Combining RCC5 Relations with Betweenness Information

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Abstract

RCC5 is an important and well-known calculus for representing and reasoning about mereological relations. Among many other applications, it is pivotal in the formalization of commonsense reasoning about natural categories. In particular, it allows for a qualitative representation of conceptual spaces in the sense of Gärdenfors. To further the role of RCC5 as a vehicle for conceptual reasoning, in this paper we combine RCC5 relations with information about betweenness of regions. The resulting calculus allows us to express, for instance, that some part (but not all) of region B is between regions A and C. We show how consistency can be decided in polynomial time for atomic networks, even when regions are required to be convex. From an application perspective, the ability to express betweenness information allows us to use RCC5 as a basis for interpolative reasoning, while the restriction to convex regions ensures that all consistent networks can be faithfully represented as a conceptual space.

1 Introduction

Many cognitive models of categorization rely on spatial representations of meaning in one way or another. In prototype theory [Rosch, 1973], for instance, objects are assigned to a given category if they are closer to the prototypes of that category than they are to the prototypes of other, contrast categories. Building on the ideas from prototype theory, Gärdenfors [2000] presents evidence that natural categories can be represented as convex regions in a multi-dimensional metric space, called a conceptual space, which is usually assumed to be Euclidean for simplicity. Each dimension in a conceptual space corresponds to a primitive cognitive feature.

Conceptual spaces are a useful metaphor, as semantic relations between categories can thus be identified with qualitative spatial relations between the corresponding regions. Mereological (or part-whole) relations are of particular interest in this context. For example, the assertion that the category *furniture* subsumes the category *table*, corresponds to the spatial relationship *table* $\mathbf{P}$ *furniture*, where we identify each category with its conceptual space representation for the ease of presentation. The notation $\mathbf{P}$ refers to the part-of relation. Table 1 summarizes the intended meaning of the spatial relations that we will consider in this paper. In addition, we will use $\mathbf{PP}^{-1}$ for the inverse of $\mathbf{PP}$, i.e. $a\,\mathbf{PP}^{-1}\,b$ iff $a\,\mathbf{PP}\,b$. The relations $\mathbf{PP}$, $\mathbf{PP}^{-1}$, $\mathbf{DR}$, $\mathbf{PO}$ and $\mathbf{EQ}$ are jointly exhaustive and pairwise disjoint (JEPD), i.e. exactly one of these relations holds between each pair of regions. These relations are the primitives of the RCC5 calculus, which is an important fragment of the region connection calculus [Cohn *et al.*, 1997].

Table 1: Mereological relations between two regions a and b with interiors $i(a)$ and $i(b)$.

Name	Symbol	Meaning
Part of	**P**	$a \subseteq b$
Proper part of	**PP**	$a \subset b$
Equals	**EQ**	$a = b$
Overlaps	**O**	$i(a) \cap i(b) \neq \emptyset$
Disjoint from	**DR**	$i(a) \cap i(b) = \emptyset$
Partially overlaps	**PO**	$i(a) \cap i(b) \neq \emptyset$, $a \not\subseteq b$, $b \not\subseteq a$

The computational properties of the RCC5 calculus are well understood. However, for reasoning about semantic relations between categories, RCC5 by itself is too limited. First, category boundaries tend to be vague. To cope with this, [Gärdenfors and Williams, 2001] proposes to use the egg-yolk calculus [Cohn and Gotts, 1996] to reason about categories. In the egg-yolk calculus, vague regions are represented as nested sets of classical regions. Spatial relations between vague regions are then defined in terms of RCC5 relations between the corresponding sets of classical regions. As such, spatial reasoning in the egg-yolk calculus straightforwardly reduces to decision procedures for the RCC5 calculus. Second, the conceptual spaces setting requires regions to be convex. Although this property is not usually considered in decision procedures for the RCC5 calculus, because conceptual spaces tend to be high-dimensional, requiring regions to be convex does not affect consistency or entailment checking procedures, not even for the more expressive RCC8 calculus [Schockaert and Li, 2012]. Third, not all semantic relations of interest correspond to part-whole relations. The notion of betweenness, in particular, plays a central role for interpolation [Kóczy and Hirota, 1993; Dubois *et al.*, 1997;

Schockaert and Prade, 2011], an important form of commonsense reasoning: knowing that b is between a and c, we can infer that properties common to a and c are also valid for b. For example, knowing that *bardolino* is between *beaujolais* and *valpolicella* and that the latter two wines are characterized by low tannins, we can infer that *bardolino* has low tannins. The validity of interpolation relies on the convexity of regions: if $b \subseteq \text{CH}(a \cup c)$, writing CH for the convex hull operator, and x is a convex region then from $a \subseteq x$ and $c \subseteq x$, we can derive $b \subseteq x$.

In applications, we may be able to learn betweenness information from analysing data; e.g. techniques such as multidimensional scaling can be used to discover the latent geometric structure of a given domain. Subsumption information can be obtained from existing taxonomies, or again from analysing data. There is a need, however, for techniques to reason about subsumption information in the presence of betweenness relations, e.g. to make explicit what can be entailed from known information or to identify inconsistencies when merging information from different sources. For example, knowing that *taiga* is between *tundra* and *temperateForest*, and that *taiga* subsumes *borealForest*, we should be able to conclude that *borealForest* is between *tundra* and *temperateForest*. To this end, in this paper we consider an extension of the RCC5 calculus in which assertions of the form $b\,\mathbf{PP}\,(a \bowtie c)$ can be expressed, where $(a \bowtie c)$ corresponds to $\text{CH}(a \cup c)$, i.e. we want to be able to express that some region is completely, partially, or not at all between two other regions.

The notion of betweenness has only had limited attention in the qualitative spatial reasoning (QSR) literature so far. One notable exception is [Clementini *et al.*, 2010], which proposes a set of 34 JEPD relations that can hold between three regions in the plane, and which are rich enough to express betweenness. A composition table is presented, but no sound and complete procedure for consistency checking is proposed. For points on the real line, the betweenness problem (or total ordering problem) is a well-known problem which consists of deciding whether a set of constraints of the form 'b is between a and c' can be satisfied, where b is between a and c iff $a < b < c$ or $c < b < a$. The problem is known to be NP-complete [Opatrny, 1979]. More generally, various authors have looked at combining RCC5 or RCC8 with relations encoding other aspects of space. For example, [Gerevini and Renz, 2002] combines RCC8 relations with size information and [Liu *et al.*, 2009] combines RCC8 relations with qualitative direction calculi. In [Wolter and Zakharyaschev, 2000], RCC8 is extended with region terms, which allow to express that e.g. a is equal to the union of b and c. The idea of using QSR for reasoning about natural language categories has been previously considered by a number of authors. Most notably, this has been discussed at length in [Gärdenfors and Williams, 2001] but also [Lehmann and Cohn, 1994] propose something along these lines.

The paper is structured as follows. In the next section, we recall some basic results about the RCC5 calculus, and introduce the notations and concepts that will be used throughout the paper. In Section 3 we introduce a number of properties that a consistent RCC5 network with betweenness information should satisfy. Subsequently we show in Section 4 that any network which satisfies these properties can be realized in a Euclidean space by using convex regions.

2 Preliminaries

Let $V = \{v_1, ..., v_n\}$ be a set of variables. An RCC5 network over V is a set of constraints of the form

$$\Theta = \{v_i\, R_{ij}\, v_j \mid (v_i, v_j) \in V^2 \text{ and } R_{ij} \subseteq \mathcal{R}_5\}$$

where $\mathcal{R}_5 = \{\mathbf{PP}, \mathbf{PP}^{-1}, \mathbf{DR}, \mathbf{PO}, \mathbf{EQ}\}$. If $R_{ij} = \{\rho\}$ is a singleton, we write $v_i\, \rho\, v_j$ as an abbreviation for $v_i\, \{\rho\}\, v_j$. When specifying an RCC5 network, we usually omit trivial constraints of the form $v_i\, \mathcal{R}_5\, v_j$. Moreover, as R_{ij} completely determines R_{ji} and vice versa, we usually only specify one of them explicitly. For convenience, we use the following abbreviations:

$$\mathbf{P} = \{\mathbf{PP}, \mathbf{EQ}\} \qquad \mathbf{O} = \{\mathbf{PP}, \mathbf{PP}^{-1}, \mathbf{PO}, \mathbf{EQ}\}$$
$$\mathbf{P}^{-1} = \{\mathbf{PP}^{-1}, \mathbf{EQ}\}$$

An RCC5 network is called *atomic* if each $R_{ij} = \{\rho_{ij}\}$ is a singleton. For atomic networks, consistency can be decided in $O(n^3)$ by checking path consistency. Specifically, it suffices to check for each i, j, k whether ρ_{ik} is among the relations in $\rho_{ij} \circ \rho_{jk}$, where the composition $\rho_{ij} \circ \rho_{jk}$ of ρ_{ij} and ρ_{jk} is defined by the RCC5 composition table, shown in Table 2. For example, $\Theta = \{v_1\, \mathbf{PP}\, v_2, v_2\, \mathbf{PO}\, v_3, v_1\, \mathbf{EQ}\, v_3\}$ is not consistent, because $\mathbf{EQ}$ is not among $\mathbf{PP} \circ \mathbf{PO} = \{\mathbf{DR}, \mathbf{PO}, \mathbf{PP}\}$.

In this paper, we will consider RCC5 networks over an extended set of variables

$$V^{\bowtie} = V \cup \{v_i \bowtie v_j \mid (v_i, v_j) \in V^2\}$$

At the semantic level, $v_i \bowtie v_j$ corresponds to the convex hull of $v_i \cup v_j$. For example, $v_1\, \mathbf{PP}\, (v_2 \bowtie v_3)$ means that v_1 is entirely between v_2 and v_3 while $(v_1 \bowtie v_2)\, \mathbf{PO}\, (v_3 \bowtie v_4)$ means that the convex hulls of $v_1 \cup v_2$ and of $v_3 \cup v_4$ overlap. To express that u is contained in the convex hull of $v_1 \cup ... \cup v_k$ we can introduce fresh variables $u_2, ..., u_k$ and express that $u_2\, \mathbf{EQ}\, (v_1 \bowtie v_2), u_3\, \mathbf{EQ}\, (u_2 \bowtie v_3), ..., u_k\, \mathbf{EQ}\, (u_{k-1} \bowtie v_k)$ and $u\, \mathbf{P}\, u_k$. Indeed, it is clear that u_i corresponds to $\text{CH}(v_1 \cup ... \cup v_i)$ for each i.

An RCC5$^{\bowtie}$ network over V is encoded as an RCC5 network over $V^{\bowtie}$. A solution to an RCC5$^{\bowtie}$ network Θ over V is an assignment $\mathcal{S}$ from the set of variables V to the set of regular closed regions in $\mathbb{R}^m$ for some $m \geq 1$, such that

- for each constraint of the form $v_i\, R_{ij}\, v_j$, the relation between $\mathcal{S}(v_i)$ and $\mathcal{S}(v_j)$ is among those in R_{ij};
- for each constraint of the form $v_i\, R_{ij}\, (v_j \bowtie v_k)$, the relation between $\mathcal{S}(v_i)$ and $\text{CH}(\mathcal{S}(v_j) \cup \mathcal{S}(v_k))$ is among those in R_{ij};
- for each constraint of the form $(v_i \bowtie v_j)\, R_{ij}\, v_k$, the relation between $\text{CH}(\mathcal{S}(v_i) \cup \mathcal{S}(v_j))$ and $\mathcal{S}(v_k)$ is among those in R_{ij};
- for each constraint of the form $(v_i \bowtie v_j)\, R_{ij}\, (v_k \bowtie v_l)$, the relation between $\text{CH}(\mathcal{S}(v_i) \cup \mathcal{S}(v_j))$ and $\text{CH}(\mathcal{S}(v_k) \cup \mathcal{S}(v_l))$ is among those in R_{ij}.

Table 2: Composition table for RCC5 relations.

$\circ$	$\mathbf{PP}$	$\mathbf{PP}^{-1}$	$\mathbf{DR}$	$\mathbf{PO}$	$\mathbf{EQ}$
$\mathbf{PP}$	$\mathbf{PP}$	$\mathcal{R}_5$	$\mathbf{DR}$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}\}$	$\mathbf{PP}$
$\mathbf{PP}^{-1}$	$\mathbf{O}$	$\mathbf{PP}^{-1}$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}^{-1}\}$	$\{\mathbf{PO},\mathbf{PP}^{-1}\}$	$\mathbf{PP}^{-1}$
$\mathbf{DR}$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}\}$	$\mathbf{DR}$	$\mathcal{R}_5$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}\}$	$\mathbf{DR}$
$\mathbf{PO}$	$\{\mathbf{PO},\mathbf{PP}\}$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}^{-1}\}$	$\{\mathbf{DR},\mathbf{PO},\mathbf{PP}^{-1}\}$	$\mathcal{R}_5$	$\mathbf{PO}$
$\mathbf{EQ}$	$\mathbf{PP}$	$\mathbf{PP}^{-1}$	$\mathbf{DR}$	$\mathbf{PO}$	$\mathbf{EQ}$

We will mainly focus on *convex solutions* in this paper, i.e. solutions $\mathcal{S}$ in which $\mathcal{S}(v)$ is a convex region for every v. In particular, in this paper we are interested in checking whether an RCC5$^{\bowtie}$ network has a convex solution. We will show that this can be decided in polynomial time for atomic networks, which means that the problem is in NP in general. As consistency checking for RCC5 is NP-hard and is not affected by the requirement that regions be convex, it follows that the problem is NP-complete.

We will say that Θ entails a constraint $v_i\, R\, v_j$, written $\Theta \models v_i\, R\, v_j$, if Θ contains a constraint $v_i\, R'\, v_j$ with $R' \subseteq R$, or if such a constraint can be derived from Θ using the RCC5 composition table (i.e. by enforcing path-consistency). To check the consistency of Θ as an RCC5$^{\bowtie}$ network, we first need to check whether Θ is consistent as an RCC5 network, i.e. when we treat expressions of the form $v_i \bowtie v_j$ simply as variables. For example if $v_i\, \mathbf{PO}\, (v_j \bowtie v_k)$ and $(v_j \bowtie v_k)\, \mathbf{PP}\, (v_s \bowtie v_t)$ are in Θ, then we should have that the relation between v_i and $(v_s \bowtie v_t)$ in Θ is either $\mathbf{PO}$ or $\mathbf{PP}$ (see Table 2). If Θ is consistent when interpreted as an RCC5 network, we will say that Θ is RCC5-consistent. Our aim is to decide when it is possible to realize $v_i \bowtie v_j$ as $\text{CH}(\mathcal{S}(v_i) \cup \mathcal{S}(v_j))$. If this is the case, we will say that Θ is RCC5$^{\bowtie}$-consistent.

3 Properties of betweenness

In this section, we introduce a number of properties an RCC5-consistent RCC5$^{\bowtie}$ network Θ should satisfy to be RCC5$^{\bowtie}$-consistent, and an additional property Θ should satisfy to have a convex solution. In the next section, we will then show that these conditions are sufficient to guarantee that Θ has a convex solution.

As betweenness is symmetric, for each v_i and v_j we should have:

$$(v_i \bowtie v_j)\, \mathbf{EQ}\, (v_j \bowtie v_i) \tag{1}$$

A region v_i is between itself and any other region:

$$v_i\, \mathbf{P}\, (v_i \bowtie v_j) \tag{2}$$

Note that (2) implies

$$v_i\, \mathbf{P}^{-1}\, (v_j \bowtie v_k) \Rightarrow v_i\, \mathbf{P}^{-1}\, v_j \tag{3}$$

$$v_i\, \mathbf{P}\, v_j \Rightarrow v_i\, \mathbf{P}\, (v_j \bowtie v_k) \tag{4}$$

Betweenness also satisfies the following monotonicity property:

$$v_i\, \mathbf{P}\, v_j \Rightarrow (v_i \bowtie v_k)\, \mathbf{P}\, (v_j \bowtie v_k) \tag{5}$$

The soundness of the conditions (1)–(5) is obvious. We can also show the following transitivity property.

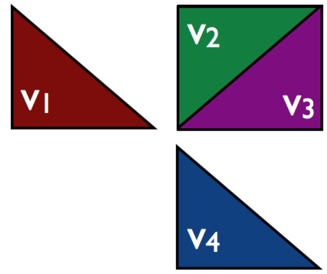

Figure 1: From $v_2\, \mathbf{P}\, (v_1 \bowtie v_3)$ and $v_3\, \mathbf{P}\, (v_2 \bowtie v_4)$ we cannot derive $v_3\, \mathbf{O}\, (v_1 \bowtie v_4)$ in general.

Proposition 1. *Any solution to Θ will satisfy:*

$$(v_i\, \mathbf{P}\, (v_k \bowtie v_l)) \wedge (v_j\, \mathbf{P}\, (v_k \bowtie v_l)) \wedge (v_s\, \mathbf{P}\, (v_i \bowtie v_j)) \Rightarrow (v_s\, \mathbf{P}\, (v_k \bowtie v_l)) \tag{6}$$

Proof. Let V_i, V_j, V_k, V_l, V_s be the realizations of v_i, v_j, v_k, v_l, v_s in $\mathbb{R}^m$ for some $m \geq 1$, and let $X = \text{CH}(V_i \cup V_j \cup V_k \cup V_l \cup V_s)$. Assume that $V_i\, \mathbf{P}\, (V_k \bowtie V_l)$, $V_j\, \mathbf{P}\, (V_k \bowtie V_l)$ and $V_s\, \mathbf{P}\, (V_i \bowtie V_j)$, then we show that also $V_s\, \mathbf{P}\, (V_k \bowtie V_l)$ must hold. Let us call a point p a vertex of X iff there exists a hyperplane H such that $H \cap X = \{p\}$. If $p \in V_s$ then because of $V_s\, \mathbf{P}\, (V_i \bowtie V_j)$ it holds that $p \in V_i$ or $p \in V_j$. Indeed, otherwise the hyperplane H separates the point p from $V_i \cup V_j$, which means that p and a fortiori V_s cannot be in $\text{CH}(V_i \cup V_j)$.

Suppose that $p \in V_i$; the case where $p \in V_j$ is entirely analogous. We then find from $v_i\, \mathbf{P}\, (v_k \bowtie v_l)$ that $p \in V_k$ or V_l, using an entirely analogous argument. It follows that every vertex of X belongs to $V_k \cup V_l$, which means that $\text{CH}(V_k \cup V_l) = X$ and $V_s \subseteq \text{CH}(V_k \cup V_l)$. □

On the other hand, note that from $v_j\, \mathbf{P}\, (v_i \bowtie v_k)$ and $v_k\, \mathbf{P}\, (v_j \bowtie v_l)$ we cannot derive $v_k\, \mathbf{P}\, (v_i \bowtie v_l)$ or even $v_k\, \mathbf{O}\, (v_i \bowtie v_l)$ in general. A counterexample in 2D is shown in Figure 1.

Conditions (1)–(6) hold for arbitrary regions. If we require regions to be convex, the following condition should additionally hold:

$$(v_i\, \mathbf{P}^{-1}\, v_j) \wedge (v_i\, \mathbf{P}^{-1}\, v_k) \Rightarrow (v_i\, \mathbf{P}^{-1}\, (v_j \bowtie v_k)) \tag{7}$$

It follows from (7) that when $v_i\, \mathbf{P}^{-1}\, v_j$ we also have $v_i\, \mathbf{P}^{-1}\, (v_i \bowtie v_j)$, which together with (2) entails:

$$v_i\, \mathbf{P}^{-1}\, v_j \Rightarrow v_i\, \mathbf{EQ}\, (v_i \bowtie v_j) \tag{8}$$

In particular, we always have

$$v_i\, \mathbf{EQ}\, (v_i \bowtie v_i) \tag{9}$$

Note that from (7) we can derive (6). Indeed, assume that $v_i\,\mathbf{P}\,(v_k \bowtie v_l)$, $v_j\,\mathbf{P}\,(v_k \bowtie v_l)$ and $v_s\,\mathbf{P}\,(v_i \bowtie v_j)$ are satisfied. By (7) we have $(v_i \bowtie v_j)\,\mathbf{P}\,(v_k \bowtie v_l)$. By the transitivity of $\mathbf{P}$, $v_s\,\mathbf{P}\,(v_k \bowtie v_l)$ follows from $v_s\,\mathbf{P}\,(v_i \bowtie v_j)$ and $(v_i \bowtie v_j)\,\mathbf{P}\,(v_k \bowtie v_l)$.

Similarly, from (1), (2) and (7) we can derive (5). Indeed, assume that $v_i\,\mathbf{P}\,v_j$ holds. From (2) we know $v_j\,\mathbf{P}\,(v_j \bowtie v_k)$, which together with $v_i\,\mathbf{P}\,v_j$ implies $v_i\,\mathbf{P}\,(v_j \bowtie v_k)$ by the transitivity of $\mathbf{P}$. From (2) we also have $v_k\,\mathbf{P}\,(v_k \bowtie v_j)$ and because of (1) also $v_k\,\mathbf{P}\,(v_j \bowtie v_k)$. Together with $v_i\,\mathbf{P}\,(v_j \bowtie v_k)$, it follows from (7) that $(v_i \bowtie v_k)\,\mathbf{P}\,(v_j \bowtie v_k)$.

However, while (7) only holds for convex regions, (5) and (6) are valid for arbitrary regions.

4 Realizing consistent RCC5$^{\bowtie}$ networks using convex regions

In this section, we show the following result.

Proposition 2. *Let Θ be an atomic RCC5$^{\bowtie}$ network which is RCC5-consistent and satisfies* (1), (2) *and* (7). *It holds that Θ has a convex solution, and a fortiori that Θ is RCC5$^{\bowtie}$-consistent.*

We then also obtain the following corollary.

Corollary 1. *It can be checked in polynomial time whether an atomic RCC5$^{\bowtie}$ network Θ has a convex solution.*

Indeed, verifying whether Θ is RCC5-consistent is known to be in P, whereas each of the conditions (1), (2) and (7) can straightforwardly be checked in polynomial time.
In the construction below, we will not explicitly consider constraints of the form $(v_i \bowtie v_j)\,R\,(v_r \bowtie v_s)$, as we can safely assume that such constraints are entailed by the other constraints in Θ. Indeed, if this were not the case, we could always introduce a new variable x and add the constraints $x\,\mathbf{EQ}\,(v_i \bowtie v_j)$ and $x\,R\,(v_r \bowtie v_s)$. We furthermore make the following assumptions about Θ:

- If Θ contains the formula $v_i\,\mathbf{PO}\,v_j$, then there are variables x, y and z in V such that Θ entails the following constraints: $x\,\mathbf{P}\,v_i$, $x\,\mathbf{DR}\,v_j$, $y\,\mathbf{P}\,v_j$, $y\,\mathbf{DR}\,v_i$, $z\,\mathbf{P}\,v_i$, $z\,\mathbf{P}\,v_j$.
- If Θ contains the formula $v_i\,\mathbf{PO}\,(v_j \bowtie v_k)$, then there are variables x, y and z in V such that Θ entails the following constraints: $x\,\mathbf{P}\,v_i$, $x\,\mathbf{DR}\,(v_j \bowtie v_k)$, $y\,\mathbf{P}\,(v_j \bowtie v_k)$, $y\,\mathbf{DR}\,v_i$, $z\,\mathbf{P}\,v_i$, $z\,\mathbf{P}\,(v_j \bowtie v_k)$.
- If Θ contains the formula $v_i\,\mathbf{PP}\,v_j$, then there is an $x \in V$ such that Θ entails the constraints $x\,\mathbf{P}\,v_j$ and $x\,\mathbf{DR}\,v_i$.
- If Θ contains the formula $v_i\,\mathbf{PP}\,(v_j \bowtie v_k)$ then there is an $x \in V$ such that Θ entails the constraints $x\,\mathbf{P}\,(v_j \bowtie v_k)$ and $x\,\mathbf{DR}\,v_i$.
- If Θ contains the formula $v_i\,\mathbf{PP}^{-1}\,(v_j \bowtie v_k)$ then there is an $x \in V$ such that Θ entails the constraints $x\,\mathbf{P}\,v_i$ and $x\,\mathbf{DR}\,(v_j \bowtie v_k)$.

Again we can make these assumptions without loss of generality: if suitable variables do not exist, we can always add new variables and extend Θ without affecting its consistency.

We now present a method to realize a solution for Θ. Our method is incremental. Initially, we assign to each variable v_i the interval $\sigma_0(v_i) = [0, 1]$. It is clear that initially the following assertions are valid ($m = 0$):

(P1) If $\Theta \models x\,\mathbf{P}\,y$ then $\sigma_m(x) \subseteq \sigma_m(y)$

(P2) If $\Theta \models x\,\mathbf{P}\,(y \bowtie z)$ then $\sigma_m(x) \subseteq \text{CH}(\sigma_m(y) \cup \sigma_m(z))$

(P3) If $\Theta \models x\,\mathbf{P}^{-1}\,(y \bowtie z)$ then $\sigma_m(x) \supseteq \text{CH}(\sigma_m(y) \cup \sigma_m(z))$

Now we consider a sequence of steps, each of which will use the mapping σ_m from the previous step to construct a mapping σ_{m+1} in a Euclidean space with one additional dimension, i.e. the mapping σ_m will assign to each variable a regular closed, convex subset of $\mathbb{R}^{m+1}$.

At each step, we consider a constraint of the form $v_i\,\mathbf{DR}\,(v_j \bowtie v_k)$ which is not yet satisfied by σ_m. We then define a new mapping σ_{m+1} satisfying **(P1)–(P3)**, the constraint $v_i\,\mathbf{DR}\,(v_j \bowtie v_k)$ as well as any constraint of the form $v_r\,\mathbf{DR}\,(v_s \bowtie v_t)$ that was already satisfied by σ_m. For the ease of presentation, let us write $\sigma_m(v_l) = R_l$ and $\sigma_{m+1}(v_l) = Q_l$. The regions Q_l will be obtained by using a fixpoint procedure. Initially, if $\Theta \models v_l\,\mathbf{P}\,v_i$, we define Q_l as

$$Q_l = \{(x_0, ..., x_{m+1}, y) \mid (x_0, ..., x_{m+1}) \in R_l, -1 \leq y \leq 0\}$$

If Θ entails $v_l\,\mathbf{P}\,(v_j \bowtie v_k)$, we define Q_l as

$$Q_l = \{(x_0, ..., x_{m+1}, y) \mid (x_0, ..., x_{m+1}) \in R_l, 0 \leq y \leq 1\}$$

Otherwise, we define Q_l as

$$Q_l = \{(x_0, ..., x_{m+1}, y) \mid (x_0, ..., x_{m+1}) \in R_l, -1 \leq y \leq 1\}$$

To ensure that the regions Q_l satisfy all formulas of the form $v_l\,\mathbf{P}\,(v_r \bowtie v_s)$, we subsequently contract these initial representations. In particular, we repeatedly apply the following updates until no further changes occur:

> For every formula of the form $v_l\,\mathbf{P}\,(v_r \bowtie v_s)$ entailed by Θ:
>
> $$Q_l \leftarrow \text{CH}(Q_r \cup Q_s) \cap Q_l \qquad (10)$$

Note that this procedure is monotonic in the sense that in each iteration the set Q_l can only become smaller. From the Knaster-Tarski theorem [Tarski, 1955], we then know that this procedure is well-defined, in the sense that there exists a unique greatest fixpoint which can be found by repeated applications of the update rule (10). Below we will show that the mapping σ_{m+1} which is thus obtained satisfies $v_i\,\mathbf{DR}\,(v_j \bowtie v_k)$, as well as all relevant relations that were already satisfied by σ_m. First, however, we need to show that the regions Q_l are convex and regular closed (and in particular that they are $(m+2)$-dimensional).

Lemma 1. *For every variable v_l, it holds that Q_l is regular closed and convex.*

Proof. It is clear that Q_l is a closed region and convex, as it is the intersection of closed and convex regions. To show that $\dim(Q_l) = n+2$, and thus that Q_l is regular closed, note that for every region R_l, it holds that

$$Q_l \supseteq \{(x_0, ..., x_{m+1}, 0) \mid (x_0, ..., x_{m+1}) \in R_l\}$$

Indeed, this is true initially (i.e. before applying the fixpoint procedure) and is invariant under updates of the form (10).

Let us call the region Q_l positive if

$$\forall(x_0, ..., x_{m+1}) \in R_l \,.\, \exists\theta > 0 \,.\, (x_0, ..., x_{m+1}, \theta) \in Q_l$$

Note that if $(x_0, ..., x_{m+1}, \theta) \in Q_l$ we also have $(x_0, ..., x_{m+1}, \theta^*) \in Q_l$ for all $\theta^* \in [0, \theta]$ since $(x_0, ..., x_{m+1}, 0) \in Q_l$ and Q_l is convex.

Similarly we call the region Q_l negative if

$$\forall(x_0, ..., x_{m+1}) \in R_l \,.\, \exists\theta < 0 \,.\, (x_0, ..., x_{m+1}, \theta) \in Q_l$$

Let $N = \{l \,|\, \Theta \models v_l \,\mathbf{P}\, v_i\}$ and $P = \{l \,|\, \Theta \models v_l \,\mathbf{P}\, (v_j \bowtie v_k)$. Initially, for each $l \in N$, the region Q_l is negative but not positive, for each $l \in P$ the region Q_l is positive but not negative, and for all other l the region Q_l is both positive and negative.

- If Q_l (at an intermediate stage) is positive, then $\text{CH}(Q_r \cup Q_s) \cap Q_l$ will be positive as soon as one of Q_r and Q_s is positive. Indeed, for $(x_0, ..., x_{m+1}) \in R_l$ there exist $(x'_0, ..., x'_{m+1}) \in R_r$ and $(x''_0, ..., x''_{m+1}) \in R_s$ such that $(x_0, ..., x_{m+1})$ is between $(x'_0, ..., x'_{m+1})$ and $(x''_0, ..., x''_{m+1})$. Now assume that Q_r is positive and thus contains some point $(x'_0, ..., x'_{m+1}, \theta')$ for $\theta' > 0$. Then clearly there is a $\theta > 0$ such that $(x_0, ..., x_{m+1}, \theta)$ is between $(x'_0, ..., x'_{m+1}, \theta')$ and $(x''_0, ..., x''_{m+1}, 0)$, the latter point being included in Q_s. It follows that for each $\theta^* \in [0, \theta]$, $(x_0, ..., x_{m+1}, \theta^*) \in \text{CH}(Q_r \cup Q_s)$. In particular, there therefore must exist some $\theta^* \in [0, \theta]$ such that $(x_0, ..., x_{m+1}, \theta^*) \in \text{CH}(Q_r \cup Q_s) \cap Q_l$.

  It follows that the only regions Q_l which are not positive must be those for which $l \in N$. Indeed, this is clearly true initially. Now assume that Q_l is positive and $v_l \,\mathbf{P}\, (v_r \bowtie v_s)$ is entailed by Θ. If $r \notin N$ or $s \notin N$ we know that $\text{CH}(Q_r \cup Q_s) \cap Q_l$ will be positive. On the other hand, if $r \in N$ and $s \in N$, we also have $l \in N$ by (7). Hence regions which are initially positive (i.e. the regions which are not in N) will remain positive throughout the procedure.

- Similar as for positive regions, we have that when Q_l is negative at any stage of the procedure, then $\text{CH}(Q_t \cup Q_s) \cap Q_l$ will be negative as soon as one of Q_t and Q_s is negative.

  Initially only regions Q_l for $l \in P$ are not negative. A region Q_l for which $l \notin P$ can only become not negative if Θ entails a constraint of the form $v_l \,\mathbf{P}\, (v_{j'} \bowtie v_{k'})$ such that $Q_{j'}$ and $Q_{k'}$ are not negative. From this observation, it is easy to see that whenever Q_l is not negative, then there must exist a set of regions $u_1, ..., u_m$ such that for each $l' \geq 3$, $u_{l'} \,\mathbf{P}\, (u_s \bowtie u_t)$ is entailed by Θ for some $s < l'$ and $t < l'$, where $u_1 = v_j$, $u_2 = v_k$ and $u_m = v_l$. From Proposition 1 we then obtain that $v_l \,\mathbf{P}\, (v_j \bowtie v_k)$ is entailed by Θ. In particular, we find that we will never arrive at a stage where Q_l is not negative when $l \in N$, since we know that $v_i \,\mathbf{DR}\, (v_j \bowtie v_k) \in \Theta$ and thus also $v_l \,\mathbf{DR}\, (v_j \bowtie v_k)$ for all $l \in N$; in particular Θ does not entail $v_l \,\mathbf{P}\, (v_j \bowtie v_k)$ for any $l \in N$.

- Note that we have not yet excluded the situation where the procedure requires an infinite number of steps to reach the fixpoint, such that one or more regions become infinitely small.

  To see why none of the regions can become infinitely small, first note that the point $(0, 0, ..., 0) \in \mathbb{R}^{m+1}$ is contained in each of the representations R_l. Now consider the point $p^+ = (0, ..., 0, 1) \in \mathbb{R}^{m+2}$. Clearly, each positive region Q_l initially contains p^+. Moreover, if Q_l is updated following (10), we know that either Q_r or Q_s is also positive. Suppose e.g. that Q_r is positive. Then $p^+ \in Q_l \cap Q_r$ and in particular $p^+ \in \text{CH}(Q_r \cup Q_s) \cap Q_l$. It follows that each of the positive regions contains p^+ in the final representations. Similarly we can show that each of the negative regions contains $p^- = (0, ..., 0, -1)$ in the final representations. Since each region Q_l contains either p^+ or p^- and $\{(x_0, ..., x_{m+1}, 0) \,|\, (x_0, ..., x_{m+1}) \in R_l\}$, clearly none of the regions can be of dimension $n + 1$.

□

Lemma 2. *For every formula of the form $v_r \,\mathbf{P}\, v_s$ entailed by Θ, it holds that*

$$Q_r \,\mathbf{P}\, Q_s$$

Proof. If $v_r \,\mathbf{P}\, v_s$ then by the transitivity of $\mathbf{P}$ we have that whenever $v_s \,\mathbf{P}\, (v_t \bowtie v_u)$ is entailed by Θ also $v_r \,\mathbf{P}\, (v_t \bowtie v_u)$ is entailed by Θ. This means that whenever Q_s is replaced by $\text{CH}(Q_t \cup Q_u) \cap Q_s$ then Q_r is replaced by $\text{CH}(Q_t \cup Q_u) \cap Q_r$. Hence it is clear that from $R_r \,\mathbf{P}\, R_s$ we obtain $Q_r \,\mathbf{P}\, Q_s$. □

Lemma 3. *If $R_r \,\mathbf{DR}\, R_s$ then we also have*

$$Q_r \,\mathbf{DR}\, Q_s$$

Proof. This is clear, since Q_r and Q_s are initially disjoint (as they are a "cylindrical" extension of disjoint regions) and the updates only contract the representations. □

Lemma 4. *If $v_l \,\mathbf{P}\, (v_r \bowtie v_s)$ is entailed by Θ, it holds that*

$$Q_l \,\mathbf{P}\, \mathit{CH}(Q_r \cup Q_s)$$

Proof. By construction, after the procedure converges, $Q_l = \text{CH}(Q_r \cup Q_s) \cap Q_l$ holds whenever $v_l \,\mathbf{P}\, (v_r \bowtie v_s)$ is entailed by Θ, which is equivalent with $Q_l \subseteq \text{CH}(Q_r \cup Q_s)$. □

Lemma 5. *If $v_l \,\mathbf{P}^{-1}\, (v_r \bowtie v_s)$ is entailed by Θ, it holds that*

$$Q_l \,\mathbf{P}^{-1}\, \mathit{CH}(Q_r \cup Q_s)$$

Proof. If $v_l \,\mathbf{P}^{-1}\, (v_r \bowtie v_s)$ is entailed by Θ, then Θ will also entail $v_l \,\mathbf{P}^{-1}\, v_r$ and $v_l \,\mathbf{P}^{-1}\, v_s$ by (3). By Lemma 2 we know that then $Q_r \,\mathbf{P}\, Q_l$ and $Q_s \,\mathbf{P}\, Q_l$. By the convexity of Q_l it then follows that $\text{CH}(Q_r \cup Q_s) \,\mathbf{P}\, Q_l$. □

Lemma 6. *If $R_l \,\mathbf{DR}\, \mathit{CH}(R_r \cup R_s)$ then also*

$$Q_l \,\mathbf{DR}\, \mathit{CH}(Q_r \cup Q_s)$$

Proof. This is clear due to the "cylindrical" nature of the initial representations of Q_l, Q_r and Q_s. □

Lemma 7. *It holds that*

$$Q_i \,\mathbf{DR}\, \mathit{CH}(Q_j \cup Q_k)$$

Proof. This follows from the observation that Q_i and $\text{CH}(Q_j \cup Q_k)$ are separated by the hyperplane

$$H = \{(x_0, ..., x_n, y) \mid y = 0\}$$

□

We can repeat the same procedure until all disjointness constraints of the form $v_i \, \mathbf{DR} \, (v_j \bowtie v_k)$ are satisfied. This also ensures that disjointness constraints of the form $v_i \, \mathbf{DR} \, v_j$ are satisfied, noting that v_j is equal to $v_j \bowtie v_j$ due to the convexity of v_j. The resulting mapping σ satisfies **(P1)**–**(P3)**, as well as

(DR1) If $\Theta \models x \, \mathbf{DR} \, y$ then $\text{int}(\sigma(x)) \cap \text{int}(\sigma(y)) = \emptyset$

(DR2) If $\Theta \models x \, \mathbf{DR} \, (y \bowtie z)$ then $\text{int}(\sigma(x)) \cap \text{int}(\text{CH}(\sigma(y) \cup \sigma(z))) = \emptyset$

It follows that σ is a solution of Θ. Indeed, we already know that constraints of the form $v_i \, \mathbf{DR} \, v_j$ and $v_i \, \mathbf{DR} \, (v_j \bowtie v_k)$ are satisfied, and we moreover have:

- If $v_i \, \mathbf{PO} \, v_j \in \Theta$ then also constraints of the form $x \, \mathbf{P} \, v_i$, $x \, \mathbf{DR} \, v_j$, $y \, \mathbf{P} \, v_j$, $y \, \mathbf{DR} \, v_i$, $z \, \mathbf{P} \, v_i$, $z \, \mathbf{P} \, v_j$ are in Θ. These latter constraints are all satisfied by the construction method, from which we derive that $\sigma(v_i) \, \mathbf{O} \, \sigma(v_j)$, $\neg(\sigma(v_i) \, \mathbf{P} \, \sigma(v_j))$ and $\neg(\sigma(v_i) \, \mathbf{P}^{-1} \, \sigma(v_j))$, i.e. σ satisfies $v_i \, \mathbf{PO} \, v_j$.
- If $v_i \, \mathbf{PO} \, (v_j \bowtie v_k) \in \Theta$ then we similarly find that this constraint is satisfied by σ.
- If $v_i \, \mathbf{PP} \, v_j \in \Theta$ then also constraints of the form $x \, \mathbf{P} \, v_j$ and $x \, \mathbf{DR} \, v_i$ are in Θ. Since $\sigma(x) \, \mathbf{P} \, \sigma v_j$ and $\sigma(x) \, \mathbf{DR} \, \sigma(v_i)$ hold, we must have $\neg(\sigma(v_i) \, \mathbf{P}^{-1} \, \sigma(v_j))$. Since we also have $\sigma(v_i) \, \mathbf{P} \, \sigma(v_j)$, it follows that σ satisfies $v_i \, \mathbf{PP} \, v_j$.
- If $v_i \, \mathbf{PP} \, (v_j \bowtie v_k)$ or $v_i \, \mathbf{PP}^{-1} \, (v_j \bowtie v_k)$ in Θ then similarly we find that this constraint is satisfied by σ.
- If $v_i \, \mathbf{EQ} \, v_j \in \Theta$ we know that $\sigma(v_i) \, \mathbf{P} \, \sigma(v_j)$ and $\sigma(v_i) \, \mathbf{P}^{-1} \, \sigma(v_j)$ hence the constraint is satisfied by σ.
- If $v_i \, \mathbf{EQ} \, (v_j \bowtie v_k) \in \Theta$ then we similarly find that this constraint is satisfied by σ.

In other words, we have constructed a mapping σ which assigns to each variable a convex and regular closed region, and which forms a solution of Θ, thus completing the proof of Proposition 2.

5 Conclusions

We have discussed the problem of deciding the consistency of RCC5-networks extended with betweenness information, where betweenness information is expressed using variables of the form $v_i \bowtie v_j$ corresponding to the convex hull of $v_i \cup v_j$. In particular, we have shown how consistency can be decided in polynomial time (for atomic networks) when regions are required to be convex. This corresponds to the problem of deciding whether RCC5-networks with betweenness constraints can be realized in a Euclidean conceptual space. These results provide an important stepping stone towards a full calculus for commonsense reasoning based on qualitative representations of conceptual spaces.

From a theoretical perspective, two areas for future work present themselves. First, we may wonder whether the consistency of RCC5-networks with betweenness information may still be decided in polynomial time when we do not require that regions be convex. Second, we may consider the problem of deciding consistency for RCC8 networks with betweenness information. Unfortunately, there seems to be no obvious way of generalizing the presented construction to the case of RCC8 networks. To illustrate this point, we can show that in any solution of a "cycle" of the form $\Theta = \{v_i \, \mathbf{DR} \, v_j \mid 0 \leq i \neq j \leq n\} \cup \{v_i \, \mathbf{PP} \, (v_{i-1} \bowtie v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{v_n \, \mathbf{PP} \, (v_{n-1} \bowtie v_0), v_0 \, \mathbf{PP} \, (v_n \bowtie v_1)\}$ there is an i such that $v_i \, \mathbf{EC} \, v_{i+1}$; in any convex solution of Θ there are at least two pairs $(i, i+1) \neq (j, j+1)$ such that $v_i \, \mathbf{EC} \, v_{i+1}$ and $v_j \, \mathbf{EC} \, v_{j+1}$ hold. These conditions are necessary but do not appear to be sufficient. In fact, it is not clear that the consistency of RCC8 networks with betweenness information can be decided in polynomial time, even for atomic networks. Note in particular that in the case of RCC8 without the requirement of convexity, reasoning about betweenness is harder than reasoning about an extension of RCC8 with a convex hull operator (given that $(v_i \bowtie v_i)$ is equal to the convex hull of v_i), which is known to be a very hard problem [Davis *et al.*, 1999].

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