

# Connecting Qualitative Spatial and Temporal Representations by Propositional Closure

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## Abstract

This paper establishes new relationships between existing qualitative spatial and temporal representations. Qualitative spatial and temporal representation (QSTR) is concerned with abstractions of infinite spatial and temporal domains, which represent configurations of objects using a finite vocabulary of relations, also called a qualitative calculus. Classically, reasoning in QSTR is based on constraints. An important task is to identify decision procedures that are able to handle constraints from a single calculus or from several calculi. In particular the latter aspect is a longstanding challenge due to the multitude of calculi proposed. In this paper we consider propositional closures of qualitative constraints which enable progress with respect to the longstanding challenge. Propositional closure allows one to establish several translations between distinct calculi. This enables joint reasoning and provides new insights into computational complexity of individual calculi. We conclude that the study of propositional languages instead of previously considered purely relational languages is a viable research direction for QSTR leading to expressive formalisms and practical algorithms.

## 1 Introduction

The field of qualitative spatial and temporal representation and reasoning (QSTR) draws its motivation from several application contexts. Qualitative representations employ a relational language to state relations between spatial or temporal entities, using a finite and often small vocabulary of concepts. Practitioners may design a qualitative representation to capture human-like concepts without being concerned about the computational effects of design decisions, while theoretical works aim to fathom the dependencies between domain representation and computational complexity of the resulting representation. Renz [2002] claimed that computational aspects in QSTR are under-explored and the claim may be regarded as valid up to today since not all representations proposed so far have been analyzed yet.

A more prevailing problem in QSTR stems from the contrast that a high specificity of a qualitative representation fos-

ters efficient reasoning algorithms, but applications often require considerations of several aspects at the same time, for example a robot reasoning about its relative orientation within its working environment as well as the arrangement of objects in the environment in terms of cardinal directions. To master this requirement, not only is a deep understanding of computational properties required, but we need to understand the interdependency of distinct formalisms.

In this paper we address the challenge from a point of view that is helpful for applications: we identify mappings between formalisms that allow statements in one formalism to be translated into another, retaining reference to the same domain objects. We particularly consider a class of mappings that rewrites relations from one formalism as Boolean combination of relations in the other formalism. This approach often does not affect the computational complexity class since most formalisms are at least NP-complete, but it allows expressive power to be increased and several translations to be established.

The approach taken in this paper is related to, but more general than the approaches of Kreuzmann and Wolter [2014] and Jonsson and Bäckström [1998] that consider translations to disjunctions of linear inequalities and the approach by Westphal and Wöfl [2009] that considers translations of qualitative constraint problems to SAT. In contrast to the previous works (i) we consider translations between any reasoning algorithm capable of handling a specific qualitative calculus and (ii) we are also involved with propositional closures, which are more expressive. We expect that studying the propositional closure of calculi is also valuable for research in spatial logics [Aiello *et al.*, 2007], which take a broader perspective on logic formalisms in spatial and temporal reasoning. In this broader view, first-order definable relations are often studied and several full complexity classifications exist in many cases, most notably the case of temporal constraints definable in  $\langle \mathbb{Q}, < \rangle$  [Bodirsky and Kára, 2010]. Thus, this paper focuses on an intermediate level between qualitative constraint calculi and first-order or algebraic constraint languages.

The remainder of this paper is organized as follows. Section 2 defines and introduces qualitative representations of space and time, their propositional closure, and introduces translations. Section 3 then establishes translations by giving proof sketches. Readers interested in the results may advance

Table 1: Overview of qualitative calculi considered in this paper

	Abbreviation, Name	Reference	Base entity	Aspect	Example statement
Temporal	PC, Point Calculus	[Vilain and Kautz, 1986]	point in $n$ -d	total order	$x < y$
	DepCalc, Dependency Calculus	[Ragni and Scivos, 2005]	point	partial order	joint past of $x$ and $y$
	IA, Allen's Interval Algebra	[Allen, 1983]	intervals in 1-d	total order	$x$ overlaps with $y$
	INDU	[Pujari <i>et al.</i> , 1999]	intervals in 1-d	order & duration	shorter $x$ meets longer $y$
	SIC	[Freksa, 1992a]	semi-intervals in 1-d	total order	$x$ ends after $y$
	GenInt, Generalized IA	[Condotta, 2000]	tuples of intervals	total order	$x$ before and after $y$
	DIA, Directed Intervals Algebra	[Renz, 2001]	directed 1-d intervals	order/orientation	$x$ overlaps from behind $= y$
Spatial	CDC, Cardinal Dir. Calc.	[Ligozat, 1998]	point in $n$ -d	total order	$x$ below $y$
	BA, Block Algebra	[Balbiani <i>et al.</i> , 1998]	$n$ -d bounding box	total order	$x$ overlaps $y$ at top
	CDR, Cardinal Direction Rel's.	[Skiadopoulos and Koubarakis, 2005]	simple regions in 2-d	direction	$x$ N and W of $y$
	RCD, Rectangle Cardinal Dir.	[Navarrete <i>et al.</i> , 2013]	simple regions in 2-d	direction	$x$ N and W of $y$
	STAR	[Renz and Mitra, 2004]	points in 2-d	direction	$x$ NNW of $y$
	SV, StarVars	[Lee <i>et al.</i> , 2013]	oriented points in 2-d	relative dir.	$x$ front left of $y$
	LR	[Scivos and Nebel, 2005]	points in 2-d	relative dir.	$x$ left of $\overline{xy}$
	1-,2-cross	[Freksa, 1992b]	points in 2-d	relative dir.	$z$ is front left of $\overline{xy}$
	OPRA	[Mossakowski and Moratz, 2012]	oriented points in 2-d	relative dir.	$x$ front left of $y$
	EOPRA, Elevated OPRA	[Moratz and Wallgrün, 2012]	oriented points in 2-d	dir. & distance	$x$ far left of $y$
	TPCC, Ternary Point Config. Calc.	[Moratz and Ragni, 2008]	points in 2-d	dir. & distance	$x$ is left, between $\overline{xy}$
	LOS, Lines of Sight	[Galton, 1994]	2-d regions	occlusion	$x$ partially occluded by $y$
	VR, Visibility Relations	[Tarquini <i>et al.</i> , 2007]	simple regions in 2-d	occlusion	$x$ in shadow of $y$ wrt. $z$
	RCC-5,-8, Region Connection Calc.	[Randell <i>et al.</i> , 1992]	regions	topology	$x$ partially overlaps $y$

to Section 4 where we summarize the results of this paper, followed by a conclusion discussing the impact on future research in the field.

## 2 Qualitative Spatial and Temporal Representations

Our work is motivated by the diversity of qualitative formalisms developed by the QSTR community so far. A central notion in this area is that of a *qualitative calculus* [Ligozat and Renz, 2004] which, in context of investigating expressiveness of formalisms can be summarized for binary relations as follows (symbolic operations usually considered for qualitative calculi are not relevant to our work and are omitted in the definition; the definition naturally generalizes to  $n$ -ary relations).

**Definition 1.** A binary *qualitative calculus*  $\mathbf{C}$  is a tuple  $\langle \mathcal{R}, \iota, D \rangle$ , where  $\mathcal{R}$  is a finite set of relational symbols that are interpreted by  $\iota : \mathcal{R} \rightarrow D \times D$  with the property that any pair  $(x, y)$  from  $D \times D$  is in exactly one relation  $R$  from  $\mathcal{R}$ . The powerset  $2^{\mathcal{R}}$  is called the set of *composite relations*. Semantics for composite relations is defined disjunctively by means of set union:  $\iota(\{R_1, \dots, R_k\}) := \iota(R_1) \cup \dots \cup \iota(R_k)$ .

To gain an overview of the distinct calculi considered in this paper, Table 1 presents a brief overview, grouping calculi by their domain and spatial or temporal aspects. We have selected the calculi with the aim to present a cross section of the variety of existing calculi and also to include calculi for which no computational properties are published so far (VR, EOPRA). For each calculus we additionally give an example statement that is meant to convey an intuitive example of what kind of knowledge can be captured with the calculus. Precise definitions of relations can be found in the references and are also given in context of translations in the next section.

Knowledge representations based on qualitative calculi utilizes constraints written  $(x R y)$ , in which  $R \in 2^{\mathcal{R}}$  is a com-

posite relation and  $x, y$  are variables ranging over domain  $D$ . A constraint is called atomic if  $|R| = 1$  holds. The constraint satisfaction problem in qualitative reasoning for a calculus  $\mathbf{C}$ , written  $\text{CSP}(\mathbf{C})$ , is to decide joint satisfiability of a set of constraints. Although composite relations can represent disjunctions, their expressiveness is restricted to a single pair of variables. For example it is not possible to state the disjunction of constraints  $(x R y)$  and  $(y R' z)$ . In this paper we relax this restriction and consider a more general representation allowing arbitrary Boolean combinations of constraints.

**Definition 2** (Propositional closure). Let  $\mathbf{C} = \langle \mathcal{R}, \iota, D \rangle$  be a qualitative calculus. The *propositional closure* of  $\mathbf{C}$  is a language  $\mathcal{L}_{\mathbf{C}}$  that involves in addition to the relations from  $\mathbf{C}$  Boolean connectives  $\neg, \wedge, \vee$  and a countably infinite set  $V = \{v_1, v_2, \dots\}$  of variables. A formula of  $\mathcal{L}_{\mathbf{C}}$ , or an  $\mathcal{L}_{\mathbf{C}}$ -formula, is a Boolean combination of relational statements of the form  $(v_i R v_j)$ .

We say  $\mathcal{L}_{\mathbf{C}}$ -formula  $\varphi$  is satisfiable if there exists a valuation  $m : V \rightarrow D$ , called model, such that  $\varphi$  evaluates to True by first applying the rewriting  $(v_i R v_j) \mapsto \text{True}$  iff  $(m(v_i), m(v_j)) \in \iota(R)$  and False otherwise, then evaluating it as a usual propositional formula. If  $m$  lets  $\varphi$  evaluate to True we write  $m \models \varphi$ .

In this paper we are concerned with languages  $\mathcal{L}_{\mathbf{C}}$  that are propositional closures of qualitative calculi  $\mathbf{C}$ . The problem of deciding whether an  $\mathcal{L}_{\mathbf{C}}$ -formula is satisfiable, written  $\text{SAT}(\mathcal{L}_{\mathbf{C}})$  is closely related to  $\text{CSP}(\mathbf{C})$ . An  $\mathcal{L}_{\mathbf{C}}$ -formula of the form  $\bigwedge_{ij} v_i R_{ij} v_j$  is called *atomic*, its satisfiability can be decided by a decision procedure for  $\text{CSP}(\mathbf{C})$ . If the decision procedure runs in (nondeterministic) polynomial-time for atomic  $\mathcal{L}_{\mathbf{C}}$ -formulas, then  $\text{SAT}(\mathcal{L}_{\mathbf{C}})$  can be solved in NP by means of a backtracking search. This typically puts  $\text{SAT}(\mathcal{L})$  in the same complexity class as  $\text{CSP}(\mathbf{C})$  which is usually at least NP-complete due to disjunctions in composite relations.

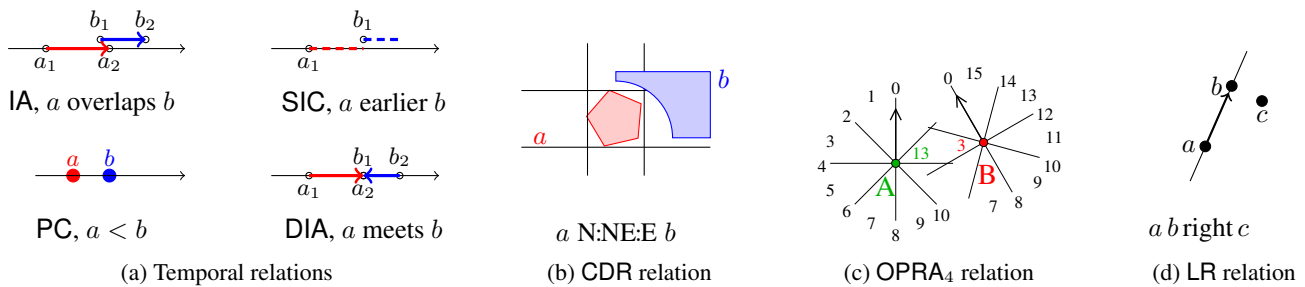


Figure 1: Some qualitative spatial and temporal representations.

An interesting aspect in the analysis of knowledge representation languages is studying their expressive power. Two qualitative spatial (or temporal) representations can differ in their domains but still allow capturing the same kind of information. Relationships that allow reasoning tasks posed in language  $\mathcal{L}_1$  to be reduced to  $\mathcal{L}_2$  teach about computational complexity and can also be exploited by practical systems which can avoid implementations of specialized procedures for  $\mathcal{L}_1$ . Applications using qualitative representations may also be interested in concrete models of formulas, for example, for the purpose of communicating decisions. To this end, we define *translations*<sup>1</sup> as reductions that respect models:

**Definition 3.** Let  $\mathcal{L}_1, \mathcal{L}_2$  be propositional closures according to Definition 2. We say that  $\mathcal{L}_1$  translates to  $\mathcal{L}_2$ , denoted  $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ , if there exists a polynomial-time mapping  $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  that presents a polynomial-time many-one reduction of  $\text{SAT}(\mathcal{L}_1)$  to  $\text{SAT}(\mathcal{L}_2)$  and a polynomial-time mapping  $g$  that translates models of  $\mathcal{L}_2$  back to  $\mathcal{L}_1$ , i.e.,  $m \models f(\varphi) \Rightarrow g(m) \models \varphi$ .

The requirement of polynomial-time model translations  $g$  is driven by a twofold motivation in practical applications:

- One might wish to jointly reason about knowledge represented by statements in either  $\mathcal{L}_1$  or  $\mathcal{L}_2$ . Existence of a reduction alone would not explain how to interpret a model computed for the resulting  $\mathcal{L}_2$ -formula in terms of  $\mathcal{L}_1$ .
- Among NP-complete qualitative calculi, existence of polynomial-time many-one reductions is already implied by NP-completeness and is thus not informative.

To give an example, let  $\mathcal{L}_1$  be a language representing visibility between 2D objects wrt. occlusion and let  $\mathcal{L}_2$  be another language representing directional relations between 2D points of interest and assume that  $\mathcal{L}_1 \rightarrow \mathcal{L}_2$  holds. A mobile robot capable of reasoning about occlusion in 2D environment for the purpose of navigation as described in [Fenelon *et al.*, 2012] may now—if  $\mathcal{L}_1 \rightarrow \mathcal{L}_2$  holds—combine this information with additional knowledge about directions in order to improve its navigation capabilities. In order to make use of a model computed for the single  $\mathcal{L}_2$ -formula obtained by combining a  $\mathcal{L}_1$ - and  $\mathcal{L}_2$ -formula, the resulting model needs to be interpreted in terms of concrete regions of visibility and in

<sup>1</sup>This relates to polynomial translatability in the sense of [Fargier *et al.*, 2013], but is tailored to our investigations.

terms of directions in a mutually consistent manner to be useful for the purpose of navigation. As we show in Section 3.3, this is possible for VR visibility relations and LR relative direction relations. Our definition allows for joint reasoning whenever interdependencies of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are expressible in  $\mathcal{L}_2$ . This includes, for example, integrations that can be handled using the bi-pathconsistency method [Wöfl and Westphal, 2009].

### 3 Connections between Qualitative Spatial and Temporal Representations

We now establish translations  $\mathcal{L}_{\mathcal{C}_1} \rightarrow \mathcal{L}_{\mathcal{C}_2}$  for calculi  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , briefly describing relations defined in the calculi to explain the constructions. Translations presented here have been selected to represent the four categories of Table 1. Results achieved are depicted graphically in Fig. 3 and further discussed in the next section. Any translation shown in the figure but not described in the following can be realized as a reduction among calculi and is described in the literature referenced in Table 1 for the respective calculus. To obtain connections that are not only reductions from some  $\text{SAT}(\mathcal{L}_{\mathcal{C}_1})$  to some  $\text{SAT}(\mathcal{L}_{\mathcal{C}_2})$  but translations which allow a  $\mathcal{L}_1$ -model to be derived from the resulting  $\mathcal{L}_{\mathcal{C}_2}$ -formula, the following proof ideas are constructive in the sense that they simulate  $\mathcal{L}_{\mathcal{C}_1}$  entities and relations using a  $\mathcal{L}_{\mathcal{C}_2}$ -formula. To this end, it is sufficient to consider atomic  $\mathcal{L}_{\mathcal{C}_1}$ -formulas.

#### 3.1 Temporal Calculi

The following relationships are well-known (see respective references in Table 1 and illustrations in Fig. 1a) but repeated here for completeness. Interval relations of IA have been defined using atomic formulas of point-based PC constraints (thus  $\text{IA} \rightarrow \text{PC}$ ), the converse direction directly follows from so-called ‘pointizable’ configurations in which intervals are not allowed to partially overlap and thus can represent points. Semi-interval relations in SIC are specified as disjunctions of IA relations, representing only the relative positions of the start points (respectively end points) of an interval. DIA extends the interval domain of IA by allowing intervals to point in either direction in time (rather 1-d space), hence refining IA relations to the relative orientation of intervals involved (same direction, head-on, departing). The additional expressivity can be simulated in IA by means of disjunctions ranging over the two possible orientations in 1-d, the other direction is

obvious. INDU refines IA by introducing an additional comparison by relative length, therefore  $IA \rightarrow INDU$  holds.

### 3.2 Cardinal Direction Calculi

All translations among cardinal direction calculi shown in Fig. 3 directly follow from the respective literature given in Table 1, except for the following translation that achieves a connection from CDR to BA and equivalent calculi.

BA is a generalization of IA to two dimensions obtained by considering the projections of regions on the x- and y-axis as intervals and applying IA to these, i.e., BA relations are pairs of IA relations. Equivalence of IA and BA with respect to translations is thus obvious.

CDR is an expressive calculus related to BA for describing the location of simple regions in 2-d using cardinal directions. Two variants of CDR have been considered: one restricted to simple regions and one addressing regions not necessarily connected. Relations in CDR list the cardinal direction relations using an 8-sector model plus proximity zone, in which one region is located with respect to another, see Fig. 1b. This leads to  $2^9$  atomic relations in case of unconnected regions and 218 in case of connected regions [Liu *et al.*, 2010]. Translations for the case of unconnected regions are straightforward by instantiating one box in BA for every sector mentioned in a CDR relation, so we focus on the case of connected regions.

#### Proposition 4. $CDR \rightarrow BA$

*Proof sketch.* Construction follows the algorithm described in Liu *et al.* [2010]. Their algorithm constructs a canonical solution by joining a set of rectangular regions represented as pixels on a bitmap of size  $2n \times 2n$  for  $n$  entities. Here we use BA entities, which are boxes. For each base relation in a CSP(CDR) we introduce a chain of BA entities and enforce each box in the chain to be connected to its successor by means of BA relations (meets,  $*$ ) or ( $*$ , meets) where  $*$  denotes the disjunction of all IA base relations. CDR base relations define a sequence of cardinal direction relations, e.g., N:NE:E as shown in Fig. 1b. We thus enforce the chain to follow these relations by restricting the  $i$ th element to the  $i$ th cardinal direction cell. It remains to be ensured that chains constructed for two different constraints but corresponding to the same CDR entity do not overlap with any direction not allowed by the CDR relations and that pairs of chains are connected, i.e., there exists an element in one overlapping an element in the other.  $\square$

### 3.3 Relative Direction Calculi

OPRA <sub>$m$</sub>  relates two oriented points with respect to their relative orientation towards each other [Mossakowski and Moratz, 2012], see Fig. 1c. An oriented point  $O$  is given by its Cartesian coordinates  $x_O, y_O \in \mathbb{R}$  and direction  $\phi_O \in [0, 2\pi]$  with respect to an absolute reference direction. OPRA <sub>$m$</sub>  calculus is suited for representation of objects that have an intrinsic front or move in a particular direction and can be abstracted as points. Relations in OPRA <sub>$m$</sub>  are sectors and linear relations equally dividing the full circle into  $2m$  planar and  $2m$  linear regions.

STAR <sub>$m$</sub>  is similar to OPRA <sub>$m$</sub> , except for the fact that all objects have the same directions (e.g., north). This has the

effect that CSPSAT(STAR <sub>$m$</sub> ) is in P while OPRA <sub>$m$</sub>  like almost any other directional calculus is NP-hard [Wolter and Lee, 2010]. Since we can construct basic axes using linear relations of OPRA <sub>$m$</sub>  objects and force the directions of all other variables to be aligned with the one of the grid axes, we have that

#### Proposition 5. $STAR_m \rightarrow OPRA_m$ .

EOPRA <sub>$m$</sub>  augments OPRA <sub>$m$</sub>  with distance constraints, retaining the same set of directional relations. Each entity in EOPRA <sub>$m$</sub>  has its own fixed threshold value  $\delta$  for determining the qualitative distance (e.g., close and far) to other objects. Thus the entities in EOPRA <sub>$m$</sub>  are defined as tuples  $(x, y, \theta, \delta) \in \mathbb{R}^2 \times [0, 2\pi) \times \mathbb{R}_0^+$ . An example of the distance constraint is illustrated in Fig. 2a. Distance constraints can however easily be simulated in OPRA <sub>$m$</sub> , thus we have that

#### Proposition 6. $OPRA_m \leftrightarrow EOPRA_m$ .

*Proof sketch.* Fig. 2a shows how a distance constraint of an EOPRA <sub>$m$</sub>  object  $P$  with respect to another EOPRA <sub>$m$</sub>  object  $Q$  can be emulated with the help of several dummy OPRA <sub>$m$</sub>  objects to construct a kite. The construction detailed in Fig. 2b is possible since EOPRA <sub>$m$</sub>  defines linear direction relations and superposition of objects with individual orientations. Two pairs of sides are of equal lengths in a kite. Its construction allows us to enforce that  $D_Q$  and  $D_R$  are in the same distance to  $P$ . Hence, by relating  $Q$  to  $D_Q$  (respectively  $R$  to  $D_R$ ) we can compare their distance to a specific distance class  $\delta$  associated with  $P$ .  $\square$

### 3.4 Occlusion Calculi

VR [Tarquini *et al.*, 2007] describes locations of convex regions in the plane with respect to visibility relations. VR employs the partition scheme depicted in Fig. 2c, distinguishing five basic classes of visibility. For example, a location  $p$  is said to be in the shadow zone (SZ) of region  $A$  with respect to region  $B$ , if every line segment connecting an interior point of  $B$  with  $p$  also coincides with an interior point of  $A$ . Visibility is thus considered to be a ternary concept. Visibility relations among regions are determined by the zones in which interior points of a region are located, for example a region may be partially overlapping SZ and LZ, the light zone determined by full visibility of the target object. Zones of partial visibility are called twilight zones and further distinguished by whether the left side (TZL), the middle (TZM), or the right side (TZR) is occluded for an imaginary observer located in the plane. By construction, all zones are derived from lines connecting tangent points of regions (see Fig. 2c). Visibility relations may be regarded as a generalization of cyclic ordering for extended objects as is also captured by the LR calculus.

LR [Scivos and Nebel, 2005] is a calculus representing ternary point configurations, see Fig. 1d for illustration. Most importantly, LR defines the relations *left* and *right*:  $ab$  left  $c$  holds iff  $c$  is positioned on the left-hand side of the straight line  $\vec{ab}$ , and  $ab$  right  $c$  iff  $c$  is positioned on the right-hand side of  $\vec{ab}$ . Additional relations defined for the case when  $a, b, c$  are collinear are not relevant to establish the following construction:

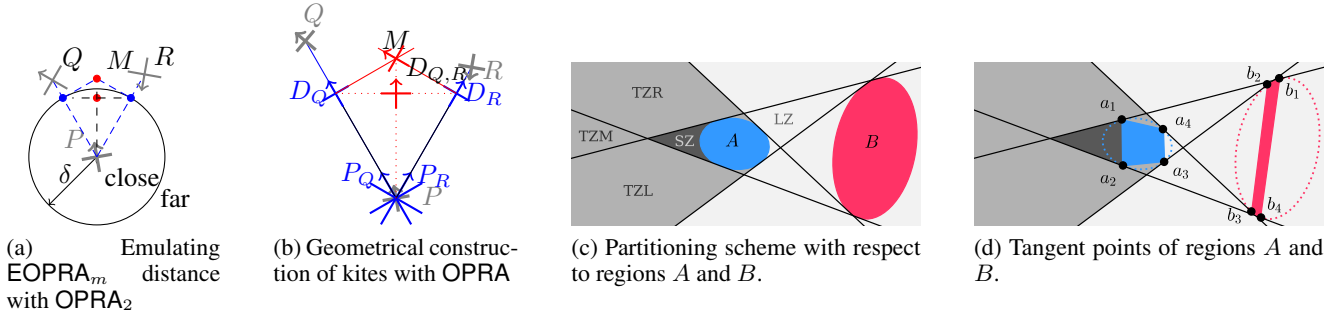


Figure 2: EOPRA<sub>m</sub> and VR

**Proposition 7.**  $VR \rightarrow LR$ .

*Proof sketch.* We first note that it suffices to consider the set of convex polygons as the domain of VR instead of the set of convex 2D regions, because only a finite number of extreme points of each region affects visibility (see Fig. 2d). More specifically, those extreme points of a convex region are determined by four tangent lines induced by this region and another region. Therefore, in a scene with  $n$  regions, a region can be represented by its  $m := 4(n - 1)$  tangent points and we have altogether  $m \cdot n$  points that determine a scene.

VR relations are ternary and we write  $ABRC$  to state that region  $C$  is located in region  $R \in \{TZL, TZM, TZR, SZ, LZ\}$  of  $A$  with respect to object  $B$ . Now we show the translation of a VR-term  $ABSZC$  into an LR-formula. Other VR-terms can be translated analogously. Note that since each VR variable is a polygon with  $m$  vertices, we can identify variables  $A, B, C$  with sequences of their vertices:  $A := (a_1, \dots, a_m), B := (b_1, \dots, b_m), C := (c_1, \dots, c_m)$ . For the resulting LR-formula we introduce quantifiers  $\forall$  and  $\exists$  for convenience; they can be replaced with  $\bigwedge$  and  $\bigvee$ , respectively.

$ABSZC$

$$\equiv \text{Conv}(A) \wedge \text{Conv}(B) \wedge \text{Conv}(C) \quad (1)$$

$$\wedge \exists a_{i_1} \exists a_{i_2} \exists a_{i_3} \exists a_{i_4} \exists b_{i_1} \exists b_{i_2} \exists b_{i_3} \exists b_{i_4} \quad (2)$$

$$(\text{Tangent}_1(a_{i_1}, b_{j_1}) \wedge \text{Tangent}_2(a_{i_2}, b_{j_2})) \quad (3)$$

$$\wedge \text{Tangent}_3(a_{i_3}, b_{j_3}) \wedge \text{Tangent}_4(a_{i_4}, b_{j_4}) \quad (4)$$

$$\wedge \forall c_i (a_{i_1} b_{j_1} \text{ right } c_i \wedge a_{i_2} b_{j_2} \text{ left } c_i) \quad (5)$$

$$a_{i_3} b_{j_3} \text{ right } c_i \wedge a_{i_4} b_{j_4} \text{ left } c_i) \quad (6)$$

$$\wedge \text{Outside}(A, c_i) \wedge a_{i_1} a_{i_4} \text{ right } c_i) \quad (7)$$

The explanation of the formula is as follows. In line (1) we impose convexity on the polygons, where

$\text{Conv}(A)$

$$\equiv a_1 a_2 \text{ left } a_3 \wedge a_2 a_3 \text{ left } a_4 \wedge \dots \wedge a_{m-1} a_m \text{ left } a_1$$

$$\wedge a_{m-1} a_m \text{ left } a_1 \wedge a_m a_1 \text{ left } a_2$$

Then we specify four pairs  $(a_{i_1}, b_{j_1}), \dots, (a_{i_4}, b_{j_4})$  of vertices of  $A$  and  $B$  that respectively determine the four tangent lines. In the case of the upper tangent from  $A$  to  $B$  the corresponding vertex pair  $(a_{i_1}, b_{j_1})$  is specified by imposing that all vertices  $a_i \in A$  with  $a_i \neq a_{i_1}$  and all vertices  $b_i \in B$  with

$b_i \neq b_{j_1}$  are on the right-hand side of the directed line  $\overrightarrow{a_{i_1} b_{j_1}}$  (in Fig. 2d the upper tangent is the directed line  $\overrightarrow{a_1 b_1}$ ). This specification is given in (2)–(4); here,  $\text{Tangent}_1$  determines the vertex pair for the upper tangent from  $A$  to  $B$ :

$$\text{Tangent}_1(a_{i_1}, b_{j_1})$$

$$\equiv \forall a_i (a_i \neq a_{i_1} \rightarrow a_{i_1} b_{j_1} \text{ right } a_i)$$

$$\wedge \forall b_j (b_j \neq b_{j_1} \rightarrow a_{i_1} b_{j_1} \text{ right } b_j)$$

The definitions for  $\text{Tangent}_2, \text{Tangent}_3, \text{Tangent}_4$  can be obtained by changing the combination of the two LR-relations (right, right) occurring in the definition of  $\text{Tangent}_1$  to combinations (right, left), (left, right), (left, left), respectively.

Finally, in (5)–(7) we specify the location of  $C$  by stating that it is inside the polygon induced by the four tangent lines  $\overrightarrow{a_{i_1} b_{j_1}}, \dots, \overrightarrow{a_{i_4} b_{j_4}}$ . Note that in (7) we impose that first,  $C$  does not overlap  $A$ , and second,  $C$  is behind  $A$ . The first condition is satisfied, if each vertex  $c_i$  of  $C$  is outside  $A$ :

$$\text{Outside}(A, c_i)$$

$$\equiv a_1 a_2 \text{ right } c_i \vee a_2 a_3 \text{ right } c_i \vee \dots \vee a_{m-1} a_m \text{ right } c_i$$

$$\vee a_{m-1} a_m \text{ right } c_i \vee a_m a_1 \text{ right } c_i$$

and the second condition is satisfied, if, in addition to the first condition, each vertex of  $C$  is to the right of directed line  $\overrightarrow{a_{i_1} a_{i_4}}$  (in Fig. 2d it is  $\overrightarrow{a_1 a_2}$ ).  $\square$

**Line-Of-Sight Calculus (LOS)** The Line-Of-Sight (LOS) calculus addresses the representation of occlusion from an observer-centered point of view. Entities in the calculus are regions in an observer's visual plane that result from the observation of physical objects and are thus assumed to be connected. Unlike RCC-8, regions in LOS cannot overlap but they are staggered in layers according to distance to the observer. LOS can be seen as a refined interpretation of RCC-8 relations in which all modes of overlap, i.e., PO, TPP(i), NTPP(i), and EQ need to distinguish which region is in front of the other. This leads to an increased set of 14 atomic relations (LOS does not define an additional equality relation, but replaces RCC-8's EQ with the pair exactly hides/ exactly hidden by).

**Proposition 8.**  $LOS \leftrightarrow RCC-8$

*Proof sketch.* Technically speaking, LOS augments RCC-8 by a (depth) order for non-discrete entities. Assume topologi-

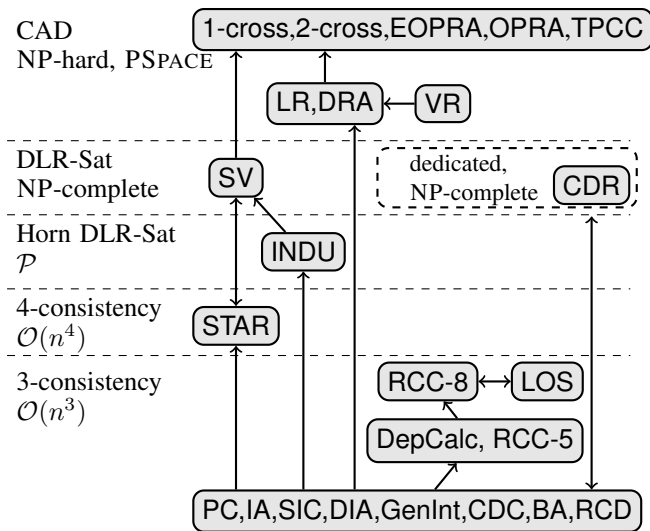


Figure 3: The complexity landscape of qualitative temporal and spatial calculi by their decision procedure for atomic formulas. A link  $A \rightarrow B$  is shown in case a translation in the sense of Def. 3 is known for the corresponding propositional closures. Calculi equivalent with respect to translations are grouped.

cal information present in LOS to be represented using RCC-8. To achieve LOS expressive power we additionally need to simulate its depth ordering: introduce a fresh depth variable  $D_{X_i}$  for every variable  $X_i$  and for every LOS constraint stating  $X_i$  to be in front of some  $X_j$  introduce the RCC-8 ordering constraint  $(D_{X_i} \text{ NTPP}, D_{X_j})$ , and, analogously, using NTPPi in case  $X_i$  is behind  $X_j$ . This procedure is sufficient since LOS does not define equality, i.e., one cannot enforce two regions to be at the same distance to the observer.  $\square$

## 4 Summary and Conclusion

This paper considers propositional closure of qualitative calculi as means for representing spatial (or temporal) knowledge. The main motivations of studying Boolean combinations of constraints is that they allow several interesting translations to be defined which cannot be exploited within the framework of qualitative calculi due to a lack of expressivity.

The main result of this paper is summarized in Fig. 3 which depicts individual calculi  $\mathcal{C}$  arranged according to the reasoning algorithm required to solve  $\text{SAT}(\mathcal{L}_{\mathcal{C}})$  for atomic formulas. The vertical arrangement in layers separated by dashed lines also reflects the computational complexity with harder languages arranged towards the top. The following basic reasoning algorithms are involved:

- enforcing 3- and 4-consistency;
- solving Disjunctive Linear Programs (DLP), both their tractable Horn subclass (e.g., using algorithms presented in [Koubarakis, 2001] or [Jonsson and Bäckström, 1998]) as well as the general case (e.g., by mixed integer linear programming solvers to encode disjunctions using integers);

- solving multivariate polynomial equations and inequalities (e.g., using Cylindrical Algebraic Decomposition (CAD) [Collins, 1974]).

By arrows we depict known translations, in case of mutual translations we group all calculi in a cluster. Clearly, translations are transitive but the figure only presents direct links to avoid visual clutter.

The translations depicted in Fig. 3 are important for two reasons: First, translations enable statements to be mixed that were originally composed using relations from two distinct calculi. This allows for joint reasoning which is important for practical applications, since qualitative calculi are highly specialized to a certain aspect of a single domain, but many tasks require several aspects (e.g., cardinal relations and visibility information) to be considered at once. To foster practical relevance we only consider translations that allow models obtained in the target languages to be mapped back.

Second, by translating one representation into another we are able to apply the same decision procedure to problem formulations originally given in two different qualitative formalisms, requiring two distinct techniques. This is useful for developers of versatile reasoning tools who wish to handle a wide range of spatial or temporal relations. The ability to translate one calculus into another language using Boolean combinations suggests a new approach in qualitative reasoning: to consider frameworks like SMT (see [Barrett *et al.*, 2008] for an overview) or ASP [Lifschitz, 2008] with solvers supporting external theories, since both offer sophisticated means to address the propositional combinations occurring. Even if the translations lead to a higher class of complexity, the resulting method may be very efficient. While Westphal and Wöfl [2009] have shown that highly optimized QSTR search techniques outperform SAT encodings in case of calculi with few relations (e.g., IA, RCC-8), they also discovered that SAT techniques become superior for calculi defining many atomic relations (e.g., CDR). SMT- or ASP-like techniques supporting specialized domain level reasoning are necessary since SAT is not sufficient to encode calculi from the CAD class which is conjectured to be beyond NP.

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## References

- [Aiello *et al.*, 2007] Marco Aiello, Ian E. Pratt-Hartmann, and Johan F.A.K. van Benthem, editors. *Handbook of Spatial Logics*. Springer, 2007.
- [Allen, 1983] James F. Allen. Maintaining knowledge about temporal intervals. *Commun. ACM*, 26(11):832–843, 1983.
- [Balbiani *et al.*, 1998] Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. A model for reasoning about bidimensional temporal relations. In *Proc. of KR-98*, pages 124–130. Morgan Kaufmann, 1998.

- [Barrett *et al.*, 2008] Clark Barrett, Roberto Sebastiani, Sanjit A. Seshia, and Cesare Tinelli. Satisfiability modulo theories. In Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*. IOS Press, 2008.
- [Bodirsky and Kára, 2010] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM*, 57(2), 2010.
- [Collins, 1974] George Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *ACM SIGSAM Bulletin*, 8(3):80–90, 1974.
- [Condotta, 2000] Jean-François Condotta. Tractable sets of the generalized interval algebra. In *Proc. of ECAI 2000*, pages 78–82. IOS Press, 2000.
- [Fargier *et al.*, 2013] H el ene Fargier, Pierre Marquis, and Alexandre Niveau. Towards a knowledge compilation map for heterogeneous representation language. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI 2013)*, 2013.
- [Fenelon *et al.*, 2012] Valquiria Fenelon, Paulo E. Santos, Hannah M. Dee, and Fabio G. Cozman. Reasoning about shadows in a mobile robot environment. *Appl. Intell.*, 22(2), 2012.
- [Freksa, 1992a] Christian Freksa. Temporal reasoning based on semi-intervals. *Artif. Intell.*, 54(1):199–227, 1992.
- [Freksa, 1992b] Christian Freksa. Using orientation information for qualitative spatial reasoning. In *Spatio-Temporal Reasoning*, volume 639 of *LNCS*, pages 162–178. Springer, 1992.
- [Galton, 1994] Antony P. Galton. Lines of sight. In *Proc. of AI and Cognitive Science '94*, pages 103–113, 1994.
- [Jonsson and B ackstr om, 1998] Peter Jonsson and Christer B ackstr om. A unifying approach to temporal constraint reasoning. *Artificial Intelligence*, 102(1):143–155, 1998.
- [Koubarakis, 2001] Manolis Koubarakis. Tractable disjunctions of linear constraints: basic results and applications to temporal reasoning. *Theoretical Computer Science*, 266:311–339, 2001.
- [Kreutzmann and Wolter, 2014] Arne Kreutzmann and Diedrich Wolter. Qualitative spatial and temporal reasoning with AND/OR linear programming. In *Proc. of ECAI 2014*, volume 263 of *FAIA*, pages 495–500. IOS Press, 2014.
- [Lee *et al.*, 2013] Jae Hee Lee, Jochen Renz, and Diedrich Wolter. StarVars—effective reasoning about relative directions. In *Proc. of IJCAI 2013*, pages 976–982. AAAI Press, 2013.
- [Lifschitz, 2008] Vladimir Lifschitz. What is answer set programming? In Dieter Fox and Carla P. Gomes, editors, *Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence, AAAI 2008, Chicago, Illinois, USA, July 13-17, 2008*, pages 1594–1597. AAAI Press, 2008.
- [Ligozat and Renz, 2004] G erard Ligozat and Jochen Renz. What is a qualitative calculus? A general framework. In *Proc. of PRICAI 2004*, pages 53–64, 2004.
- [Ligozat, 1998] G erard Ligozat. Reasoning about cardinal directions. *J. Vis. Lang. Comput.*, 9(1):23–44, 1998.
- [Liu *et al.*, 2010] Weiming Liu, Xiaotong Zhang, Sanjiang Li, and Mingsheng Ying. Reasoning about cardinal directions between extended objects. *Artif. Intell.*, 174(12–13):951–983, 2010.
- [Moratz and Ragni, 2008] Reinhard Moratz and Marco Ragni. Qualitative spatial reasoning about relative point position. *J. Vis. Lang. Comput.*, 19(1):75–98, 2008.
- [Moratz and Wallgr un, 2012] Reinhard Moratz and Jan Oliver Wallgr un. Spatial reasoning with augmented points: Extending cardinal directions with local distances. *J. Spatial Information Science*, 5(1):1–30, 2012.
- [Mossakowski and Moratz, 2012] Till Mossakowski and Reinhard Moratz. Qualitative reasoning about relative direction of oriented points. *Artif. Intell.*, 180-181:34–45, 2012.
- [Navarrete *et al.*, 2013] Isabel Navarrete, Antonio Morales, Guido Sciavicco, and M. Antonia C ardenas Viedma. Spatial reasoning with rectangular cardinal relations – the convex tractable subalgebra. *Ann. Math. Artif. Intell.*, 67(1):31–70, 2013.
- [Pujari *et al.*, 1999] Arun K. Pujari, G. Vijaya Kumari, and Abdul Sattar. INDU: An interval and duration network. In *Proc. of AI 1999*, volume 1747 of *LNCS*, pages 291–303. Springer, 1999.
- [Ragni and Scivos, 2005] Marco Ragni and Alexander Scivos. Dependency calculus: Reasoning in a general point relation algebra. In *Proc. of KI 2005*, volume 3698 of *LNCS*, pages 49–63. Springer, 2005.
- [Randell *et al.*, 1992] David A. Randell, Zhan Cui, and Anthony G. Cohn. A spatial logic based on regions and “Connection”. In *Proc. of KR 1992*, pages 165–176. Morgan Kaufmann, 1992.
- [Renz and Mitra, 2004] Jochen Renz and Debasis Mitra. Qualitative direction calculi with arbitrary granularity. In *Proc. of PRICAI 2004*, pages 65–74, 2004.
- [Renz, 2001] Jochen Renz. A spatial odyssey of the interval algebra: 1. directed intervals. In *Proc. of IJCAI 2001*, pages 51–56. Morgan Kaufmann, 2001.
- [Renz, 2002] Jochen Renz. *Qualitative Spatial Reasoning with Topological Information*, volume 2293 of *LNCS*. Springer, Berlin, 2002.
- [Scivos and Nebel, 2005] Alexander Scivos and Bernhard Nebel. The Finest of its Class: The Natural Point-Based Ternary Calculus LR for Qualitative Spatial Reasoning. In *Spatial Cognition IV. Reasoning, Action, Interaction*, pages 283–303. Springer Berlin Heidelberg, January 2005.
- [Skiadopoulos and Koubarakis, 2005] Spiros Skiadopoulos and Manolis Koubarakis. On the consistency of cardinal direction constraints. *Artif. Intell.*, 163(1):91–135, 2005.
- [Tarquini *et al.*, 2007] Francesco Tarquini, Giorgio De Felice, Paolo Fogliaroni, and Eliseo Clementini. A qualitative model for visibility relations. In *Proc. of KR 07*, volume 4667 of *LNCS*, pages 510–513. Springer, 2007.
- [Vilain and Kautz, 1986] Marc B. Vilain and Henry A. Kautz. Constraint propagation algorithms for temporal reasoning. In *Proc. of AAAI 1986*, pages 377–382. Morgan Kaufmann, 1986.
- [Westphal and W olfl, 2009] Matthias Westphal and Stefan W olfl. Qualitative CSP, finite CSP, and SAT: Comparing methods for qualitative constraint-based reasoning. In *Proceedings of the 21th International Joint Conference on Artificial Intelligence (IJCAI 2009)*, 2009.
- [W olfl and Westphal, 2009] Stefan W olfl and Matthias Westphal. On combinations of binary qualitative constraint calculi. In *Proceedings of the 21th International Joint Conference on Artificial Intelligence (IJCAI 2009)*, 2009.
- [Wolter and Lee, 2010] Diedrich Wolter and Jae Hee Lee. Qualitative reasoning with directional relations. *Artificial Intelligence*, 174(18):1498–1507, 2010.