Truthfulness of a Proportional Sharing Mechanism in Resource Exchange

Yukun Cheng[1], Xiaotie Deng[2], Qi Qi[3], and Xiang Yan[2]
[1]Zhejiang University of Finance and Economics, Hangzhou China
ykcheng@amss.ac.cn
[2]Shanghai Jiao Tong University, Shanghai China
dengxiaotie@sjtu.edu.cn, xyansjtu@163.com
[3]The Hongkong University of Science and Technology, Hong Kong
kaylaqi@ust.hk

Abstract

In this paper, we consider the popular proportional sharing mechanism and discuss the incentives and opportunities of an agent to lie for personal gains in resource exchange game. The main result is a proof that an agent manipulating the proportional sharing mechanism by misreporting its resource amount will not benefit its own utility eventually. This result establishes a strategic stability property of the resource exchange protocol. We further illustrate and confirm the result via network examples.

1 Introduction

The rapid growth of the wireless and mobile Internet has provided an opportunity for wide applications of exchanging resources or services over networks, which go beyond the peer-to-peer (P2P) bandwidth sharing idea. In a traditional P2P network, each peer is both a supplier and a consumer of resources. Resources (such as processing power, disk storage or network bandwidth) are shared among multiple interconnected peers who use a mechanism to make a portion of their resources directly available to other network participants in a distributed management system [Schollmeier, 2001].

The successes of the resource exchange networks highly depend on the efficient resource utilization. A P2P system can be modeled as a pure exchange economy in which each user brings its own divisible resources to the market and exchanges its resources with its neighbors to derive utility. Formally, the system is modeled as an undirected graph G, where each node i represents a peer with w_i units of divisible resources (or weight) to be distributed among its neighbors. The utility u_i is determined by the total amount of resources obtained from its neighbors. An efficient allocation in such an exchange economy can be characterized by the market equilibrium. On the other hand, to encourage users to contribute more, a fair allocation rule is essential. Ideally a user should receive as much as it gives. However, such perfect reciprocation may not be feasible due to the network structure. Then a proportional fairness mechanism is a natural approximation. Under this mechanism, each peer provides each neighbor a portion of its contribution in proportion to what it receives from its neighbors. Recently, Wu and Zhang [Wu and Zhang, 2007] showed that the market equilibrium obtained by a combinatorial method, called bottleneck decomposition is a proportional fair solution. They also showed the proportional dynamics converged to this equilibrium for linear utility functions under mild conditions.

However, agents are strategic. The market equilibrium is determined by agents' reported information rather than their true information. An agent may have the incentive to misreport its information if it is profitable. Hurwicz [Hurwicz, 1972] showed that it is impossible to design a truthful mechanism that guarantees a market equilibrium outcome in general. Even in a Fisher market game with linear utility, it was shown that an agent may derive a better payoff [Adsul *et al.*, 2010], more specifically, may double its benefit by strategic behaviors [Chen *et al.*, 2014]. The resource exchange game studied in a P2P setting is a special case of Arrow-Debreu market where agents are only interested in their neighbors' resources and only care about the total amount of resources received. Hurwicz's impossibility theorem may not hold for restricted settings. A recent work [Cheng *et al.*, 2015] discussed the issue of agents' manipulations by cheating on their connectivity information. The authors proved that the proportional sharing mechanism is robust to such manipulations. As utilities are determined by allocation that depends on agents' weights and the network structure, the strategic manipulations of an agent can only be misreporting its weight or/and misreporting its connection edges in the network. This work leaves a clear open question whether an agent could lie on the amount of resources it can offer. In this paper, we solve this open problem and prove the truthfulness of the proportional sharing mechanism. It shows that the proportional sharing mechanism is truthful, fair and Pareto efficient.

Technical Contributions

Our approach builds on a network bottleneck decomposition structure initially designed for the analysis of the connection of fairness (proportional sharing) and competitiveness (market equilibrium) [Wu and Zhang, 2007] and then the nonmanipulability [Cheng *et al.*, 2015]. For a given subset of agents, the total resources of their neighbors has a key importance on their eventual gains with respect to their own total resources. The concept of bottleneck captures the ratio of those two by finding a subset with this ratio at the minimum value. Recursively applying this operation, the bottleneck decomposition of G is obtained.

The bottleneck decomposition of G and the utility of agent u depends on the reported weight x_u by u, given the other agents' weights. With the increasing of x_u, we characterize the change process of bottleneck decomposition and prove the monotonically nondecreasing property of the utility function of u. Therefore, the desirable truthfulness of the proportional sharing mechanism follows.

Related Works

The classical economists and algorithmic game theorists have made an extensive study of competitive equilibrium [Arrow and Debreu, 1954], in terms of computation for prices and allocation [Lange, 1967], complexity and approximation [Papadimitriou, 2001; Deng *et al.*, 2002; Devanur *et al.*, 2002; Jain, 2007; Ye, 2008; Duan *et al.*, 2016]. Those works have started to have an influence in resource allocation among multiple agents, especially in the important implementations for the Internet enabled economic, management and social activities. How to fairly redistribute and share those resources have become an important issue with more and more online platforms which facilitate the exchange of commodities and services [GetRidApp, ; Homeexchang, ; Nestia, ; Swap,]

The automated process through information and communication technology for Internet applications in such problems has made it possible for the agents to make their best moves. Hence manipulating one's own private information becomes a possibility. Some of the recent theoretical studies have been in this direction such as recent studies on agent incentives in the Fisher market equilibrium for linear market [Adsul *et al.*, 2010; Chen *et al.*, 2014] and for constant elasticity of substitution market [Branzei, 2014].

The proportional sharing mechanism has been extensively studied in the exchange economy model. Compared to other allocation mechanism, the proportional sharing mechanism is simple, computational efficient and fair. Some of previous work [Feldman *et al.*, 2009; Zhang, 2005] focus on the price-taking scheme. But in the resource exchange/sharing model, money is no longer needed as a medium and the non-price-taking scheme is considered in our paper and [Wu and Zhang, 2007; Cheng *et al.*, 2015].

2 Preliminaries

In this section, we model the resource exchange setting by an undirected graph $G = (V, E; w)$. Each vertex $u \in V$ represents an agent with an upload resource amount (weight) $w_u > 0$ to be exchanged with its neighbors. Let $\Gamma(u)$ be the neighborhood of u, i.e. the set of vertices adjacent to u in G. And f_{uv} is denoted as the fraction of resource u allocated to its neighbor v. Obviously, $0 \leq f_{uv} \leq 1$ and the resource vertex u provides to v is $w_u f_{uv}$. $F = (f_{uv})_{u,v \in E}$ is called a *feasible allocation* if $\sum_{v \in \Gamma(u)} f_{uv} = 1$, that means u allocates all its resource out. The utility of agent u is defined as $U_u(F) = \sum_{v \in \Gamma(u)} f_{vu} w_v$, i.e. all received resource from its neighborhood $\Gamma(u)$.

2.1 Bottleneck Decomposition

Consider an undirected connected graph $G = (V, E; w)$ with weight $w : V \to R_+$. For any set $S \subseteq V$, we define $w(S) = \sum_{u \in S} w(u)$ and $\Gamma(S) = \cup_{u \in S} \Gamma(u)$. It is possible that $S \cap \Gamma(S) \neq \emptyset$. Define $\alpha(S) = w(\Gamma(S))/w(S)$, referred to as the inclusive expansion ratio of S, or the α*-ratio* of S for short.

Definition 1 (Maximal Bottleneck) *A vertex subset $B \subseteq V$ is called a* bottleneck *of G if $\alpha(B) = \min_{S \subseteq V} \alpha(S)$. B is a maximal bottleneck if for any subset $\widetilde{B}$, $B \subset \widetilde{B} \subseteq V$ implies $\alpha(\widetilde{B}) \geq \alpha(B)$. $(B, \Gamma(B))$ is called the maximal bottleneck pair in V.*

Definition 2 (Bottleneck Decomposition) *Given $G = (V, E; w)$. Start with $V_1 = V$, $G_1 = G$ and $i = 1$. Find the maximal bottleneck B_i of G_i and let G_{i+1} be the induced subgraph on the vertex set $V_{i+1} = V_i - (B_i \cup C_i)$, where $C_i = \Gamma(B_i) \cap V_i$, the neighbor set of B_i in the subgraph G_i. Repeat if $G_{i+1} \neq \emptyset$ and set $k = i$ if $G_{i+1} = \emptyset$. Then we call $\mathcal{B} = \{(B_1, C_1), \cdots, (B_k, C_k)\}$ the bottleneck decomposition of G, α_i the i-th* α-ratio *and $< \alpha_i = \frac{w(C_i)}{w(B_i)} : i = 1, 2, \cdots, k >$ the* α-ratio vector.

As stated in [Wu and Zhang, 2007], the problem of computing the bottleneck decomposition can be solved by the *parametric maximum flow algorithm* in a polynomial time. Further, they also derived some properties of the bottleneck decomposition as follows.

Proposition 3 ([Wu and Zhang, 2007]) *Given graph G, the bottleneck decomposition of G is unique and*
(1) $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq 1$;
(2) if $\alpha_i = 1$, then $i = k$ and $B_i = C_i$; otherwise B_i is an independent set and $B_i \cap C_i = \emptyset$;
(3) if $B \subseteq V_i$ and $C = \Gamma(B) \cap V_i$, then $w(C) \setminus w(B) \geq \alpha_i$, where if the equality holds, then $B \subseteq B_i$ and $C \subseteq C_i$.

For the third claim in Proposition 3, we name such a pair of (B, C) with $B \subseteq V_i$ and $C = \Gamma(B) \cap V_i$ as a *candidate pair* in V_i for convenience.

2.2 BD Mechanism

Given bottleneck decomposition, an allocation mechanism [Wu and Zhang, 2007] can be determined by distinguishing three cases. For convenience, we call such an allocation mechanism BD Mechanism.

BD Mechanism:

- For $\alpha_i < 1$, consider the bipartite graph $\widehat{G}_i = (B_i, C_i; E_i)$ where $E_i = (B_i \times C_i) \cap E$. By Proposition 3-(2), B_i is an independent set in G. Let $\widehat{f}_{uv}$ be the amount of bandwidth that vertex $u \in B_i$ upload to $v \in C_i$ along edge $(u, v) \in E_i$. By the max-flow min-cut theorem, there exists flow $\widehat{f}_{uv} \geq 0$ for $u \in B_i$ and $v \in C_i$ such that $\sum_{v \in \Gamma(u) \cap C_i} \widehat{f}_{uv} = w_u$ and $\sum_{u \in \Gamma(v) \cap B_i} \widehat{f}_{uv} = w_v / \alpha_i$. Let $\widehat{f}_{vu} = \alpha_i \widehat{f}_{uv}$ which means that $\sum_{u \in \Gamma(v) \cap B_i} \widehat{f}_{vu} = w_v$.
- When $B_k = C_k$ with $\alpha_k = 1$, similarly we construct a bipartite graph $\widehat{G} = (B_k, B'_k; E'_k)$ such that

B'_k is a copy of B_k. There is an edge $(u, v') \in E'_k$ if and only if $(u, v) \in E[B_k]$. By Hall's theorem, for any edge $(u, v) \in E[B_k]$, there exists flow $\widehat{f}_{uv'}$ such that $\sum_{v' \in \Gamma(u) \cap B'_k} \widehat{f}_{uv'} = w_u$ and $\widehat{f}_{uv'} = \widehat{f}_{vu'}$. Let $\widehat{f}_{uv} = \widehat{f}_{uv'}$.

- For any other edge, $(u, v) \notin B_i \times C_i$, $i = 1, 2, \cdots, k$, define $\widehat{f}_{uv} = 0$

It is clear that BD Mechanism assigns all resource of each agent to its neighbors from the same pair, that is all available resources exchanged along edges in $B_i \times C_i$, $i = 1, \cdots, k$. In terms of "fairness" and "efficiency", BD Mechanism is exactly a *proportional sharing mechanism*.

Definition 4 (Proportional Sharing Mechanism) *For each vertex u, the allocation $(f_{uv} : v \in \Gamma(u))$ of its resource w_u is proportional to what it receives from its neighbors $(w_v \cdot f_{vu} : \Gamma(u))$. That is $f_{uv} = \frac{f_{vu} w_v}{\sum_{k \in \Gamma(u)} f_{ku} w_k}$.*

Proposition 5 ([Wu and Zhang, 2007]) *BD Mechanism is a proportional sharing mechanism.*

As stated before, the P2P system can be modeled an exchange economy where each agent u sells its own divisible resource and use the money earned through trading to buy its neighbors' resource. An efficient allocation in such an exchange economy can be characterized by the *market equilibrium*. Given a bottleneck decomposition, if a price vector P is well defined as: $p_u = \alpha_i w_u$, if $u \in B_i$; and $p_u = w_u$ otherwise, then such a price vector combining the allocation F obtained from BD Mechanism satisfying

Proposition 6 ([Wu and Zhang, 2007]) *(P,F) is a market equilibrium, where $F = (f_{uv})$ is obtained from BD Mechanism.*

2.3 Resource Exchange Game

From a system design point of view, though BD Mechanism shall allocate resources among interconnected participants fairly and efficiently it is unknown whether an agent is willing to follow such a distributed network mechanism at the execution level. Can agents make strategic moves for gains in their utilities? Specific to the resource exchange model, the resources all agents have are their private information. They may manipulate BD Mechanism by misreporting the resource they own. We call such a problem with incentive factors the *resource exchange game*.

In the resource exchange game, let the resource amount reported by agent u be $x_u \in (0, w_u]$. The reason why x_u cannot exceed the true bandwidth w_u is that each agent must upload all its reported resource to its neighbors. The collection $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is referred to as the *weight profile* and $\mathbf{x}_{-u} = (x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_n)$ is the weight profile without agent u. Thus $\mathbf{x} = (x_u, \mathbf{x}_{-u})$. Let the bottleneck decomposition be $\mathcal{B}(\mathbf{x}) = \{(B_1(\mathbf{x}), C_1(\mathbf{x})), \cdots, (B_k(\mathbf{x}), C_k(\mathbf{x}))\}$ and BD Mechanism outputs an allocation $F(\mathbf{x})$ based on the given weight profile $\mathbf{x}$. The utility of agent u written as $U_u(\mathbf{x})$ is [Wu and Zhang, 2007]:

$$U_u(\mathbf{x}) = \begin{cases} x_u \cdot \alpha_i(\mathbf{x}), & u \in B_i(\mathbf{x}); \\ x_u / \alpha_i(\mathbf{x}), & u \in C_i(\mathbf{x}), \end{cases} \tag{1}$$

where $\alpha_i(\mathbf{x})$ is the i-th α ratio according to weight profile $\mathbf{x}$.

For mechanism design, the notion of *truthfulness* is perhaps the most important concept.

Definition 7 *A mechanism is truthful if no agent can benefit strictly from misreporting its resource amount irrespective of what is reported by other agents. Formally, given agent u and profile $\mathbf{x} = (x_u, \mathbf{x}_{-u})$, it holds that for any $\mathbf{x}_{-u}$*

$$U_u(w_u, \mathbf{x}_{-u}) \geq U_u(x_u, \mathbf{x}_{-u}). \tag{2}$$

It is obvious that truthfully reporting resource amount w_u is the dominant strategy for each agent u in the resource exchange game, if a mechanism is truthful.

3 Truthfulness of BD Mechanism

As we know, the bottleneck decomposition of G depends on the structure of network and the resource all agents have. So for any agent u and any weight profile $\mathbf{x}_{-u}$, the bottleneck decomposition $\mathcal{B}$ of G shall change with the reported resource x_u. Thus $\mathcal{B}$ can be viewed as a function of x_u if $\mathbf{x}_{-u}$ is fixed. On the other hand, we observe that bottleneck decomposition of G could be the same when x_u falls in an interval. Based on such an observation, we partition interval $(0, w_u]$ into several disjoint subintervals $\{\langle a_i, b_i \rangle\}_i$ and construct a series of bottleneck decompositions $\{\mathcal{B}^i\}_i$ such that when $x_u \in \langle a_i, b_i \rangle$, $\mathcal{B}(x_u, \mathbf{x}_{-u}) = \mathcal{B}^i = \{(B^i_1, C^i_1), \cdots, (B^i_{k^i}, C^i_{k^i})\}$. W.l.o.g., assume that $0 < a_i \leq b_i = a_{i+1} \leq b_{i+1}$. We use symbol "$\langle\rangle$" to denote the subinterval, because $\langle a_i, b_i \rangle$ could be one of five forms $[a_i, b_i]$, $[a_i, b_i)$, $(a_i, b_i]$, (a_i, b_i) and $a_i = b_i$. If $a_i = b_i$, then $\langle a_i, b_i \rangle$ contains exactly one point which can be viewed as a special closed interval. Further, in order to keep the one-to-one correspondence between the subintervals and the bottleneck decompositions, we use a_i and b_i to denote the left and right endpoints of the i-th subinterval, respectively. Thus for any two adjacent interval $\langle a_i, b_i \rangle$ and $\langle a_{i+1}, b_{i+1} \rangle$, there are only two cases: 1. $\langle a_i, b_i]$ and $(a_{i+1}, b_{i+1}\rangle$; 2. $\langle a_i, b_i)$ and $[a_{i+1}, b_{i+1}\rangle$; depending on which interval contains the break point $a_{i+1} = b_i$.

In the sequel, we first characterize the pairs agent u belongs to in adjacent $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ in Subsection 3.1. Then the monotonicity of utility functions is derived by such useful characterizations in Subsection 3.2.

3.1 Characterization of $\{\mathcal{B}^i\}$

In order to show the structure properties of bottleneck decomposition $\mathcal{B}$ in-depth, we redefine the bottleneck decomposition in more detail below.

Definition 8 (Bottleneck Decomposition) *Let $\mathcal{B}^i = \{(B^i_1, C^i_1), \cdots, (B^i_{k^i}, C^i_{k^i})\}$ be the bottleneck decomposition of graph G when $x_u \in \langle a_i, b_i \rangle$ and let the α-ratio of (B^i_j, C^i_j) be $\alpha^i_j = w(C^i_j)/w(B^i_j)$, $j = 1, \cdots, k^i$. For pair (B^i_j, C^i_j) with $\alpha^i_j < 1$, each vertex in B^i_j (or C^i_j) is called a B-class (or C-class) vertex. For the special case $B^i_{k^i} = C^i_{k^i}$, i.e., $\alpha^i_{k^i} = 1$, all vertices in $B^i_{k^i}$ are categorized as both B-class and C-class. Define $V^i_1 = V$, $V^i_{j+1} = V^i_j - (B^i_j \cup C^i_j)$ for $j = 1, \cdots, k^i - 1$ and G^i_j for the induced subgraph on V^i_j.*

Note that a vertex in $V^i_{k^i}$ with $\alpha^i_{k^i} = 1$ could simultaneously be B-class and C-class, in the case $B^i_{k^i} = C^i_{k^i}$. Moreover to highlight the α-ratio of pairs (B^i_j, C^i_j) containing agent u, we denote it $\alpha^i_j(x_u)$. Further, if $V^i_h = V^{i+1}_h$ for some index h, then the maximal bottleneck pair in V^i_h or V^{i+1}_h just is a candidate pair in the other. So Proposition 3-(3) can be applied on those candidate pairs. In the subsequent discussions, such a technique will be used repeatedly.

In the following, we assume that $u \in B^i_j \cup C^i_j$ and $u \in B^{i+1}_\ell \cup C^{i+1}_\ell$ in $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ respectively. As we know, the bottleneck decomposition of G changes because of the reported resource of agent u. Agent u's strategic move is only able to influence such pairs which are decomposed after pairs that u is in. But those with index less than j and ℓ still keep the same. So

Lemma 9 *For any agent u, if $u \in B^i_j \cup C^i_j$ and $u \in B^{i+1}_\ell \cup C^{i+1}_\ell$ in $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ respectively, then $V^i_h = V^{i+1}_h$ for each $h = 1, 2, \cdots, \min\{j, \ell\}$.*

Since agent u may be a B-class vertex or C-class vertex in $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ respectively, there are totally 4 cases: $u \in B^i_j \cap C^{i+1}_\ell$, $u \in C^i_j \cap B^{i+1}_\ell$, $u \in B^i_j \cap B^{i+1}_\ell$ and $u \in C^i_j \cap C^{i+1}_\ell$. Specially for the first two cases, we assume that $\alpha^i_j(x_u) < 1$ if $x_u \in \langle a_i, b_i\rangle$ and $\alpha^{i+1}_\ell(x_u) < 1$ if $x_u \in \langle a_{i+1}, b_{i+1}\rangle$. This assumption is reasonable. For example, if $u \in B^i_j \cap C^{i+1}_\ell$ and $\alpha^i_j(x_u) = 1$ when $x_u \in \langle a_i, b_i\rangle$, then u can be viewed as a C-class vertex and B^i_j can be rewritten as C^i_j, which makes such a case be that of $u \in C^i_j \cap C^{i+1}_\ell$.

Lemma 10 *For bottleneck decomposition $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$, it is impossible that $u \in B^i_j \cap C^{i+1}_\ell$ and $u \in C^i_j \cap B^{i+1}_\ell$.*

Proof: (sketch) For simplicity, we only show the impossibility of $u \in B^i_j \cap C^{i+1}_\ell$ and discuss the case that the corresponding intervals of $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ have the forms of $\langle a_i, b_i]$ and $(a_{i+1}, b_{i+1}\rangle$. There are two cases that $j < \ell$ and $j \geq \ell$. If $j < \ell$, then $V^i_j = V^{i+1}_j$ by Lemma 9. If $x_u = b_i$, then the bottleneck decomposition is $\mathcal{B}^i$ and (B^{i+1}_j, C^{i+1}_j) is a candidate pair of V^i_j, which induces $w(C^{i+1}_j)/w(B^{i+1}_j) \geq w(C^i_j)/(w(B^i_j \setminus \{u\}) + b_i)$ by Proposition 3-(3). But if x_u increases to $b_i + \epsilon \in (a_{i+1}, b_{i+1}\rangle$ for any small $\epsilon > 0$, then the bottleneck decomposition changes to be $\mathcal{B}^{i+1}$ and (B^i_j, C^i_j) is a candidate pair of V^{i+1}_j. So we have $w(C^i_j)/(w(B^i_j \setminus \{u\}) + b_i + \epsilon) \geq w(C^{i+1}_j)/w(B^{i+1}_j)$. Combining such two inequalities, $w(C^i_j)/(w(B^i_j \setminus \{u\}) + b_i + \epsilon) \geq w(C^i_j)/(w(B^i_j \setminus \{u\}) + b_i)$. It's a contradiction. The proof for case $j \geq \ell$ is similar. □

Lemma 10 demonstrates the impossibility of cases $u \in B^i_j \cap C^{i+1}_\ell$ and $u \in C^i_j \cap B^{i+1}_\ell$. So our focus turns to those that u is in the same classes in adjacent $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$.

Lemma 11 *For bottleneck decomposition $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ corresponding to intervals $\langle a_i, b_i\rangle$ and $\langle a_{i+1}, b_{i+1}\rangle$,*
1. if $u \in B^i_j$ and $u \in B^{i+1}_\ell$, then $\ell \leq j$;
2. if $u \in C^i_j$ and $u \in C^{i+1}_\ell$, then $\ell \geq j$.

Proof:(sketch) For the first claim, we suppose to the contrary that $\ell > j$ which guarantees $V^i_j = V^{i+1}_j$ by Lemma 9. So (B^i_j, C^i_j) is a candidate pair in V^{i+1}_j if $\widehat{x}_u \in \langle a_{i+1}, b_{i+1}\rangle$ and we can get the first inequality of (3). Similarly, the role of candidate of (B^{i+1}_j, C^{i+1}_j) in V^i_j if $x_u \in \langle a_i, b_i\rangle$ promises the second inequality of (3)

$$\frac{w(C^i_j)}{\widehat{x}_u + w(B^i_j \setminus \{u\})} \geq \frac{w(C^{i+1}_j)}{w(B^{i+1}_j)} \geq \frac{w(C^i_j)}{x_u + w(B^i_j \setminus \{u\})} \quad (3)$$

It's a contradiction since $\widehat{x}_u > x_u$. □

Lemma 11 shows that the relationship between the indexes of pairs which contain u, if u is in the same classes in adjacent $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$. Furthermore, the following claims will concretely describe the relation between B^i_j and B^{i+1}_ℓ, C^i_j and C^{i+1}_ℓ. The key of proofs is how to analyze the transient changes of bottleneck decompositions at the break point $x_u = b_i = a_{i+1}$.

Lemma 12 *For bottleneck decomposition $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ and $u \in B^i_j \cap B^{i+1}_\ell$,*
1. if corresponding adjacent intervals have the forms as $\langle a_i, b_i]$ and $(a_{i+1}, b_{i+1}\rangle$, then $u \in B^i_\ell$, $B^i_\ell = B^{i+1}_\ell \cup B^{i+1}_{\ell+1}$ and $C^i_\ell = C^{i+1}_\ell \cup C^{i+1}_{\ell+1}$.
2. if corresponding adjacent intervals have the forms as $\langle a_i, b_i)$ and $[a_{i+1}, b_{i+1}\rangle$, then $u \in B^i_{\ell+1}$, $B^{i+1}_\ell = B^i_\ell \cup B^i_{\ell+1}$ and $C^{i+1}_\ell = C^i_\ell \cup C^i_{\ell+1}$.

Proof: Here we prove the first claim in two steps: 1. $u \in B^i_\ell$; and 2. $B^i_\ell = B^{i+1}_\ell \cup B^{i+1}_{\ell+1}$. The proof for the second one is similar. Lemma 11 guarantees that index $\ell \leq j$ which implies $V^i_\ell = V^{i+1}_\ell$ by Lemma 9. Now Suppose to the contrary that $u \notin B^i_\ell$. Let us focus on the break point that $x_u = b_i$. At this point, the bottleneck decomposition of G is $\mathcal{B}^i$ and the candidate role of $(B^{i+1}_\ell, C^{i+1}_\ell)$ in V^i_ℓ keeps that

$$\alpha^{i+1}_\ell(b_i) = \frac{w(C^{i+1}_\ell)}{w(B^{i+1}_\ell \setminus \{u\}) + b_i} \geq \frac{w(C^i_\ell)}{w(B^i_\ell)}. \quad (4)$$

If the strict inequality in (4) holds, then there must exist a positive number $\epsilon > 0$ such that $b_i + \epsilon = a_{i+1} + \epsilon \in (a_{i+1}, b_{i+1}\rangle$ and

$$\alpha^{i+1}_\ell(b_i + \epsilon) = \frac{w(C^{i+1}_\ell)}{w(B^{i+1}_\ell \setminus \{u\}) + (b_i + \epsilon)} > \frac{w(C^i_\ell)}{w(B^i_\ell)}. \quad (5)$$

As we know, once weight x_u increases up to $b_i + \epsilon = a_{i+1} + \epsilon \in (a_{i+1}, b_{i+1}\rangle$, the bottleneck decomposition shall change to be $\mathcal{B}^{i+1}$. As a candidate pair in V^{i+1}_ℓ, the ratio of (B^i_ℓ, C^i_ℓ) can not be strictly less than $\alpha^{i+1}_\ell(b_i + \epsilon)$ which induces the impossibility of (5) and the strict inequality of (4). So

$$\alpha^{i+1}_\ell(b_i) = \frac{w(C^{i+1}_\ell)}{w(B^{i+1}_\ell \setminus \{u\}) + b_i} = \frac{w(C^i_\ell)}{w(B^i_\ell)}. \quad (6)$$

Then $B^{i+1}_\ell \subseteq B^i_\ell$ by Proposition 3-(3) and $u \in B^i_\ell$, because $u \in B^{i+1}_\ell$. Furthermore, we confirm that $B^{i+1}_\ell \subset B^i_\ell$. Otherwise the pairs containing u are the same in $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ which leads to $\mathcal{B}^i = \mathcal{B}^{i+1}$. It's contradicts the partition of $(0, w_u]$.

Next, we shall show that $B_\ell^i = B_\ell^{i+1} \cup B_{\ell+1}^{i+1}$. For this purpose, let us define $B^c = B_\ell^i - B_\ell^{i+1} \neq \emptyset$ and $C^c = C_\ell^i - C_\ell^{i+1} \neq \emptyset$. We can see that (B^c, C^c) is a candidate pair in $V_{\ell+1}^{i+1}$. Moreover, we have

$$\frac{w(C_\ell^i)}{w(B_\ell^i)} = \frac{w(C_\ell^{i+1})}{w(B_\ell^{i+1} \setminus \{u\}) + b_i} = \frac{w(C^c)}{w(B^c)} \geq \frac{w(C_{\ell+1}^{i+1})}{w(B_{\ell+1}^{i+1})}, \tag{7}$$

where the first equality comes from (6), the second one is from a simple arithmetic calculation and Proposition 3-(3) promises the last inequality. Now let us discuss the inequality

$$\frac{w(C_\ell^{i+1})}{w(B_\ell^{i+1} \setminus \{u\}) + b_i} \geq \frac{w(C_{\ell+1}^{i+1})}{w(B_{\ell+1}^{i+1})} = \alpha_{\ell+1}^{i+1}. \tag{8}$$

If the strict inequality of (8) is right, then we can also find a positive number $\epsilon > 0$ such that $b_i + \epsilon = a_{i+1} + \epsilon \in (a_{i+1}, b_{i+1}\rangle$ and

$$\alpha_\ell^{i+1}(b_i + \epsilon) = \frac{w(C_\ell^{i+1})}{w(B_\ell^{i+1} \setminus \{u\}) + (b_i + \epsilon)} > \frac{w(C_{\ell+1}^{i+1})}{w(B_{\ell+1}^{i+1})}. \tag{9}$$

It is easy to see that the bottleneck decomposition of G is $\mathcal{B}^{i+1}$ when $x_u = b_i + \epsilon = a_{i+1} + \epsilon \in (a_{i+1}, b_{i+1}\rangle$. So by the increasing monotonicity property of α-ratios with indexes, we have $\alpha_\ell^{i+1}(b_i + \epsilon) < \alpha_{\ell+1}^{i+1}$ which illustrates the incorrectness of (9). So the equality of (8) holds and

$$\frac{w(C^c)}{w(B^c)} = \frac{w(C_\ell^{i+1})}{w(B_\ell^{i+1} \setminus \{u\}) + b_i} = \frac{w(C_{\ell+1}^{i+1})}{w(B_{\ell+1}^{i+1})}.$$

Therefore $B^c \subseteq B_{\ell+1}^{i+1}$ by Proposition 3-(3). On the other hand, in order to keep the maximality of B_ℓ^i's size, there is only unique possibility that $B^c = B_{\ell+1}^{i+1}$ and $B_\ell^i = B_\ell^{i+1} \cup B_{\ell+1}^{i+1}$. □

Note that, with the increasing of x_u from $\langle a_i, b_i\rangle$ to $\langle a_{i+1}, b_{i+1}\rangle$, we can visualize the operations as *split* and *combine* to describe the changes of pairs containing u. Naturally, we can imagine that there are similar results for the case that $u \in C_j^i \cap C_\ell^{i+1}$.

Lemma 13 *For bottleneck decomposition $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ and $u \in C_j^i \cap C_\ell^{i+1}$,*
1. if corresponding adjacent intervals have the forms as $\langle a_i, b_i]$ and $(a_{i+1}, b_{i+1}\rangle$, then $u \in C_{j+1}^{i+1}$, $B_j^i = B_j^{i+1} \cup B_{j+1}^{i+1}$ and $C_j^i = C_j^{i+1} \cup C_{j+1}^{i+1}$.
2. if corresponding adjacent intervals have the forms as $\langle a_i, b_i)$ and $[a_{i+1}, b_{i+1}\rangle$, then $u \in C_j^{i+1}$, $B_j^{i+1} = B_j^i \cup B_{j+1}^i$ and $C_j^{i+1} = C_j^i \cup C_{j+1}^i$.

3.2 Monotonicity of utility function on $(0, w_u]$

For the sake of convenience, we use $U_u(x_u)$ and $\alpha_u(x_u)$ to denote the utility function and the α-ratio function of agent u for any given $\mathbf{x}_{-u}$ respectively, where $x_u \in (0, w_u]$. As before shown in (1), $U_u(x_u) = x_u \cdot \alpha_u(x_u)$ if u is in B-class; otherwise $U_u(x_u) = x_u/\alpha_u(x_u)$.

First, we discuss the monotone property of $U_u(x_u)$ in any single subinterval $\langle a_i, b_i\rangle$. Then the bottleneck decomposition of G is $\mathcal{B}^i = \{(B_1^i, C_1^i), \cdots, (B_{k^i}^i, C_{k^i}^i)\}$ when $x_u \in \langle a_i, b_i\rangle$. So agent u may be in B_j^i or C_j^i, $j = 1, 2, \cdots, k^i$. To be specific, its α-ratio is

$$\alpha_u(x_u) = \alpha_j^i(x_u) = \begin{cases} \frac{w(C_j^i)}{x_u + w(B_j^i \setminus \{u\})}, & u \in B_j^i; \\ \frac{x_u + w(C_j^i \setminus \{u\})}{w(B_j^i)}, & u \in C_j^i. \end{cases} \tag{10}$$

It is easy to observe that the α-ratio of u is monotonically increasing if it is a C-class vertex; and monotonically decreasing, otherwise. The utility function of agent u is

$$U_u(x_u) = \begin{cases} \frac{x_u w(C_j^i)}{x_u + w(B_j^i \setminus \{u\})}, & u \in B_j^i; \\ \frac{x_u w(B_j^i)}{x_u + w(C_j^i \setminus \{u\})}, & u \in C_j^i, \end{cases}$$

which is continuous on $\langle a_i, b_i\rangle$ and derivable on (a_i, b_i). Furthermore, the derivative function of $U_u(x_u)$ can be computed easily, which is nonnegative on $\langle a_i, b_i\rangle$. Combining the continuity of $U_u(x_u)$ on $\langle a_i, b_i\rangle$, we have

Lemma 14 *For any agent u, any weight profile $\mathbf{x}_{-u}$, and any single interval $\langle a_i, b_i\rangle \subset (0, w_v]$, utility function $U_u(x_u)$ is monotonically nondecreasing on $x_u \in \langle a_i, b_i\rangle$.*

Next we shall challenge the monotonicity of $U_u(x_u)$ on the whole interval $(0, w_u]$. One of the biggest difficulties we are facing is that the α-ratio function and the utility function of agent u may vary with the bottleneck decompositions. If we can investigate the continuity of α-ratio and utility function of agent u at each break point, the monotone property of $U_u(x_u)$ can be achieved by Lemma 14.

Lemma 15 *For bottleneck decomposition $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ and $u \in B_j^i \cap B_\ell^{i+1}$ or $u \in C_j^i \cap C_\ell^{i+1}$, the utility function $U_u(x_u)$ is continuous at break point.*

Proof: Here we only show the continuity of $U_u(x_u)$ at break point $x_u = b_i$ for the case that $u \in B_j^i \cap B_\ell^{i+1}$ and the adjacent intervals have the forms as $\langle a_i, b_i]$ and $(a_{i+1}, b_{i+1}\rangle$. Under this case, Lemma 12 promises $u \in B_\ell^i$. So $\alpha_u(x_u) = \alpha_\ell^i(x_u)$ if $x_u \in \langle a_i, b_i]$ and $\alpha_u(x_u) = \alpha_\ell^{i+1}(x_u)$ if $x_u \in (a_{i+1}, b_{i+1}\rangle$. Furthermore, equation (6) in the proof of Lemma 12 tells us that if $x_u = b_i$

$$\alpha_\ell^i(b_i) = \frac{w(C_\ell^i)}{w(B_\ell^i \setminus \{u\}) + b_i} = \frac{w(C_\ell^{i+1})}{w(B_\ell^{i+1} \setminus \{u\}) + b_i} = \alpha_\ell^{i+1}(b_i)$$

Therefore, the continuity of α-ratio function at break point $x_u = b_i$ holds as

$$\begin{aligned} \lim_{\epsilon \to 0^+} \alpha_u(b_i + \epsilon) &= \lim_{\epsilon \to 0^+} \alpha_\ell^{i+1}(b_i + \epsilon) = \alpha_\ell^{i+1}(b_i) \\ &= \alpha_\ell^i(b_i) = \alpha_u(b_i), \end{aligned}$$

And the utility function is also continuous at $x_u = b_i$,

$$\begin{aligned} \lim_{\epsilon \to 0^+} U_u(b_i + \epsilon) &= \lim_{\epsilon \to 0^+} (b_i + \epsilon) \cdot \alpha_\ell^{i+1}(b_i + \epsilon) \\ &= b_i \cdot \alpha_\ell^i(b_i) = U_u(b_i). \end{aligned}$$

□

Combining the monotone property in each subinterval and the continuity at each break point, the monotonicity of $U_u(x_u)$ on whole interval $(0, w_u]$ is derived directly.

Corollary 16 *For any agent u and any given weight profile $\mathbf{x}_{-u}$, utility function $U_u(x_u)$ is monotonically nondecreasing on $(0, w_u]$.*

Obviously, the monotone property of $U_u(x_u)$ in Corollary 16 for any given $\mathbf{x}_{-u}$ ensures the correctness of $U_u(w_u, \mathbf{x}_{-u}) \geq U_u(x_u, \mathbf{x}_{-u})$, where $x_u \in (0, w_u]$. So the main result of this paper is deduced directly.

Theorem 17 *BD Mechanism is truthful for the resource sharing game.*

Remarks: The strategy of weight misreporting cannot be replaced by the strategy of edge cutting [Cheng *et al.*, 2015]. Let us consider an example of a triangle in which three agents have the same weights. Obviously if one edge is cut, the resources can only be allocated along the remaining edges. But if one agent plays the strategy of weight misreporting, the resources are still be allocated on each edge whatever $x_u \in (0, w_u]$.

4 Numerical Example

In this section, we analyze a representative numerical example to have an intuitive understanding on the above results. Consider the network of Fig. 1. Each cycle represents a vertex and the number in the cycle is depicted the vertex's weight. As stated before, the bottleneck decomposition of

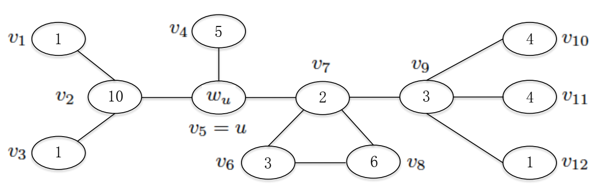

Figure 1: A network with 12 vertices where v_5 is the agent u who has a strategic move to misreport its own resource.

G shall change with agent u's reporting weight x_u for any given $\mathbf{x}_{-u}$. So the interval $(0, w_u]$ is partitioned into different disjoint subintervals, each $\langle a_i, b_i\rangle$ corresponding to one decomposition $\mathcal{B}^i$. For the example shown in Fig. 1, Table 1 lists all subintervals and all pairs containing u in different subintervals.

From Table 1, there are some important informations worthy of note. First, u is in the same classes in adjacent $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$ that verifies Lemma 10. The relation between adjacent B-sets or C-sets is that of containing and contained which has been proved in Lemma 12 and 13. Second in non-adjacent decompositions, u may be in different classes. For example, when $x_u \in [1, 3)$, u is in C-class with $\alpha_u(x_u) = (2 + x_u)/15 < 1$. And when $x_u \in [16, 32]$, then u is a B-class vertex in B_2 with $\alpha_u(x_u) = 20/(8 + x_u) < 1$. So there must be a crucial decomposition $\mathcal{B}^i$. In such a decomposition, u can be viewed as a B-class and a C-class vertex simultaneously which means that $\alpha_u(x_u) = 1$ if $x_u \in \langle a_i, b_i\rangle$. Obviously in Table 1, $x_u = 13$ just is the turning point and the corresponding decomposition is the crucial one. Third, if the class containing u changes with $x_u \in (0, w_u]$, then the changing process must be from C-class to B-class with the increasing of x_u, not vice versa.

x_u	B	C
$(0, 1)$	$\{v_4\}$	$\{\mathbf{u}\}$
$[1, 3)$	$\{v_2, v_4\}$	$\{v_1, v_3, \mathbf{u}\}$
3	$\{v_2, v_4, v_{10}, v_{11}, v_{12}\}$	$\{v_1, v_3, \mathbf{u}, v_9\}$
(3, 10.5)	$\{v_2, v_4\}$	$\{v_1, v_3, \mathbf{u}\}$
10.5	$\{v_2, v_4, v_8\}$	$\{v_1, v_3, \mathbf{u}, v_6, v_7\}$
(10.5, 13)	$\{v_2, v_4\}$	$\{v_1, v_3, \mathbf{u}\}$
13	$\{v_1, v_2, v_3, v_4, \mathbf{u}\}$	$\{v_1, v_2, v_3, v_4, \mathbf{u}\}$
$(13, 16)$	$\{v_1, v_3, \mathbf{u}\}$	$\{v_2, v_4\}$
$[16, 32]$	$\{v_1, v_3, \mathbf{u}, v_8\}$	$\{v_2, v_4, v_6, v_7\}$
$(32, 49)$	$\{v_1, v_3, \mathbf{u}\}$	$\{v_2, v_4, v_7\}$
49	$\{v_1, v_3, \mathbf{u}, v_{10}, v_{11}, v_{12}\}$	$\{v_2, v_4, v_7, v_9\}$
$(49, \infty)$	$\{v_1, v_3, \mathbf{u}\}$	$\{v_2, v_4, v_7\}$

Table 1: All subintervals, each corresponding one decomposition. The second and third column represent the pair (B, C) which u is in, where u is written in bold.

The left of Fig. 2 well illustrates the property of continuity and monotonically nondecreasing property of $U_u(x_u)$. But for α-ratio shown in the right of Fig. 2, $\alpha_u(x_u)$ is monotonically increasing when $x_u < 13$ and it is monotonically decreasing when $x_u > 13$. This result just coincides with the facts that u is in C-class when $x_u < 13$ and in B-class when $x_u > 13$ shown in Table 1. In addition, $\alpha_u(x_u)$ reaches its peak at $x_u = 13$ as $\alpha_u(x_u) \leq 1$.

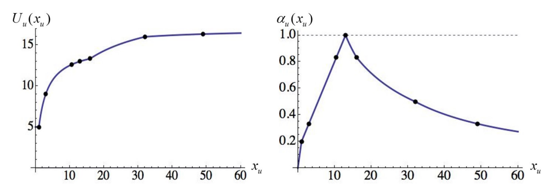

Figure 2: The figures of utility function $U_u(x_u)$ and α-ratio function $\alpha_u(x_u)$.

5 Conclusion

In this article, we discuss the issue of possible strategic manipulations of agents with respect to BD Mechanism for the application of resource exchange. Our work resolves an open problem on the strategic stability of a resource exchange protocol (i.e., the BD mechanism) from the mechanism design perspective.

We show that, no agent could gain by misreporting its resource amount in a BD Mechanism. Combining the work of Cheng, et al., [Cheng *et al.*, 2015], we establish a strong incentive stability result that an agent could not improve its utility by cutting off any incident edge or by reporting less resource. We note that an agent could not benefit even from the combination of the above two strategies. If an agent does so, it is equivalent to a two-stage strategy. At the first stage, the agent cuts some of the adjacent edges. At the second stage, it decreases its weight in the updated network. The results in [Cheng *et al.*, 2015] and in our paper show that the utility of this agent would be non-increasing. So this completes the research on truthfulness of the popular proportional mechanism on resource exchange.

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