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# Efficient Paraconsistent Reasoning with Ontologies and Rules*

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## Abstract

Description Logic (DL) based ontologies and nonmonotonic rules provide complementary features whose combination is crucial in many applications. In hybrid knowledge bases (KBs), which combine both formalisms, for large real-world applications, often integrating knowledge originating from different sources, inconsistencies can easily occur. These commonly trivialize standard reasoning and prevent us from drawing any meaningful conclusions. When restoring consistency by changing the KB is not possible, paraconsistent reasoning offers an alternative by allowing us to obtain meaningful conclusions from its consistent part.

In this paper, we address the problem of efficiently obtaining meaningful conclusions from (possibly inconsistent) hybrid KBs. To this end, we define two paraconsistent semantics for hybrid KBs which, beyond their differentiating properties, are faithful to well-known paraconsistent semantics as well as the non-paraconsistent logic they extend, and tractable if reasoning in the DL component is.

## 1 Introduction

In this paper, we address the problem of dealing with inconsistent knowledge bases consisting of ontologies and nonmonotonic rules, following a paraconsistent reasoning approach with a focus on efficiency.

*Description Logics* (*DLs*) and *Logic Programs* (*LPs*) provide different strengths when used for *Knowledge Representation and Reasoning*. While DLs employ the *Open World Assumption* and are suited for defining ontologies, LPs adopt the *Closed World Assumption* and are able to express nonmonotonic rules with exceptions and preference orders. Combining features of both formalisms has been actively pursued over the last few years, resulting in different proposals with different levels of integration and complexity: while some extend DLs with rules [Horrocks and Patel-Schneider, 2004; Krötzsch *et al.*, 2011], others follow a hybrid combination of ontologies with nonmonotonic rules, either providing a modular approach where rules and ontologies use their own semantics, and allowing limited interaction between them [Eiter *et al.*, 2008], or defining a unifying framework for both components [Motik and Rosati, 2010; Knorr *et al.*, 2011]. Equipped with semantics that are faithful to their constituting parts, these proposals allow for the specification of so-called *hybrid knowledge bases* (*hybrid KBs*) either from scratch, benefiting from the added expressivity, or by combining existing ontologies and rule bases.

The complex interactions between the ontology component and the rule component of these hybrid KBs – even more so when they result from combining existing ontologies and rule bases independently developed – can easily lead to contradictions, which, under classical semantics, trivialize standard reasoning and prevent us from drawing any meaningful conclusions, ultimately rendering these hybrid KBs useless.

One way to deal with this problem is to employ some method based on belief revision (e.g. [Leite, 2003; Osorio and Cuevas, 2007; Slota and Leite, 2012a; 2014; Delgrande *et al.*, 2013] for LPs, [Flouris *et al.*, 2008; Calvanese *et al.*, 2010; Kharlamov *et al.*, 2013] for DLs, and [Slota *et al.*, 2011; Slota and Leite, 2012b] for hybrid KBs) to regain consistency so that standard reasoning services can be used, or some method based on repairing (e.g. [Arenas *et al.*, 1999] for LPs and [Haase *et al.*, 2005] for DLs) where hypothetical belief revision is employed for consistent query answering, without actually changing the KB. However, this is not always feasible e.g. because we may not have permission to change the KB – as for instance in [Alberti *et al.*, 2011] where the KB encodes laws and norms – or because the usual high complexity of belief revision and repairing methods simply renders their application prohibitive. When these methods are not possible or not feasible, paraconsistent reasoning services, typically based on some many-valued logics, offer an alternative by being able to draw meaningful conclusions in the presence of contradiction. Whereas paraconsistent reasoning has been extensively studied in the context of each individual component of hybrid KBs (see Related Work in Sect. 5), it is still a rather unexplored field in the context of hybrid KBs. Notable exceptions are [Huang *et al.*, 2011; 2014; Fink, 2012], yet their computation is not tractable in general even if reasoning in the DL component is.

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In this paper, we investigate efficient paraconsistent semantics for hybrid KBs. We adopt the base framework of [Motik and Rosati, 2010] because of its generality and tight integration between the ontology and the rules – c.f. [Motik and Rosati, 2010] for a thorough discussion – under the semantics of [Knorr *et al.*, 2011] because of its computational properties. We extend such semantics with additional truth values to evaluate contradictory pieces of knowledge, following two common views on how to deal with contradictory knowledge bases. According to one view, contradictions are dealt with locally, in a minimally intrusive way, such that a new truth value is introduced to model inconsistencies, while consistent pieces of information whose derivation depends on inconsistent information are still considered to be true in the classical sense. This view is adopted in paraconsistent semantics for DLs, e.g. [Maier *et al.*, 2013], LPs, e.g. [Sakama, 1992; Sakama and Inoue, 1995], and hybrid KBs [Huang *et al.*, 2011; Fink, 2012]. The alternative view is to distinguish truth which depends on the inconsistent part of a KB, from truth which is derivable without involving any contradictory knowledge. This view, commonly referred to as *Suspicious Reasoning*, is adopted in paraconsistent semantics for LPs, e.g. [Alferes *et al.*, 1995; Sakama, 1992; Sakama and Inoue, 1995] and hybrid KBs [Huang *et al.*, 2014].

We present solutions following both views through the definition of a five-valued and a six-valued paraconsistent semantics for hybrid KBs, the latter implementing Suspicious Reasoning, both of which enjoy the following properties:

- *Soundness* w.r.t. the three-valued semantics for consistent hybrid KBs by [Knorr *et al.*, 2011];
- *Faithfulness* w.r.t. semantics for its base formalisms;
- *Computability* by means of a sound and complete fixpoint algorithm;
- *Tractability* when a tractable DL is used for the ontology.

The paper is organized as follows: in Sect. 2, we present the formal background; in Sect. 3, we present both semantics, starting with common parts, and proceeding with the five-valued semantics in Sect. 3.1, and the six-valued one in Sect. 3.2; in Sect. 4, we investigate the properties of both semantics and discuss related work and conclude in Sect. 5.

## 2 Preliminaries

In this section, we introduce hybrid knowledge bases, and also recall the syntax of MKNF formulas, originating from the logic of minimal knowledge and negation as failure (MKNF) [Lifschitz, 1991], into which the former are embedded. For reasons of space, some details, e.g., on DLs or the semantics for MKNF formulas, are omitted here. These can still be found in [Knorr *et al.*, 2011].

The syntax of MKNF formulas is defined over a function-free first-order signature $\Sigma = (\Sigma_c, \Sigma_p)$, where the sets $\Sigma_c$ and $\Sigma_p$ contain, resp., all constants and all predicates including the binary equality predicate $\approx$. Given a predicate $P$ and *terms* $\mathbf{s}$ over $\Sigma_c$ and a set of variables, an *atom* $P(\mathbf{s})$ is an *MKNF formula*. If $\varphi$, $\varphi_1$ and $\varphi_2$ are MKNF formulas, then $\neg\varphi$, $\exists x : \varphi$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \supset \varphi_2$, $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ also are, and $\varphi_1 \vee \varphi_2$, $\forall x : \varphi$, $\top$, and $\bot$ represent the usual syntactic short-cuts. Replacing free variables $\mathbf{x}$ in $\varphi$ by terms $\mathbf{s}$ is represented by $\varphi[\mathbf{s}/\mathbf{x}]$. A *closed* MKNF formula contains no free variables. Formulas of the form $\mathbf{K}\,\varphi$ and $\mathbf{not}\,\varphi$ are called, resp., $\mathbf{K}$-*atoms* and $\mathbf{not}$-*atoms*. Intuitively, $\mathbf{K}\,\varphi$ asks if $\varphi$ is known, while $\mathbf{not}\,\varphi$ checks if $\varphi$ does not hold, which allows one to draw conclusions from the absence of information.

Hybrid KBs combine a set of MKNF rules and a DL ontology $\mathcal{O}$, which is translatable into a function-free first order formula with equality $\pi(\mathcal{O})$ and for which checking of satisfiability and instances are decidable [Baader *et al.*, 2003]. An *MKNF rule* $r$ is of the given form where $H$, $A_i$, $B_i$ are atoms:

$$\mathbf{K}\,H \leftarrow \mathbf{K}\,A_1, \ldots, \mathbf{K}\,A_n, \mathbf{not}\,B_1, \ldots, \mathbf{not}\,B_m. \quad (1)$$

$\mathbf{K}\ H$ is called the *rule head*, and the sets $\{\mathbf{K}\ A_i\}$ and $\{\mathbf{not}\,B_j\}$ are called the *positive body* and the *negative body*, respectively. A rule $r$ is *positive* if $m = 0$, and $r$ is a *fact* if $n = m = 0$. A *program* $\mathcal{P}$ is a finite set of MKNF rules, and *positive* if all rules in it are positive. A *hybrid knowledge base* (*hybrid KB*) $\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$. Given such $\mathcal{K} = (\mathcal{O}, \mathcal{P})$, $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ is a *ground* hybrid KB where $\mathcal{P}_G$ denotes the grounding of $\mathcal{P}$ using all constants occurring in $\mathcal{K}$ as usual. As hybrid KBs can be translated into MKNF formulas using $\pi(\mathcal{O})$ and the match between $\rightarrow$ and $\supset$ for the MKNF rules, their semantics is derived from that of MKNF formulas, and we abuse notation and refer to such translation $\pi(\mathcal{K})$ by $\mathcal{K}$. To achieve decidability of reasoning, hybrid KBs are restricted to be *DL-safe*, basically requiring that variables in rules appear at least once in the positive body under a predicate which does not occur in $\mathcal{O}$, thus limiting the applicability of constants in rules to those in $\mathcal{P}$. From now on, we only consider DL-safe hybrid KBs as it can always be ensured [Ivanov *et al.*, 2013].

**Example 1.** Consider the following simplified ground hybrid KB $\mathcal{K}_G$ for assessing the risk of goods at a port.

$$HasCertifiedSender \sqsubseteq \neg IsMonitored \quad (2)$$

$$\mathbf{K}IsMonitored(g) \leftarrow \mathbf{K}risk(g). \quad (3)$$

$$\mathbf{K}risk(g) \leftarrow \mathbf{not}isLabelled(g). \quad (4)$$

$$\mathbf{K}isLabelled(g) \leftarrow \mathbf{not}risk(g). \quad (5)$$

$$\mathbf{K}resolvedRisk(g) \leftarrow \mathbf{K}IsMonitored(g). \quad (6)$$

$$\mathbf{K}HasCertifiedSender(g) \leftarrow \quad (7)$$

(4) and (5) state that good $g$ is either a risk ($r$) or it is labeled ($iL$). Any risk is monitored ($IM$) (3), thus a resolved risk ($rR$) (6). As $g$ has a certified sender ($HCS$) (7), it can be proven by means of axiom (2) that it is not monitored. Thus, $g$ can be derived to be monitored and not monitored at the same time if it is considered to be a risk, which can only be modeled by means of a paraconsistent evaluation.

Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid KB. The *set of* $\mathbf{K}$-*atoms of* $\mathcal{K}_G$, written $\mathsf{kA}(\mathcal{K}_G)$, is the smallest set that contains (i) all ground $\mathbf{K}$-atoms occurring in $\mathcal{P}_G$, and (ii) a $\mathbf{K}$-atom $\mathbf{K}\,\xi$ for each $\mathbf{not}$-atom $\mathbf{not}\,\xi$ occurring in $\mathcal{P}_G$. For a subset $S$ of $\mathsf{kA}(\mathcal{K}_G)$, the *objective knowledge* of $S$ w.r.t. $\mathcal{K}_G$ is the set of first-order formulas $\mathsf{ob}_{\mathcal{O},S} = \{\pi(\mathcal{O})\} \cup \{\xi \mid \mathbf{K}\xi \in S\}$.

## 3 Paraconsistent MKNF Semantics

In this section, we define two paraconsistent semantics for hybrid KBs, namely 5- and 6-models, their main difference being whether Suspicious Reasoning is supported in the rules of the hybrid KB or not. This requires the integration of different concepts and assumptions w.r.t. paraconsistency, independently developed for each of the base formalisms. E.g. Suspicious Reasoning has not been considered in DLs, so developing a unified semantics that is faithful to the paraconsistent semantics of the two base formalisms, thus limiting Suspicious Reasoning to inconsistencies from the LP, is highly non-trivial. Finding a model theory corresponding to some LP fixed-point based semantics is also very challenging. All this while maintaining faithfulness to the three-valued semantics [Knorr *et al.*, 2011], including properties such as *Coherence*, i.e. false (first-order) formulas are also default false. We start with common notions of both semantics.

First, we introduce p-interpretations, which extend first-order interpretations[1] with the ability to represent that certain pieces of information are true and false at the same time.

**Definition 1.** Given two first-order interpretations $I$ and $I_1$ with $I_1 \subseteq I$, the pair $\mathcal{I} = \langle I, I_1 \rangle$ is called a *p-interpretation*.

Intuitively, $I$ indicates what is *true* ($\mathbf{t}$) and *false* ($\mathbf{f}$), while the additional interpretation $I_1$ only designates for each (*true*) element in $I$ if it is actually *inconsistent* ($\mathbf{b}$ - for *both*) or not. No fourth value is assigned, arguably resulting in a simpler paraconsistent semantics for first-order formulas, and at the same time in a stronger consequence relation [Priest, 1979].

P-interpretations are the basis for defining structures used in both semantics, subsequently defined, for interpreting MKNF formulas, to which hybrid KBs can be translated.

**Definition 2.** A *6-structure* $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ consists of a p-interpretation $\mathcal{I}$ and two pairs $\mathcal{M} = \langle M, N \rangle$ and $\mathcal{N} = \langle M_1, N_1 \rangle$ of sets of p-interpretations. A *5-structure* is a 6-structure where $M_1 \subseteq M$ and $N_1 \subseteq N$.

Thus, a 5-structure is a special case of a 6-structure.

As common with MKNF semantics, $\mathcal{M}$ and $\mathcal{N}$ will be used, resp., to evaluate **K**- and **not**-atoms, while the p-interpretation $\mathcal{I}$ is used to evaluate first-order expressions. In particular, the latter applies in both semantics to atoms, according to the intuition given after Def. 1.

**Definition 3.** Given 5- or 6-structure $(\mathcal{I}, \mathcal{M}, \mathcal{N})$, atom $P(\mathbf{s})$:

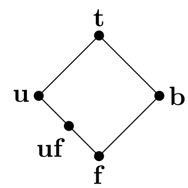
$$(\langle I, I_1 \rangle, \mathcal{M}, \mathcal{N})(P(\mathbf{s})) = \begin{cases} \mathbf{b} & \text{iff } \mathbf{s}^I \in P^I, \mathbf{s}^{I_1} \in P^{I_1} \\ \mathbf{t} & \text{iff } \mathbf{s}^I \in P^I, \mathbf{s}^{I_1} \notin P^{I_1} \\ \mathbf{f} & \text{iff } \mathbf{s}^I \notin P^I, \mathbf{s}^{I_1} \notin P^{I_1} \end{cases}$$

The evaluation of complex MKNF formulas differs for 5- and 6-structures, and is therefore spelled out separately.

Also based on p-interpretations are the notions that represent potential model candidates for each of the semantics.

**Definition 4.** A *6-pair* $(M, N)$ consists of two non-empty sets $M$, $N$ of p-interpretations. A *5-pair* $(M, N)$ is a 6-pair where $N \subseteq M$. If $M = N$, then $(M, N)$ is called *total*.

[1] In MKNF semantics, commonly the standard name assumption (SNA) is imposed on interpretations $I$, i.e., (1) the universe $\Delta$ of $I$ contains $\Sigma_c$ and an infinite supply of additional constants, (2) constants are mapped to themselves, and (3) equality is interpreted as a congruence relation. As first-order consequences under SNA and the standard first-order semantics are not distinguishable [Motik and Rosati, 2010], we also assume SNA here and in the remainder.




| $\supset$ | $\mathbf{b}$ | $\mathbf{t}$ | $\mathbf{f}$ | $\mathbf{uf}$ | $\mathbf{u}$ |
|---|---|---|---|---|---|
| $\mathbf{b}$ | $\mathbf{b}$ | $\mathbf{t}$ | $\mathbf{f}$ | $\mathbf{uf}$ | $\mathbf{u}$ |
| $\mathbf{t}$ | $\mathbf{b}$ | $\mathbf{t}$ | $\mathbf{f}$ | $\mathbf{uf}$ | $\mathbf{u}$ |
| $\mathbf{f}$ | $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{t}$ |
| $\mathbf{uf}$ | $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{uf}$ | $\mathbf{t}$ | $\mathbf{t}$ |
| $\mathbf{u}$ | $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{u}$ | $\mathbf{t}$ | $\mathbf{t}$ |

Figure 1: The lattice $\mathcal{FIVE}$ and evaluation of the operator $\supset$.

It is possible to define a so-called knowledge order $\succeq_k$ on 5-pairs and 6-pairs, with the intuition that elements which are greater allow one to derive more true and false knowledge.

**Definition 5.** For two 5- or 6-pairs $(M_1, N_1)$ and $(M_2, N_2)$, $(M_1, N_1) \succeq_k (M_2, N_2)$ iff $M_1 \subseteq M_2$ and $N_1 \supseteq N_2$.

Some 5- and 6-pairs will turn out to be 5- and 6-models as defined next in Sects. 3.1 and 3.2, and the order $\succeq_k$ will be used to compare such 5- and 6-models, and single out those that are most skeptical.

**Definition 6.** Let $\varphi$ be a closed MKNF formula and $(M, N)$ a 5-model (6-model) of $\varphi$ such that $(M_1, N_1) \succeq_k (M, N)$ for all 5-models (6-models) $(M_1, N_1)$ of $\varphi$. Then $(M, N)$ is a *well-founded 5-model* (*well-founded 6-model*) of $\varphi$.

In Sects. 3.1 and 3.2, we will now provide a model-based account of 5- and 6-models and a procedural characterization of the (unique) well-founded 5- and 6-model.

## 3.1 Five-Valued Semantics

We begin with the five-valued semantics, whose motivation stems from the fact that, for the three-valued MKNF semantics [Knorr *et al.*, 2011], with truth values *true, false* and *undefined* ($\mathbf{t}$, $\mathbf{f}$ and $\mathbf{u}$ resp.), two kinds of inconsistencies are identified. Namely, either some piece of information is simultaneously considered *true* and *false*, or *undefined* and *false*. We handle this by introducing for each of the two cases a further truth value, namely *inconsistent* ($\mathbf{b}$) and *undefined false* ($\mathbf{uf}$), respectively. The resulting lattice $\mathcal{FIVE}$ (Fig. 1) extends the well-known lattice $\mathcal{FOUR}$, and is not symmetric simply because no inconsistencies can occur between $\mathbf{t}$ and $\mathbf{u}$ in [Knorr *et al.*, 2011], as $\mathbf{t}$ always prevails in this case.

The evaluation of closed MKNF formulas in 5-structures for the truth values in $\mathcal{FIVE}$ is shown in Fig. 2 and, in the following, we will give intuitions and necessary notions. First, $\neg$ behaves for all values in $\mathcal{FOUR}$ as expected. The remaining case follows the intuition that $\mathbf{uf}$ behaves under negation like $\mathbf{f}$. The implication $\supset$ is defined (Fig. 1) like internal implication by [Maier *et al.*, 2013] for $\{\mathbf{b}, \mathbf{t}, \mathbf{f}, \mathbf{u}\}$, apart from the case $\mathbf{u} \supset \mathbf{f}$, which is no longer mapped to $\mathbf{t}$, as it does correspond to the kind of inconsistency between $\mathbf{u}$ and $\mathbf{f}$ in [Knorr *et al.*, 2011], for which we introduced $\mathbf{uf}$ in the first place. Hence, $\mathbf{u} \supset \mathbf{uf}$ is mapped to $\mathbf{t}$, which also corresponds to the idea that $\mathbf{uf}$ behaves under $\supset$ like a special case of $\mathbf{u}$.

For $\wedge$ and $\vee$, $min$ and $max$ are used like in [Knorr *et al.*, 2011] instead of the join and meet operations, more common for paraconsistent semantics, and this originates from the fact that $\mathbf{b}$ and $\mathbf{uf}$ should behave like special cases of $\mathbf{t}$ and $\mathbf{u}$ respectively, that are only necessary if there is an explicit occurrence of an inconsistency. This means that, if a rule body

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\neg\varphi) = \begin{cases} \mathbf{b} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{b} \\ \mathbf{t} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) \in \{\mathbf{f},\mathbf{uf}\} \\ \mathbf{f} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{t} \\ \mathbf{u} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{u} \end{cases}$$

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1 \wedge \varphi_2) = (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1) \wedge (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_2)$$
$$(\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1 \supset \varphi_2) = (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1) \supset (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_2)$$
$$(\mathcal{I},\mathcal{M},\mathcal{N})(\exists x : \varphi) = \bigvee_{\alpha\in\Delta} (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi[\alpha/x])$$

$$(I,\mathcal{M},\mathcal{N})(\mathbf{K}\varphi) = \begin{cases} \mathbf{b} & \text{iff } \bigcap_M(\varphi) = \mathbf{b} \\ \mathbf{t} & \text{iff } \bigcap_M(\varphi) = \mathbf{t} \\ \mathbf{f} & \text{iff } \bigcap_N(\varphi) = \mathbf{f} \\ \mathbf{uf} & \text{iff } \bigcap_M(\varphi) = \mathbf{f} \text{ and } \bigcap_N(\varphi) = \mathbf{b} \\ \mathbf{u} & \text{otherwise} \end{cases}$$

$$(I,\mathcal{M},\mathcal{N})(\mathbf{not}\varphi) = \begin{cases} \mathbf{b} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{b} \\ \mathbf{t} & \text{iff } (\bigcap_{N_1}(\varphi) = \mathbf{f}) \text{ or} \\ & (\bigcap_{M_1}(\varphi) = \mathbf{f} \text{ and } \bigcap_{N_1}(\varphi) = \mathbf{b}) \\ \mathbf{f} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{t} \\ \mathbf{u} & \text{otherwise} \end{cases}$$

Figure 2: Recursive five-valued evaluation of a closed MKNF formula in a 5-structure $(\mathcal{I},\mathcal{M},\mathcal{N})$, given that $\varphi$, $\varphi_1$, and $\varphi_2$ are MKNF formulas, and that $\mathcal{M} = \langle M, N\rangle$ and $\mathcal{N} = \langle M_1, N_1\rangle$. The operator $\supset$ is evaluated as shown in Fig. 1. The operators $\wedge$ and $\vee$ are defined respectively as $min$ and $max$ on the order $\mathbf{f} < \mathbf{u} < \mathbf{uf} < \mathbf{t} < \mathbf{b}$.

is $\mathbf{b}$, then its head should be $\mathbf{t}$, unless we can explicitly derive the negation of the head elsewhere, and similarly for a rule body that is $\mathbf{uf}$ whose head should be $\mathbf{u}$, unless we can explicitly derive its negation (or alternatively that it is $\mathbf{t}$ or $\mathbf{b}$ which would prevail over the derivation from this rule). Thus, the order $\mathbf{f} < \mathbf{u} < \mathbf{uf} < \mathbf{t} < \mathbf{b}$ implicitly reduces to $\mathbf{f} < \mathbf{u} < \mathbf{t}$ in [Knorr *et al.*, 2011]. This means that if a rule body contains (a conjunction of) two elements that are $\mathbf{b}$ and $\mathbf{u}$, then the head is $\mathbf{u}$ from this rule alone, and not $\mathbf{f}$, and, for $\vee$, that $\exists x : \varphi$ should be $\mathbf{t}$ if there is one replacement of $x$ which makes it $\mathbf{b}$, one that is $\mathbf{u}$ (or $\mathbf{uf}$), and none that is $\mathbf{t}$.

For the evaluation of $\mathbf{K}$ and $\mathbf{not}$, we employ intersections over sets of p-interpretations present in 5- and 6-structures for which we introduce specific notation. Namely, we can intersect p-interpretations component-wise to obtain the pieces of information on which they coincide. Given a set $M$ of p-interpretations $\mathcal{I}_i = \langle I_i, I'_i\rangle$, we can define $\bigcap M = \langle \bigcap I_i, \bigcap I'_i\rangle$, and it can be shown that $\bigcap M$ is indeed also a p-interpretation. In addition, we abbreviate $(\bigcap_{\mathcal{J}\in X} \mathcal{J}, \langle M, N\rangle, \langle M_1, N_1\rangle)(\varphi)$ for any $\varphi$, $M$, $N$, $M_1$, and $N_1$, and $X \in \{M, N, M_1, N_1\}$, with $\bigcap_X(\varphi)$.

Regarding the actual evaluation, $\mathbf{K}\varphi$ being $\mathbf{b}$ can be seen as a special case of being $\mathbf{t}$ in the sense that no $J \in M$ can map $\varphi$ to $\mathbf{t}$, and likewise for $\mathbf{uf}$ and $\mathbf{u}$ w.r.t. $M_1$, i.e. $\mathbf{uf}$ behaves like (a special case of) $\mathbf{u}$. The evaluation of $\mathbf{not}\,\varphi$ is symmetric to that of $\mathbf{K}\varphi$, for all but $\mathbf{uf}$, which, again, behaves under negation like $\mathbf{f}$. Using intersections in this evaluation slightly deviates, formally, from [Knorr *et al.*, 2011], but the differences are of no impact for hybrid KBs, and this choice results in a simpler notation.

With the evaluation in 5-structures in place, we can define 5-satisfaction for 5-pairs as follows.

**Definition 7.** Given a closed MKNF formula $\varphi$, a 5-pair $(M, N)$ *5-satisfies* $\varphi$, written $(M, N) \models_5 \varphi$, if and only if $(\mathcal{I}, \langle M, N\rangle, \langle M, N\rangle)(\varphi) \in \{\mathbf{b}, \mathbf{t}\}$ for each $\mathcal{I} \in M$.

For defining 5-models, an additional preference order over 5-pairs is required, minimizing knowledge under operator $\mathbf{K}$.

**Definition 8.** Any 5-pair $(M, N)$ is a *5-model* for a given closed MKNF formula $\varphi$ iff

(1) $(M, N) \models_5 \varphi$ and
(2) for each 5-pair $(M', N')$ with $M \subseteq M'$ and $N \subseteq N'$, where at least one of the inclusions is proper, there is $\mathcal{I}' \in M'$ such that $(\mathcal{I}', \langle M', N'\rangle, \langle M, N\rangle)(\varphi) \notin \{\mathbf{b}, \mathbf{t}\}$.

If $\varphi$ has a 5-model, then $\varphi$ is *5-consistent*.

The evaluation of **not**-atoms is fixed before checking whether $M$ and $N$ are maximal, i.e. whether the evaluation of $\mathbf{K}$-atoms is minimal w.r.t. the order $\mathbf{f} < \mathbf{u} < \mathbf{uf} < \mathbf{t} < \mathbf{b}$.

**Example 2.** Consider $\mathcal{K}_G$ from Ex. 1. For (4) and (5) alone, there are three 5-models since it is not determined whether $g$ is labelled or a risk: a), $\mathbf{K}r$ is $\mathbf{t}$ and $\mathbf{K}iL$ is $\mathbf{f}$, b), vice-versa, or c), both are $\mathbf{u}$. Both being $\mathbf{b}$ would also 5-satisfy (4) and (5), but with the evaluation of $\mathbf{not}$-atoms fixed to $\mathbf{b}$ in such a 5-pair $(M, N)$, the rule heads can be $\mathbf{t}$ to satisfy the rules, and there is $(M', N')$ as described in (2) of Def. 8.

For 5-satisfying (3)-(5), $\mathbf{K}\,IM$ has to be $\mathbf{t}$ or $\mathbf{b}$ for a) as $\mathbf{K}\,r$ is $\mathbf{t}$, can have any truth value for b) as $\mathbf{K}\,r$ is $\mathbf{f}$, and not be $\mathbf{f}$ for c) as $\mathbf{K}\,r$ is $\mathbf{u}$ (cf. Fig. 1). However, e.g. any 5-pair mapping $\mathbf{K}r$ to $\mathbf{f}$ and $\mathbf{K}\,IM$ not to $\mathbf{f}$ is not a 5-model since it is not minimal by (2) of Def. 8. Accordingly, $\mathbf{K}\,IM$ is minimized to $\mathbf{t}$ for a), to $\mathbf{f}$ for b), and to $\mathbf{u}$ for c).

Taking all of $\mathcal{K}_G$, there is a conflict between (2) and (3) if $\mathbf{K}r$ is $\mathbf{t}$ (model a)) or $\mathbf{u}$ (model c)) since the classical negation of $IM$ is also derivable. Thus, for the three possible 5-models for (4) and (5) alone, $\mathbf{K}\,IM$ and $\mathbf{K}\,rR$ are resp. minimized to $\mathbf{b}$ and $\mathbf{t}$ for model a), to $\mathbf{f}$ and $\mathbf{f}$ for model b), and to $\mathbf{uf}$ and $\mathbf{u}$ for c). Note that the head of (6) is evaluated as if the body was $\mathbf{t}$ and $\mathbf{u}$ respectively, w.r.t. models a) and c). Hence, neither kind of inconsistency is propagated, i.e., for a), $\mathbf{K}IM$ is $\mathbf{b}$, yet $\mathbf{K}rR$ is $\mathbf{t}$, and for c) $\mathbf{K}IM$ is $\mathbf{uf}$, yet $\mathbf{K}rR$ is $\mathbf{u}$.

As reasoning – entailment from hybrid KBs – with models that are usually infinite would be unfeasible, we adopt a common technique for reasoning with hybrid KBs [Motik and Rosati, 2010; Knorr *et al.*, 2011] and provide a finite representation of 5-models and the well-founded 5-model in particular, which can be directly used for query answering w.r.t. entailed information. Similarly to [Knorr *et al.*, 2011], this finite representation is obtained via a fixpoint construction.

To introduce this fixpoint construction for the five-valued semantics, we first define a variant of the doubling of hybrid KBs in [Alferes *et al.*, 2013], for which we introduce a new predicate $A^d$ for each predicate $A$ appearing in $\mathcal{K}_G$. Here, we abuse notation and denote also with $A^d$ the atom resulting from replacing the original predicate $A$ with $A^d$. It will be clear from the context whether $A^d$ is a predicate or an atom.

**Definition 9.** Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid KB. We introduce a new predicate $A^d$ for each predicate $A$ appearing in $\mathcal{K}_G$, and we constructively define

1. $\mathcal{O}^d$ by substituting each predicate $A$ in $\mathcal{O}$ by $A^d$;
2. $\mathcal{P}^d_G$ by transforming each rule of the form (1) into:
   (a) $\mathbf{K}H \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n, \mathbf{not}\, B^d_1, \ldots, \mathbf{not}\, B^d_m$,
   (b) $\mathbf{K}H^d \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n, \mathbf{not}\, B^d_1, \ldots, \mathbf{not}\, B^d_m$;
3. and the *ground doubled hybrid KB* $\mathcal{K}^d_G = (\mathcal{O}, \mathcal{O}^d, \mathcal{P}^d_G)$.

We now define a monotonic immediate consequence operator $T_{\mathcal{K}^d_G, \mathcal{C}}$. It collects the consequences from the program component and from the ontology, and subtracts those doubled $\mathbf{K}$-atoms of which the classical negation is derivable.

**Definition 10.** Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground positive hybrid KB. The operators $R_{\mathcal{K}^d_G}$, $D_{\mathcal{K}^d_G}$, $C_{\mathcal{K}^d_G,\mathcal{C}}$, and $T_{\mathcal{K}^d_G,\mathcal{C}}$ are defined on $\mathcal{K}^d_G$ and subsets $S$ and $C$ of $\mathsf{kA}(\mathcal{K}^d_G)$ as follows:

$$\begin{aligned}
R_{\mathcal{K}^d_G}(S) &= \{\mathbf{K}H \mid \mathcal{P}^d_G \text{ contains a rule of the form} \\
&\qquad \mathbf{K}H \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n \text{ s.t. } \mathbf{K}A_i \in S\} \\
D_{\mathcal{K}^d_G}(S) &= \{\mathbf{K}\xi \mid \mathbf{K}\xi \in \mathsf{kA}(\mathcal{K}_G), \mathsf{ob}_{\mathcal{O},S} \models_5 \xi\} \cup \\
&\qquad \{\mathbf{K}\xi^d \mid \mathbf{K}\xi \in \mathsf{kA}(\mathcal{K}_G), \mathsf{ob}_{\mathcal{O}^d,S} \models_5 \xi^d\} \\
C_{\mathcal{K}^d_G,\mathcal{C}}(S) &= \{\mathbf{K}\xi^d \mid \mathbf{K}\xi \in \mathsf{kA}(\mathcal{K}_G), \mathsf{ob}_{\mathcal{O},\mathcal{C}} \models_5 \neg\xi\} \\
T_{\mathcal{K}^d_G,\mathcal{C}}(S) &= (R_{\mathcal{K}^d_G}(S) \cup D_{\mathcal{K}^d_G}(S)) \setminus C_{\mathcal{K}^d_G,\mathcal{C}}(S)
\end{aligned}$$

Since $T_{\mathcal{K}^d_G,\mathcal{C}}$ is only defined for positive hybrid KBs, a standard transformation for general hybrid KBs is introduced.

**Definition 11.** Let $R$ be a set of ground MKNF rules and $S$ a set of ground $\mathbf{K}$-atoms. The *transform* $R/S$ contains all rules $\mathbf{K}H \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n$ for which there exists a rule of the form (1) in $R$ with $\mathbf{K}B_j \notin S$ for each $1 \leq j \leq m$.

Additionally, let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid KB. The *transforms* $\mathcal{K}_G/S$ and $\mathcal{K}^d_G/S$ are defined as $\mathcal{K}_G/S = (\mathcal{O}, \mathcal{P}_G/S)$ and $\mathcal{K}^d_G/S = (\mathcal{O}, \mathcal{O}^d, \mathcal{P}^d_G/S)$ respectively.

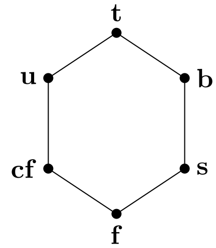
We can now define an antitonic operator that computes the least fixpoint of $T_{\mathcal{K}^d_G,\mathcal{C}}$ w.r.t. the resulting hybrid KB.

**Definition 12.** Let $\mathcal{K}_G$ be a ground hybrid KB and $S \subseteq \mathsf{kA}(\mathcal{K}^d_G)$. We define the operator $\Gamma_{\mathcal{K}^d_G}(S) = T_{\mathcal{K}^d_G/S,S} \uparrow \omega$ and two sequences $\mathbf{P}^d_i$ and $\mathbf{N}^d_i$ as follows.

$$\begin{aligned}
\mathbf{P}^d_0 &= \emptyset & \mathbf{N}^d_0 &= \mathsf{kA}(\mathcal{K}^d_G) \\
\mathbf{P}^d_{n+1} &= \Gamma_{\mathcal{K}^d_G}(\mathbf{N}^d_n) & \mathbf{N}^d_{n+1} &= \Gamma_{\mathcal{K}^d_G}(\mathbf{P}^d_n) \\
\mathbf{P}^d_\omega &= \bigcup \mathbf{P}^d_i & \mathbf{N}^d_\omega &= \bigcap \mathbf{N}^d_i
\end{aligned}$$

The sequence of $\mathbf{P}^d_i$ is in fact monotonically increasing, maximizing the set of *true* and *inconsistent* non-doubled $\mathbf{K}$-atoms, while that of $\mathbf{N}^d_i$ is monotonically decreasing, minimizing the set of non-doubled $\mathbf{K}$-atoms that are not *false*. It can be shown that both sequences are finite, in fact, even of polynomial length, so that both $\mathbf{P}^d_\omega$ and $\mathbf{N}^d_\omega$ exist for every ground hybrid KB and are unique.

**Theorem 1 (Soundness and completeness).** *Let $\mathcal{K}_G$ be a 5-consistent, ground hybrid KB, and $\mathbf{P}^d_\omega$ and $\mathbf{N}^d_\omega$ the fixpoints of the corresponding sequences. Then $(I_P, I_N)$ is the well-founded 5-model of $\mathcal{K}_G$, where $I_P = \{\mathcal{I} \mid \mathcal{I} \models_5 \mathsf{ob}_{\mathcal{O},\mathbf{P}^d_\omega \cap \mathsf{kA}(\mathcal{K}_G)}\}$ and $I_N = \{\mathcal{I} \mid \mathcal{I} \models_5 \mathsf{ob}_{\mathcal{O},\mathbf{N}^d_\omega \cap \mathsf{kA}(\mathcal{K}_G)}\}$.*




| $\supset$ | b | s | t | f | cf | u |
|---|---|---|---|---|---|---|
| **b** | b | t | t | f | f | f |
| **s** | b | t | t | f | f | f |
| **t** | b | f | t | f | f | f |
| **f** | t | t | t | t | t | t |
| **cf** | t | t | t | t | t | t |
| **u** | t | f | t | f | t | t |

Figure 3: The lattice $\mathcal{SIX}$ and evaluation of the operator $\supset$.

By Theorem 1, $\mathbf{P}^d_\omega$ and $\mathbf{N}^d_\omega$ constitute a finite representation of the well-founded 5-model and can, for example, be used for query answering w.r.t. entailed information. Hence, by computing the two fixpoints for $\mathcal{K}_G$ of Example 1, the truth values of all $\mathbf{K}$-atoms appearing in $\mathsf{kA}(\mathcal{K}_G)$ w.r.t. its well-founded 5-model can be determined.

**Example 3.** Recall $\mathcal{K}_G$ from Ex. 1. Then, $\mathbf{P}^d_\omega$ contains only $\mathbf{K}HCS$ (and $\mathbf{K}HCS^d$), while $\mathbf{N}^d_\omega = \mathsf{kA}(\mathcal{K}^d_G) \setminus \{\mathbf{K}IM^d\}$. Thus, model c) in Ex. 2 is the well-founded 5-model.

### 3.2 Six-Valued Semantics

The motivation for the six-valued semantics can be described as extending the five-valued semantics with the capability for Suspicious Reasoning. This results in two important changes. First, in order to implement Suspicious Reasoning, we introduce a further truth value *suspicious* ($\mathbf{s}$), which will be assigned to $\mathbf{K}$-atoms that are not explicitly inconsistent, i.e. not derivable to be *true* and *false* at the same time, but whose truth (in the five-valued semantics) is only derivable from a contradiction in the program (i.e. its derivation depends on an inconsistency in the program). Second, we replace $\mathbf{uf}$ with $\mathbf{cf}$ (*classically false*), which will be assigned to $\mathbf{K}$-atoms that are neither $\mathbf{t}$ nor $\mathbf{b}$, but whose classical negation is derivable (from the ontology). Differently from $\mathbf{uf}$ for the five-valued semantics, $\mathbf{cf}$ can be viewed as a special case of $\mathbf{f}$ (rather than $\mathbf{u}$), and no further *undefined* knowledge can be derived from it. Thus, classical falsity of $\mathbf{K}$-atoms is propagated in the semantics. All this results in the lattice $\mathcal{SIX}$ (Fig. 3).

The evaluation of closed MKNF formulas in 6-structures for $\mathcal{SIX}$ is shown in Fig. 4 and, subsequently, we will again provide intuitions and necessary notions. First, $\neg$ behaves for values in $\mathcal{FIVE}$ (replacing $\mathbf{uf}$ by $\mathbf{cf}$) as in the five-valued semantics, and the new value $\mathbf{s}$ mirrors the behavior of $\mathbf{b}$.

The implication $\supset$ is defined (Fig. 3) for all values in $\mathcal{FIVE}$ (again replacing $\mathbf{uf}$ by $\mathbf{cf}$) as in the five-valued semantics, only that all non-designated truth values have been mapped to $\mathbf{f}$ for simplicity. The only differing case is $\mathbf{cf} \supset \mathbf{f}$ which is now $\mathbf{t}$ (while $\mathbf{uf} \supset \mathbf{f}$ is $\mathbf{uf}$ in the five-valued semantics). This is justified by the fact that $\mathbf{cf}$ is understood as a special case of $\mathbf{f}$ in general (and not of $\mathbf{u}$). For, $\mathbf{s} \supset \mathbf{x}$, $\mathbf{s}$ behaves like $\mathbf{b}$ for any $\mathbf{x} \in \mathcal{SIX}$, which is motivated by the idea that the propagation of an inconsistency will itself also be propagated. For $\mathbf{x} \supset \mathbf{s}$, $\mathbf{s}$ behaves like $\mathbf{t}$ for all $\mathbf{x} \in \mathcal{SIX}$ but $\mathbf{t}$ and $\mathbf{u}$. These two cases are mapped to $\mathbf{f}$, based on the intuition that a consequent of an implication with *true* (*undefined*) antecedent does not depend on a contradiction since it is independently derivable to be *true* (*undefined*).

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\neg\varphi) = \begin{cases} \mathbf{b} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{b} \\ \mathbf{s} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{s} \\ \mathbf{t} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) \in \{\mathbf{f},\mathbf{cf}\} \\ \mathbf{f} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{t} \\ \mathbf{u} & \text{iff } (I,\mathcal{M},\mathcal{N})(\varphi) = \mathbf{u} \end{cases}$$

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1 \wedge \varphi_2) = (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1) \wedge (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_2)$$
$$(\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1 \supset \varphi_2) = (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_1) \supset (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi_2)$$
$$(\mathcal{I},\mathcal{M},\mathcal{N})(\exists x : \varphi) = \bigvee_{\alpha\in\Delta} (\mathcal{I},\mathcal{M},\mathcal{N})(\varphi[\alpha/x])$$

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\mathbf{K}\varphi) = \begin{cases} \mathbf{b} & \text{iff } \bigcap_M(\varphi) = \mathbf{b} \\ \mathbf{s} & \text{iff } \bigcap_M(\varphi) = \mathbf{t} \text{ and } \bigcap_N(\varphi) = \mathbf{f} \\ \mathbf{t} & \text{iff } \bigcap_M(\varphi) = \mathbf{t} \text{ and } \bigcap_N(\varphi) \neq \mathbf{f} \\ \mathbf{f} & \text{iff } \bigcap_M(\varphi) = \mathbf{f} \text{ s.t. } \exists M(\varphi) = \mathbf{t} \\ & \text{and } \bigcap_N(\varphi) = \mathbf{f} \\ \mathbf{cf} & \text{iff } \bigcap_M(\varphi) = \mathbf{f} \text{ s.t. } \nexists M(\varphi) = \mathbf{t} \\ \mathbf{u} & \text{iff } \bigcap_M(\varphi) = \mathbf{f} \text{ s.t. } \exists M(\varphi) = \mathbf{t} \\ & \text{and } \bigcap_N(\varphi) \neq \mathbf{f} \end{cases}$$

$$(\mathcal{I},\mathcal{M},\mathcal{N})(\mathbf{not}\varphi) = \begin{cases} \mathbf{b} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{b} \\ \mathbf{s} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{t} \text{ and } \bigcap_{N_1}(\varphi) = \mathbf{f} \\ \mathbf{t} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{f} \text{ s.t. } [\nexists M_1(\varphi) = \mathbf{t}] \\ & \text{or } [\exists M_1(\varphi) = \mathbf{t} \text{ and } \bigcap_{N_1}(\varphi) = \mathbf{f})] \\ \mathbf{f} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{t} \text{ and } \bigcap_{N_1}(\varphi) \neq \mathbf{f} \\ \mathbf{u} & \text{iff } \bigcap_{M_1}(\varphi) = \mathbf{f} \text{ s.t. } \exists M_1(\varphi) = \mathbf{t} \\ & \text{and } \bigcap_{N_1}(\varphi) \neq \mathbf{f} \end{cases}$$

Figure 4: Recursive evaluation of an MKNF formula in a 6-structure $(\mathcal{I},\mathcal{M},\mathcal{N})$, given that $\varphi$, $\varphi_1$, and $\varphi_2$ are MKNF formulas, and that $\mathcal{M} = \langle M, N\rangle$ and $\mathcal{N} = \langle M_1, N_1\rangle$. The operator $\supset$ is evaluated as shown in Fig. 3. The operators $\wedge$ and $\vee$ are defined respectively to be the join and meet operation in the lattice $\mathcal{SIX}$.

For $\wedge$ and $\vee$, the join and meet operations on $\mathcal{SIX}$ are used, which are both standard for paraconsistent semantics.

For the evaluation of the operators $\mathbf{K}$ and **not**, we may refer to the (non-)existence of p-interpretations in sets of them. For readability, we abbreviate $\exists\mathcal{J} \in X$ (resp. $\nexists\mathcal{J} \in X$) s.t. $(\mathcal{J}, \langle M, N\rangle, \langle M_1, N_1\rangle)(\varphi) = \mathbf{y}$ for any $\varphi$, $M$, $N$, $M_1$, and $N_1$, $X \in \{M, N, M_1, N_1\}$ and truth value $\mathbf{y}$, with $\exists X(\varphi) = \mathbf{y}$ (resp. $\nexists X(\varphi) = \mathbf{y}$).

Regarding the actual evaluation, the observations for $\mathbf{K}$ and **not** from Sec. 3.1 persist. Then, $\mathbf{s}$ can be seen as a second special case of $\mathbf{t}$ (different from $\mathbf{b}$). The changes w.r.t. the cases for $\mathbf{f}$, $\mathbf{cf}$, and $\mathbf{u}$ occur basically because a) for 6-pairs $(M, N)$ $N \subseteq M$ does no longer hold, and b) more importantly, $\mathbf{cf}$ is now to be understood as a special case of $\mathbf{f}$ and not $\mathbf{u}$.

Since $N \subseteq M$ does not hold for 6-pairs (Def. 4), we have to check all p-interpretations in both $M \cup N$ for p-satisfaction.

**Definition 13.** Given a closed MKNF formula $\varphi$, a 6-pair *6-satisfies* $\varphi$, written $(M, N) \models_6 \varphi$, if and only if $(\mathcal{I}, \langle M, N\rangle, \langle M, N\rangle)(\varphi) \in \{\mathbf{b}, \mathbf{t}\}$ for each $\mathcal{I} \in M \cup N$.

The definition of a *6-model* only applies to hybrid KBs, so we can refer directly to $\mathbf{K}$-atoms in the program in a new third condition ensuring that certain 6-models that contain unjustified evaluations to $\mathbf{b}$ or $\mathbf{cf}$ are removed.

**Definition 14.** Let $(M, N)$ be a 6-pair and $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ a hybrid KB. Any 6-pair $(M, N)$ is a *6-model* of $\mathcal{K}$ iff

(1) $(M, N) \models_6 \mathcal{K}$,
(2) for every 6-pair $(M', N')$ with $M \subseteq M'$ and $N \subseteq N'$ where at least one of the inclusions is proper, there is $\mathcal{I}' \in M' \cup N'$ s.t. $(\mathcal{I}', \langle M', N'\rangle, \langle M, N\rangle)(\mathcal{K}) \notin \{\mathbf{b}, \mathbf{t}\}$, and
(3) for every $\mathbf{K}\ \xi \in \mathsf{kA}(\mathcal{K}_G)$ it holds that $(*, \langle M, N\rangle, \langle M, N\rangle)(\mathbf{K}\ \xi) \in \{\mathbf{b}, \mathbf{cf}\}$ if and only if $\mathsf{ob}_{\mathcal{O},\{\mathbf{K}\,\xi' \mid (*,\langle M,N\rangle,\langle M,N\rangle)(\mathbf{K}\,\xi') \in \{\mathbf{b},\mathbf{s},\mathbf{t}\}\}} \models_6 \neg\xi$.

If $\varphi$ has a 6-model, then $\varphi$ is *6-consistent*.

**Example 4.** Recall $\mathcal{K}_G$ (Ex. 1) and the three different evaluations of $\mathbf{K}r$ and $\mathbf{K}iL$ in the 5-models a)–c) in Ex. 2. Without axiom (2), $\mathbf{K}IM$ could also be minimized to $\mathbf{cf}$ if $\mathbf{K}r$ is $\mathbf{u}$ (model c)), but this would be prevented by (3) of Def. 14 as $\neg IM$ would not be derivable. Condition (3) also removes all 6-models where $\mathbf{K}HCS$ is $\mathbf{b}$, so it has to be $\mathbf{t}$.

With the contradiction from (2), $\mathbf{K}IM$ has to be $\mathbf{b}$ for a) where $\mathbf{K}r$ is $\mathbf{t}$, and $\mathbf{K}IM$ has to be $\mathbf{cf}$ for b) if $\mathbf{K}r$ is $\mathbf{f}$ or c) where $\mathbf{K}r$ is $\mathbf{u}$, due to the definition of $\supset$ and (3) of Def. 14. In the first case a), $\mathbf{K}\,rR$ can be minimized to $\mathbf{s}$ such that support on contradiction can be detected. In the other cases b) and c), $\mathbf{K}rR$ is $\mathbf{f}$. Thus, unlike Ex. 2 where $\mathbf{K}rR$ is $\mathbf{u}$ if $\mathbf{K}IM$ is $\mathbf{uf}$, the falsity is propagated to $\mathbf{K}rR$ if $\mathbf{K}IM$ is $\mathbf{cf}$, so no undefined knowledge can be derived from $\mathbf{K}IM$.

As before, we introduce a fixpoint construction which will provide a finite representation of the well-founded 6-model.

We define two operators $T_{\mathcal{K}_G}$ and $T'_{\mathcal{K}_G,\mathcal{C}}$, where $\mathbf{K}$-atoms whose classical negation is derivable are removed in $T'_{\mathcal{K}_G,\mathcal{C}}$.

**Definition 15.** Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground positive hybrid KB. The operators $R_{\mathcal{K}_G}$, $D_{\mathcal{K}_G}$, $T_{\mathcal{K}_G}$ and $T'_{\mathcal{K}_G,\mathcal{C}}$ are defined on $\mathcal{K}_G$ and subsets $S$ and $C$ of $\mathsf{kA}(\mathcal{K}_G)$ as follows.

$$\begin{aligned} R_{\mathcal{K}_G}(S) &= \{\mathbf{K}H \mid \mathcal{P}_G \text{ contains a rule of the form} \\ &\qquad \mathbf{K}H \leftarrow \mathbf{K}A_1, \ldots, \mathbf{K}A_n \text{ s.t. } \mathbf{K}A_i \in S\} \\ D_{\mathcal{K}_G}(S) &= \{\mathbf{K}\xi \mid \mathbf{K}\xi \in \mathsf{kA}(\mathcal{K}_G), \mathsf{ob}_{\mathcal{O},S} \models_p \xi\} \\ T_{\mathcal{K}_G}(S) &= R_{\mathcal{K}_G}(S) \cup D_{\mathcal{K}_G}(S) \\ T'_{\mathcal{K}_G,\mathcal{C}}(S) &= (R_{\mathcal{K}_G}(S) \cup D_{\mathcal{K}_G}(S)) \setminus \\ &\qquad \{\mathbf{K}\xi \mid \mathbf{K}\xi \in \mathsf{kA}(\mathcal{K}_G), \mathsf{ob}_{\mathcal{O},\mathcal{C}} \models_p \neg\xi\} \end{aligned}$$

Reusing the transform (Def. 11), we define two operators.

**Definition 16.** Let $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ be a ground hybrid KB and $S \subseteq \mathsf{kA}(\mathcal{K}_G)$. We define the two operators $\Gamma_{\mathcal{K}_G}(S) = T_{\mathcal{K}_G/S} \uparrow \omega$, and $\Gamma'_{\mathcal{K}_G}(S) = T'_{\mathcal{K}_G/S,S} \uparrow \omega$ and two sequences $\mathbf{P}_i$ and $\mathbf{N}_i$ as follows.

$$\begin{aligned} \mathbf{P}_0 &= \emptyset & \mathbf{N}_0 &= \mathsf{kA}(\mathcal{K}_G) \\ \mathbf{P}_{n+1} &= \Gamma_{\mathcal{K}_G}(\mathbf{N}_n) & \mathbf{N}_{n+1} &= \Gamma'_{\mathcal{K}_G}(\mathbf{P}_n) \\ \mathbf{P}_\omega &= \bigcup \mathbf{P}_i & \mathbf{N}_\omega &= \bigcap \mathbf{N}_i \end{aligned}$$

In the alternating fixpoint, $\Gamma_{\mathcal{K}_G}$ is used in the sequence of $\mathbf{P}_i$, and $\Gamma'_{\mathcal{K}_G}$ in that of $\mathbf{N}_i$. This ensures that $\mathbf{K}$-atoms whose classical negation is derivable are not contained in the latter sequence that minimizes the set of $\mathbf{K}$-atoms that are $\mathbf{t}$ or $\mathbf{u}$.

As for the five-valued semantics, the unique well-founded 6-model can be obtained from the fixpoints $\mathbf{P}_\omega$ and $\mathbf{N}_\omega$.

**Theorem 2 (Soundness and completeness).** *Let $\mathcal{K}_G$ be a 6-consistent, ground hybrid KB, and $\mathbf{P}_\omega$ and $\mathbf{N}_\omega$ the fixpoints* *of the corresponding sequences. Then* $(I_P, I_N)$ *is the well-founded 6-model of* $\mathcal{K}_G$*, where* $I_P = \{\mathcal{I} \mid \mathcal{I} \models_p \mathsf{ob}_{\mathcal{O},\mathbf{P}_\omega}\}$ *and* $I_N = \{\mathcal{I} \mid \mathcal{I} \models_p \mathsf{ob}_{\mathcal{O},\mathbf{N}_\omega}\}$.

So, similar to Theorem 1, Theorem 2 provides the desired finite representation of the well-founded 6-model for a hybrid KB, and, thus, allows to determine the truth values of the atoms in Example 1 w.r.t. its well-founded 6-model.

**Example 5.** Recall $\mathcal{K}_G$ from Ex. 1. Then, $\mathbf{P}_\omega = \{\mathbf{K}HCS\}$, while $\mathbf{N}_\omega = \{\mathbf{K}r, \mathbf{K}iL\}$. Thus, the 6-model c) in Ex. 4 is the well-founded 6-model (where $\mathbf{K}r$ and $\mathbf{K}iL$ are $\mathbf{u}$).

## 4 Properties

In this section, we show several important properties of our two semantics that demonstrate that both are well-defined.

First, both semantics generalize the three-valued semantics for hybrid KBs [Knorr *et al.*, 2011], in the sense that for each so-called 3-model, a corresponding 5- and 6-model exists.

**Theorem 3 (Faithfulness w.r.t. the three-valued MKNF semantics).** *Let* $\mathcal{K}_G = (\mathcal{O}, \mathcal{P}_G)$ *be a ground hybrid KB, and* $(M, N)$ *a 3-model of* $\mathcal{K}_G$*. Then, there exists a corresponding 5- and 6-model* $(M', N')$ *of* $\mathcal{K}_G$*, and both coincide.*

This 5- and 6-model can in fact be constructed, yet the converse does not hold, i.e. there are 5- and 6-models of $\mathcal{K}_G$ without a corresponding 3-model (due to inconsistencies) which is also why no general correspondence on the unique well-founded models for the three semantics exists.

Both semantics are also faithful w.r.t. $\mathcal{ALC}4$ [Ma *et al.*, 2007], provided that, in $\mathcal{ALC}4$, no *gaps* are admitted, i.e., no truth value $\mathbf{u}$, as well as $\top$ and $\bot$ are represented by short-cuts as shown in [Maier *et al.*, 2013], which is what we assume.

**Theorem 4 (Faithfulness w.r.t. $\mathcal{ALC}$4).** *Let* $\mathcal{O}$ *be an* $\mathcal{ALC}4$ *ontology. Models of* $\mathcal{O}$ *in* $\mathcal{ALC}4$ *without gaps are in a bijection with p-interpretations that evaluate* $\pi(\mathcal{O})$ *to* $\mathbf{t}$ *or* $\mathbf{b}$.

Moreover, the six-valued semantics also covers $WFSX_p$ [Alferes *et al.*, 1995], which is defined for LPs with explicit negation and implements Suspicious Reasoning, using a so-called MKNF-translation. Our result applies whenever classical negation in LPs is limited to unary program atoms.

**Theorem 5 (Faithfulness of the six-valued semantics w.r.t. $WFSX_p$).** *Let* $\Pi$ *be an LP with classical negation limited to unary atoms and* $\mathcal{K}_G^\Pi$ *its MKNF-translation. There is a one-to-one correspondence between the well-founded 6-model of* $\mathcal{K}_G^\Pi$ *and the unique well-founded paraconsistent model assigned by* $WFSX_p$.

No similar result for the five-valued semantics exists since no corresponding well-founded semantics for LPs exists.

Data complexity of computing the finite representations of the well-founded 5- and 6-models depends on the DL used.

**Theorem 6 (Data Complexity).** *Let* $\mathcal{K} = (\mathcal{O}, \mathcal{P})$ *be a hybrid KB where entailment of ground atoms in the DL of* $\mathcal{O}$ *is decidable with data complexity* $\mathcal{C}$*. Then, computing the fixpoints corresponding to the well-founded 5- and 6-model of* $\mathcal{K}$ *is in* $P^{\mathcal{C}}$ *w.r.t. the data complexity.*

Notably, data complexity for paraconsistent reasoning does not increase when compared to [Knorr *et al.*, 2011], and, if the considered DL-fragment is polynomial, then computing the fixpoints is in $P$.

## 5 Related Work and Conclusions

We have introduced two novel paraconsistent semantics for hybrid KBs. They differ in whether inconsistencies and classical falsity are propagated in the program component of the KB. We have provided faithfulness results for both semantics and have shown that they are efficiently computable if the employed ontology language is tractable.

Paraconsistent reasoning has been extensively studied in both base formalisms of hybrid KBs. For DLs, most work [Patel-Schneider, 1989; Straccia, 1997; Ma *et al.*, 2007; Zhang *et al.*, 2009; Maier *et al.*, 2013] focuses on four-valued semantics, varying which classical rules of inferences they satisfy. The approach to which both our semantics are faithful, [Ma *et al.*, 2007; Maier *et al.*, 2013], is most general as it covers $\mathcal{SROIQ}$, the DL behind OWL 2, considers tractable subclasses and truth value removals, and permits re-using classical reasoners. Also considered are three-valued semantics for DLs [Zhang *et al.*, 2010] and measuring the degree of inconsistency in *DL-Lite* [Zhou *et al.*, 2012]. For LPs, the survey [Damásio and Pereira, 1998] discusses e.g. a four-valued semantics without default negation [Blair and Subrahmanian, 1989], a four-, six-, and nine-valued semantics [Sakama and Inoue, 1995] for answer sets [Gelfond and Lifschitz, 1991], and a seven- [Sakama, 1992] and nine-valued [Alferes *et al.*, 1995] well-founded semantics [Gelder *et al.*, 1991]. Notably, the latter, to which our six-valued semantics is faithful, admits both the Coherence Principle and Suspicious Reasoning. More recently, a very general framework for arbitrary bilattices of truth values [Alcântara *et al.*, 2005] and paraconsistent Datalog [de Amo and Pais, 2007] have been considered.

Only two paraconsistent semantics for combinations of DLs and LPs directly relate to ours. Both build on answer sets, so their computation is not tractable even with polynomial DLs, and unlike ours, their first-order semantics is four-valued, resulting in a weaker consequence relation. The first, [Huang *et al.*, 2011], is defined for hybrid KBs in MKNF like ours, but extends [Motik and Rosati, 2010] and is four-valued and faithful to [Sakama and Inoue, 1995; Ma *et al.*, 2007]. A nine-valued extension to cover Suspicious Reasoning is considered in [Huang *et al.*, 2014]. The paraconsistent hybrid semantics [Fink, 2012] is a nine-valued extension of *semi-equilibrium semantics* [Eiter *et al.*, 2010], and faithful to [Sakama and Inoue, 1995; Maier *et al.*, 2013], but Suspicious Reasoning is not considered.

In the future, our fixpoint computations can be used to adapt the Protégé plug-in *NoHR* [Ivanov *et al.*, 2013] to also consider reasoning with our paraconsistent semantics. Future work also includes investigating Suspicious Reasoning in paraconsistent DLs, both standalone and as parts of hybrid KBs.

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