neg q \neg r$
$\pi_{q\wedge r}$	0.8	0	0	0
$\pi_{q\wedge \neg r}$	0	0.4	0	0
$\pi_{\neg q}$	0	0	1	0.7143
$\max\{\pi_{q\wedge r}, \pi_{q\wedge\neg r}, \pi_{\neg q}\}$	0.8	0.4	1	0.7143

The following proposition shows that hybrid conditioning with uncertain input μ can be viewed as a disjunctive combination of possibility distributions associated with each element of μ:

Proposition 1 *Let $\mu = \{(\lambda_i, a_i, \diamond_i), i = 1, \dots, n\}$ be an uncertain input. Let π_{λ_i} be a possibility distribution given by Definition 5. Then, $\forall \omega \in \Omega$, $\pi(\omega \mid_h \mu) = \max_{(\lambda_i, a_i, \diamond_i) \in \mu} \pi_{\lambda_i}(\omega)$, where $\pi(\cdot \mid_h \mu)$ is given in Definition 4.*

Example 4 *One can easily check that, in Example 3,*

$$\pi(\omega \mid_h \mu) = \max\{\pi_{q\wedge r}(\omega), \pi_{q\wedge\neg r}(\omega), \pi_{\neg q}(\omega)\},$$

where $\pi(\omega \mid_h \mu)$ was computed in Example 2.

The following two propositions give the characterisation of π_{λ_i} using elementary operations. The two cases considered in Definition 4 are handled separately.

Proposition 2 *Let π be an initial possibility distribution. Let $(\lambda_i, a_i, \diamond_i)$ be an element of the input μ, where $\diamond_i = \min$. We denote by π' the result of modifying π using successively the following three steps :*

1. *apply unnormalized conditioning of π on λ_i;*
2. *apply threshold a_i-discounting;*
3. *apply best a_i-enhancing;*

Then, $\forall \omega \in \Omega$, $\pi'(\omega) = \pi_{\lambda_i}(\omega)$, where π_{λ_i} is a possibility distribution given by Definition 5.

Intuitively, when $\diamond_i = \min$, applying subnormalized conditioning allows to declare all countermodels of λ_i as impossible. Of course, they may be declared as possible by other π_{λ_j}'s, with $j \neq i$ (recall that π_{λ_j}'s are combined using the maximum operation; see Proposition 1). In a case where $\Pi(\lambda_i) \geq a_i$, applying a_i-threshold discounting yields $\Pi_{\lambda_i}(\lambda_i) = a_i$. In this case, applying best-a_i-enhancing has no effect. Clearly, we recover the first item of Definition 4. the semantic definition of hybrid conditioning. Now, in a case where $\Pi(\lambda_i) < a_i$, applying a_i-threshold discounting has no effect, since no model in $[\lambda_i]$ has a degree greater than a_i. However, applying best-a_i-enhancing allows to promote the best models of λ_i to have a possibility degree equal to a_i. Again, we recover the first item of Definition 4.

A similar result holds for product-based conditioning:

Proposition 3 *Let π be an initial possibility distribution. Let $(\lambda_i, a_i, \diamond_i)$ be an element of the input μ, where $\diamond_i =$ product. We denote by π' the result of modifying π using successively the following three steps :*

1. *apply unnormalized conditioning of π on λ_i;*
2. *apply proportional a_i-discounting;*
3. *apply proportional $\Pi(\lambda_i)$-enhancing.*

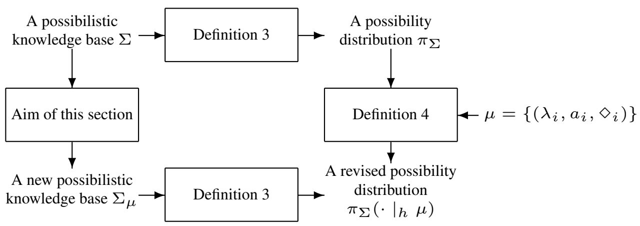
Then, $\forall \omega \in \Omega$, $\pi'(\omega) = \pi_{\lambda_i}(\omega)$, where π_{λ_i} is a possibility distribution given by Definition 5.

Table 1: Elementary operations on a possibility distribution π.

Elementary operation on π	Result of modifying π
Unnormalized conditioning	$\pi_\phi(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi; \\ 0 & \text{otherwise.} \end{cases}$
Proportional a-discounting	$\mathrm{PD}(\pi(\omega), a) = \pi(\omega) \cdot a$
Threshold a-discounting	$\mathrm{TD}(\pi(\omega), a) = \begin{cases} \pi(\omega) & \text{if } \pi(\omega) \le a; \\ a & \text{otherwise.} \end{cases}$
Proportional a-enhancing	$\mathrm{PE}(\pi(\omega), a) = \max(1, \pi(\omega)/a)$
Best threshold a-enhancing	$\mathrm{TE}(\pi(\omega), a) = \begin{cases} \pi(\omega), & \text{if } \pi(\omega) < h(\pi); \\ \max(a, h(\pi)), & \text{otherwise.} \end{cases}$

6 Syntactic Hybrid Conditioning

The following diagram summarizes the aim of this section:



The syntactic computation of hybrid conditioning is decomposed in three steps:

1. Provide the syntactic counterpart of each elementary operation on possibility distributions given in Table 1.
2. Provide the syntactic counterpart of each π_{λ_i} associated to each element $(\lambda_i, a_i, \diamond_i)$ of the input.
3. Use the syntactic counterpart of Propositions 1, 2, and 3 to give syntactic counterpart of hybrid conditioning.

6.1 Syntactic Elementary Operations

We now provide the syntactic counterparts of the elementary operations on possibility distributions given in Table 1.

Such syntactic characterizations are obtained in polynomial size of original bases. The results of this section will be used to define the syntactic counterpart of hybrid possibilistic conditioning under uncertain inputs given by Definition 4.

Table 2 summarizes these syntactic elementary operations, where Σ is a possibilistic knowledge base, π_Σ is its associated possibility distribution according to Definition 3, ϕ is a propositional formula, and $h(\pi_\Sigma) = 1 - \max\{b : \Sigma_{>b}$ is consistent$\}$.

The following proposition establishes the equivalence between the operations of Table 2 and those of Table 1.

Proposition 4 *Let Σ be a possibilistic knowledge and π_Σ its associated possibility distribution. Let π_E and Σ_E be a result of modifying π and Σ using an elementary operation E given in Tables 1 and 2 respectively. Then, $\forall \omega \in \Omega$, we have:*

$$\pi_{\Sigma_E}(\omega) = \pi_E(\omega),$$

where π_{Σ_E} is the possibility distribution associated with Σ_E.

For lack of space, we only give the proof of proportional a-discounting and threshold a-discounting.

- The proof of the syntactic counterpart of proportional a-discounting is as follows. Let ω be an interpretation.

  $\pi_{\Sigma_{\mathrm{PD}(a)}}(\omega) =$
  $= 1 - \max(1 - a, \max\{1 - (1 - b_i) \cdot a : (\phi_i, b_i) \in \Sigma$ and $\omega \not\models \phi_i\})$
  $= \min(a, 1 - \max\{1 - (1 - b_i) \cdot a : (\phi_i, b_i) \in \Sigma$ and $\omega \not\models \phi_i\})$
  $= \min(a, \min\{(1 - b_i) \cdot a : (\phi_i, b_i) \in \Sigma$ and $\omega \not\models \phi_i\})$
  $= \min(a, a \cdot \min\{(1 - b_i) : (\phi_i, b_i) \in \Sigma$ and $\omega \not\models \phi_i\})$
  $= \min(a, a \cdot \pi_\Sigma(\omega))$
  $= a \cdot \pi_\Sigma(\omega)$.

- To see the proof of threshold a-discounting, let us first decompose Σ into two subbases:

  $$\Sigma_1 = \{(\phi_i, b_i) : (\phi_i, b_i) \in \Sigma \text{ and } b_i > 1 - a\},$$
  $$\Sigma_2 = \{(\phi_i, b_i) : (\phi_i, b_i) \in \Sigma \text{ and } b_i \le 1 - a\}.$$

  Let ω be an interpretation. Assume that ω is a model of all formulas of Σ_1. Then it is easy to check that $\pi_\Sigma(\omega) \ge a$ and hence $\mathrm{TD}(\pi_\Sigma(\omega), a) = a$ by definition of Threshold a-discounting. Besides, using Definition 3, we also have $\pi_{\Sigma_{\mathrm{TD}(a)}}(\omega) = a$ (since ω trivially falsifies $(\bot, 1 - a)$), hence $\pi_{\Sigma_{\mathrm{TD}(a)}}(\omega) = \mathrm{TD}(\pi_\Sigma(\omega), a)$.
  Similarly, if ω falsifies some formulas of Σ_1, then clearly $\pi_\Sigma(\omega) = \pi_{\Sigma_1}(\omega) \le a$ hence by definition of Threshold a-discounting we have $\mathrm{TD}(\pi_\Sigma(\omega), a) = \pi_\Sigma(\omega)$. Besides one can easily check that $\pi_{\Sigma_{\mathrm{TD}(a)}}(\omega) = \pi_{\Sigma_1}(\omega)$. Hence, again we have $\pi_{\Sigma_{\mathrm{TD}(a)}}(\omega) = \mathrm{TD}(\pi_\Sigma(\omega), a)$.

6.2 Putting It All Together

We are now able to show how hybrid conditioning is performed (syntactically) on a possibilistic knowledge base.

Let Σ be a possibilistic knowledge base and π be its associated possibility distribution. Proposition 1 suggests that, in order to compute the possibilistic knowledge base Σ_μ associated with $\pi(\cdot \mid_h \mu)$, it is enough to first compute the possibilistic knowledge bases associated with each π_{λ_i}, then apply the syntactic counterpart of the disjunctive combination mode given in Proposition 5.

The syntactic counterpart of the disjunction of two possibility distributions, using maximum operator, has been studied in [Benferhat *et al.*, 2010b]:

Proposition 5 *Let Σ_1 and Σ_2 be two possibilistic bases. Let π_1 and π_2 be their associated possibility distributions. Then $\Sigma_\vee = \{(\phi_i \vee \psi_j, \min\{a_i, b_j\}) : (\phi_i, a_i) \in \Sigma_1, (\psi_j, b_j) \in \Sigma_2\}$ is associated with the disjunctive combination $\pi_\vee$ of π_1 and π_2.*

It is obvious that the disjunctive combination mode is associative and commutative, which means that these combination modes can be extended to more than two possibilistic

Table 2: syntactic counterpart of each elementary operation on possibility distribution given in Table 1.

Elementary operations	Result of modifying a possibilistic base Σ
Unnormalized conditioning	$\pi_{\Sigma_U^\phi}(\omega) = \pi(\omega \mid_U \phi)(\omega)$
Proportional a-discounting	$\Sigma_{\mathrm{PD}(a)} = \{(\phi_i, 1-(1-b_i)\cdot a) : (\phi_i, b_i) \in \Sigma\} \cup \{(\bot, 1-a)\}$
Threshold a-discounting	$\Sigma_{\mathrm{TD}(a)} = \{(\phi_i, b_i) : (\phi_i, b_i) \in \Sigma \text{ and } b_i > 1-a\}$ $\cup\{(\bot, 1-a)\}$.
Proportional a-enhancing	$\Sigma_{\mathrm{PE}(a)} = \{(\phi_i, 1-(1-b_i)/a) : (\phi_i, b_i) \in \Sigma \text{ and } (1-b_i)/a \leq 1\}$
Best threshold a-enhancing	$\Sigma_{\mathrm{TE}(a)} = \begin{cases} \Sigma & \text{if } h(\pi_\Sigma) \geq a \\ \{(\phi_i, 1-a) : (\phi_i, b_i) \in \Sigma \text{ and } b_i \leq 1-h(\pi_\Sigma)\} & \text{otherwise.} \\ \cup\{(\phi_i, b_i) : (\phi_i, b_i) \in \Sigma \text{ and } b_i > 1-h(\pi_\Sigma)\} & \end{cases}$

knowledge bases. In terms of complexity in space, the way $\Sigma_\vee$ is defined suggests that its size is the product of the sizes of Σ_1 and Σ_2. In fact, we can provide a linear representation of $\Sigma_\vee$ by introducing one new variable per possibilistic knowledge base. More precisely, let A_1 and A_2 be two new propositional symbols that are associated with Σ_1 and Σ_2 respectively. Then, we redefine the syntactic counterpart of disjunctive combination modes [Benferhat *et al.*, 2010b] as

$$\begin{aligned} \Sigma'_\vee \;=\; & \{(\neg A_1 \vee \phi_i, a_i) : (\phi_i, a_i) \in \Sigma_1\} \cup \\ & \{(\neg A_2 \vee \psi_j, b_j) : (\psi_j, b_i) \in \Sigma_2\} \cup \\ & \{(A_1 \vee A_2, 1)\}. \end{aligned}$$

Of course, this way of defining the disjunctive counterpart is more compact than the one given in Proposition 5, since the size of $\Sigma'_\vee$ is polynomial w.r.t. the sizes of Σ_1 and Σ_2. It is easy to show that $\pi_{\Sigma_\vee}(\omega) = \pi_{\Sigma'_\vee}(\omega)$, where the interpretation ω does not contain the new symbols A_1 and A_2.

To sum up, to compute the syntactic conditioning of a possibilistic base Σ under uncertain input $\mu = \{(\lambda_i, a_i, \diamond_i)\}$, we first compute each Σ_{λ_i} using the syntactic elementary operations, then we take their disjunction.

Example 5 *Let us continue with Example 2: recall that we are to revise knowledge base* $\Sigma = \{(q, 0.3), (q \vee r, 0.5)\}$ *of Example 1 for the uncertain input* $\mu = \{(q\wedge r, 0.8, \min), (q \wedge \neg r, 0.4, \min), (\neg q, 1, \cdot)\}$. *Now, we compute* Σ_μ *as follows:*

1. *Compute the unnormalized conditional bases* $\Sigma_U^{\lambda_i}$*:*
$$\begin{aligned} \Sigma_U^{q\wedge r} &= \{(q, 0.3), (q \vee r, 0.5), (q \wedge r, 1)\}, \\ \Sigma_U^{q\wedge\neg r} &= \{(q, 0.3), (q \vee r, 0.5), (q \wedge \neg r, 1)\}, \\ \Sigma_U^{\neg q} &= \{(q, 0.3), (q \vee r, 0.5), (\neg q, 1)\}. \end{aligned}$$
2. *Compute the discounted bases:*
$$\begin{aligned} \Sigma_{\mathrm{TD}(0.8)}^{q\wedge r} &= \{(q, 0.3), (q \vee r, 0.5), (q \wedge r, 1), (\bot, 0.2)\}, \\ \Sigma_{\mathrm{TD}(0.4)}^{q\wedge\neg r} &= \{(q \wedge \neg r, 1), (\bot, 0.6)\}, \\ \Sigma_{\mathrm{PD}(1)}^{\neg q} &= \{(q, 0.3), (q \vee r, 0.5), (\neg q, 1)\}. \end{aligned}$$
3. *Compute the enhanced bases:*
$$\begin{aligned} \Sigma_{\mathrm{TE}(0.8)}^{q\wedge r} &= \{(q, 0.3), (q \vee r, 0.5), (q \wedge r, 1), (\bot, 0.2)\}, \\ \Sigma_{\mathrm{TE}(0.4)}^{q\wedge\neg r} &= \{(q \wedge \neg r, 1), (\bot, 0.6)\}, \\ \Sigma_{\mathrm{PE}(0.7)}^{\neg q} &= \{(q \vee r, 0.2857), (\neg q, 1)\}. \end{aligned}$$
4. *Compute the disjunctive combination of the three enhanced bases: we begin by computing*
$$\begin{aligned} & \Sigma_{\mathrm{TE}(0.4)}^{q\wedge\neg r} +_{dm} \Sigma_{\mathrm{PE}(0.7)}^{\neg q} = \\ & \quad \{(q \vee r, 0.2857), (\neg q \vee \neg r, 1), (\neg q, 0.6)\}, \end{aligned}$$
because $(q \wedge \neg r) \vee q \vee r = q \vee r$ *and* $(q \wedge \neg r) \vee q = \neg q \vee \neg r$*; finally, we compute the disjunctive combination of the result with* $\Sigma_{\mathrm{TE}(0.8)}^{q\wedge r}$ *in tabular form as follows:*

	$(q \vee r, 0.2857)$	$(\neg q \vee \neg r, 1)$	$(\neg q, 0.6)$
$(q, 0.3)$	$(q \vee r, 0.2857)$	$(\top, 0.3)$	$(\top, 0.3)$
$(q \vee r, 0.5)$	$(q \vee r, 0.2857)$	$(\top, 0.5)$	$(\top, 0.5)$
$(q \wedge r, 1)$	$(q \vee r, 0.2857)$	$(\top, 1)$	$(\neg q \vee r, 0.6)$
$(\bot, 0.2)$	$(q \vee r, 0.2)$	$(\neg q \vee \neg r, 0.2)$	$(\neg q, 0.2)$

which, after simplification, yields

$$\Sigma_\mu = \{(q \vee r, 0.2857), (\neg q \vee r, 0.6), (\neg q, 0.2)\}.$$

It is now easy to verify that π_{Σ_μ}*, the possibility distribution corresponding to* Σ_μ*, is identical to* $\pi(\omega \mid_h \mu)$ *as computed in Example 2.*

7 Conclusion

We have studied syntactic computations of a new form of possibilistic conditioning where both min-based and product-based conditioning can be used for revising possibility distributions under uncertain inputs. Our hybrid conditioning extends the possibilistic counterparts of Jeffrey's rule [Dubois and Prade, 1993], heterogeneous conditioning [Benferhat *et al.*, 2012], and standard min-based and product-based conditioning [Dubois and Prade, 1998] in presence of a sure piece of information. We then proposed a syntactic characterization of a set of basic operations that may be applied on possibility distributions. These operations were then used to define a syntactic computation of hybrid conditioning that revises possibilistic knowledge bases. An important result is that the computational complexity is the same as the one of standard min-based and product-based conditioning. Hence, we enriched the definitions of possibilistic conditioning without extra computational cost. These results are very important for belief revision. Indeed, in [Benferhat *et al.*, 2010b] it has been proved, at the semantic level, that many revision approaches (such as adjustment [Williams, 1994], natural belief revision [Boutilier, 1993], drastic belief revision [Nayak, 1994], and/or revision of an epistemic state by another epistemic state) can be captured using possibilistic Jeffrey's rule. Syntactic counterparts of all these major revision operations studied in [Benferhat *et al.*, 2010b] can now have their syntactic counterparts using results of this paper.

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