

# Preference Orders on Families of Sets — When Can Impossibility Results Be Avoided?

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## Abstract

Lifting a preference order on elements of some universe to a preference order on *subsets* of this universe is often guided by postulated properties the lifted order should have. Well-known impossibility results pose severe limits on when such liftings exist if *all* non-empty subsets of the universe are to be ordered. The extent to which these negative results carry over to other families of sets is not known. In this paper, we consider families of sets that induce connected subgraphs in graphs. For such families, common in applications, we study whether lifted orders satisfying the well-studied axioms of dominance and (strict) independence exist for every or, in another setting, only for some underlying order on elements (*strong* and *weak orderability*). We characterize families that are strongly and weakly orderable under dominance and strict independence, and obtain a tight bound on the class of families that are strongly orderable under dominance and independence.

## 1 Introduction

When agents, individually or as a group, make decisions to select one of several options, they refer to their preference orders (or rankings) of the available choices. In a single-agent setting, the agent simply selects an option that she prefers the most. In a group setting, the agents submit their preference orders, or votes, to a voting rule, which determines the option to select.

In many situations, having a preference order on individual objects is not enough for decision making, and the ability to compare *sets* of alternatives is needed. For instance, when an agent is to select a set of objects subject to some constraints, as in the *knapsack* problem, that agent must have a preference order on the family of feasible sets. Similarly, in the problem of fair allocation of indivisible goods [Bouveret *et al.*, 2016], knowing how agents rank sets of goods is necessary to ensure that the goods are distributed fairly. Some problems involving strategic behaviors in voting [Barberà, 1977; Fishburn, 1977; Bossert, 1989; Brandt and Brill, 2011; Brandt *et al.*, 2018] and when determining optimal matchings and assignments [Roth and Sotomayor, 1990] also require the knowledge of

agents' preferences on collection of objects. Finally, preferences on sets of outcomes are needed in decision making, when there is uncertainty about the consequences of an action [Larbi *et al.*, 2010].

However, the size of families of subsets of a given set of available objects makes explicit enumerations of preference orders or rankings infeasible. To circumvent this problem, researchers proposed that agents' true preference order on sets be approximated by an order that can be derived from their preference order on individual objects. If such an order is given as a utility function on objects, the utility function (preference) on *sets* of objects can be derived assuming, say, some form of additivity. This is a common setting, used for instance in the knapsack problem and fair division.

An alternative and more abstract framework, known as the *ordinal* setting, assumes that preferences on objects in some set  $X$  are represented by an order relation on  $X$ . The objective is to *lift* this order to an order on a family of non-empty subsets of  $X$ . The problem of lifting an order relation on  $X$  to an order on the family of all non-empty subsets of  $X$  has been extensively studied. The paper by Barberà *et al.* [2004] provides an excellent extensive overview of this research area. The results can roughly be divided into two groups, those concerned with properties of specific ways to lift an order on objects to an order on sets of objects, and those following the “axiomatic” approach, where one postulates desirable properties a lifted order should have and seeks conditions that would guarantee the existence of such a lifting [Barberà, 1977; Barberà *et al.*, 1977; Moretti and Tsoukiàs, 2012]. The most striking results in this latter group are known as impossibility theorems. They say that some natural desiderata are inherently incompatible and cannot be achieved together [Kannai and Peleg, 1984; Barberà *et al.*, 2004; Geist and Endriss, 2011].

The impossibility results mentioned above seek liftings to the family of *all* non-empty subsets of a set. This is a very strong requirement. Often we are only interested in comparing sets from much smaller families of sets.

Indeed, if the set of indivisible goods are offices and labs in a new research building and agents are research groups, it is natural to only consider allocations that form topologically contiguous areas. For instance, if the building consists of a single long hall of rooms, legal allocations are only those that split this hall into segments. In such situations, only pref-

erences that research groups may have on contiguous segments of rooms need to be taken into account [Bouveret *et al.*, 2017]. For another example, we might consider a problem of farmland fragmentation, where individual farms consist of many small non-contiguous plots of land as the result of divisions of farms among heirs, and acquiring ownership through marriage [King and Burton, 1982]. Land consolidation was proposed as a method to improve economic performance. The objective of land consolidation is to reallocate the plots so that they form large contiguous land areas. In both cases, the topology of the set of goods can be modeled by a graph and valid sets of goods are those that induce in this graph a connected subgraph.

The question we are concerned here is whether the impossibility results still hold when the goal is to lift an order on a set  $X$  (of goods) to specific collections of subsets of  $X$ , namely those determined by the condition of connectivity in a given graph on  $X$ . More precisely, we seek characterizations of graphs (topologies) when the impossibility results still hold and those, for which lifting to orders satisfying prescribed postulates is possible. To this end, we introduce the key notions of strongly and weakly orderable graphs. A graph is *strongly* (resp. *weakly*) orderable wrt a set of axioms if every (resp. at least one) linear order on  $X$  can be lifted to an order on the collection of all sets inducing in the graph connected subgraphs. Our contributions are as follows.

1. We show that the disjoint union of orderable graphs yields an orderable graph as well. This enables us to fully describe strong and weak orderability by characterizing the two concepts for connected graphs.
2. We fully characterize orderable connected graphs wrt the axioms of dominance and strict independence. For these two axioms, the class of strongly orderable graphs is that of trees and the class of weakly orderable graphs is that of connected bipartite graphs. This also holds if, in addition, the axiom of (strong) extension is required.
3. We show that weakening strict independence to independence has minimal effect on strong orderability. In combination with strong extension, we show that the only additional connected strongly orderable graph that arises is the complete graph  $K_3$ . If we do not require extension, we can give a nearly complete picture: all unicyclic graphs become strongly orderable but, with only some exceptions, graphs containing two cycles are not.

An interesting implication of our results is that weakening strict independence to independence results only in a modest extension of the class of strongly orderable graphs. It points to strict independence being perhaps more essential for the concept of strong orderability than its weaker and more commonly studied version.

Although we have focused here on graphs to represent the sets of elements to be ordered in an implicit and compact way, we believe that our work has impact on further aspects of ongoing research on preferential reasoning in the field of AI. Indeed, implicit preference models are important for representing, eliciting and using preferences in practical applications. As an example, we mention here the work on preferences in Answer-Set Programming (see e.g. [Brewka *et al.*, 2003;

Faber *et al.*, 2013]) where logic programs compactly represent the sets to be ordered and languages as the one by Brewka *et al.* [2003] allow to express preferences over the individual elements, i.e. the atoms in the program. To this date, it is unclear whether the rankings obtained by such formalisms satisfy desirable properties as the ones discussed above. Our work thus can also be seen as a starting point for more general investigations on (im)possibility results in formalisms from the areas in AI and KR.

## 2 Background

All sets we consider in the paper are finite. A binary relation is called an *order*<sup>1</sup> if it is reflexive, transitive and total. An order is *linear* if it is also antisymmetric. If  $\preceq$  is an order on a set  $X$ , the corresponding *strict* order  $\prec$  on  $X$  is defined by  $x \prec y$  if  $x \preceq y$  and  $y \not\preceq x$ , where  $x, y$  are arbitrary elements of  $X$ ; the corresponding *equivalence* or *indifference* relation  $\sim$  is defined by  $x \sim y$  if  $x \preceq y$  and  $y \preceq x$ . If  $\preceq$  is linear then  $x \sim y$  only if  $x = y$ .

For a linear order  $\preceq$  on a set  $A$ , we write  $\max_{\preceq}(A)$  for the maximal element of  $A$  with respect to  $\preceq$ . Similarly, we write  $\min_{\preceq}(A)$  for the minimal element of  $A$  wrt  $\preceq$ . If no ambiguity arises, we drop the reference to the relation from the notation.

Given a set  $X$  and a linear order  $\leq$  on  $X$ , the *order lifting* problem consists of deriving from  $\leq$  an order  $\preceq$  on a family  $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$  of non-empty *subsets* of  $X$ , guided by axioms formalizing some natural desiderata<sup>2</sup> for such lifted orders. If all axioms in consideration enforce strict preferences we additionally expect the lifted order  $\preceq$  to be a linear order. Otherwise, we expect the lifted order  $\preceq$  to be just an order. In this work, independence is the only axiom not enforcing strict preferences, hence we expect the order to be linear if and only if we do not consider independence. We recall several axioms below. They are natural extensions of the versions of those axioms considered in the case when  $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$  (cf. [Barberà *et al.*, 2004]). The extensions consist of adding conditions of the form  $Y \in \mathcal{X}$  not needed in the original formulations (cf. [Maly and Woltran, 2017]).

**Axiom 1** (Extension Rule). For all  $x, y \in X$ , such that  $\{x\}, \{y\} \in \mathcal{X}$ :

$$x < y \text{ implies } \{x\} \prec \{y\}.$$

**Axiom 2** (Strong Extension). For all  $A, B \in \mathcal{X}$ :

$$\max(A) < \max(B) \text{ implies } A \prec B.$$

**Axiom 3** (Dominance). For all  $A \in \mathcal{X}$  and all  $x \in X$ , such that  $A \cup \{x\} \in \mathcal{X}$ :

$$y < x \text{ for all } y \in A \text{ implies } A \prec A \cup \{x\};$$

$$x < y \text{ for all } y \in A \text{ implies } A \cup \{x\} \prec A.$$

**Axiom 4** (Independence). For all  $A, B \in \mathcal{X}$  and for all  $x \in X \setminus (A \cup B)$ , such that  $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ :

$$A \prec B \text{ implies } A \cup \{x\} \preceq B \cup \{x\}.$$

<sup>1</sup>Often also called weak order or total preorder.

<sup>2</sup>Observe that this does not mean that every axiom is desirable in every situation. For a more nuanced discussion on the applicability of these axioms see [Barberà *et al.*, 2004].

**Axiom 5** (Strict Independence). For all  $A, B \in \mathcal{X}$  and for all  $x \in X \setminus (A \cup B)$ , such that  $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ :

$$A \prec B \text{ implies } A \cup \{x\} \prec B \cup \{x\}.$$

Note that strong extension implies the extension rule. One could also define a dual version of strong extension based on the minima of  $A$  and  $B$ . Because all problems in this paper are symmetric, we can use either version without loss of generality. These two axioms are strict versions of the well known Hoare and Smyth axioms (see [Brewka *et al.*, 2010]) restricted to linear orders.

**Example 1.** Take  $X = \{1, 2, 3, 4\}$  with the usual linear order  $\leq$  and

$$\mathcal{X} = \{\{2\}, \{4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 4\}\}.$$

The axioms impose constraints on any lifted order  $\preceq$  on  $\mathcal{X}$ . In particular, the extension rule implies  $\{2\} \prec \{4\}$ , while strong extension additionally implies  $\{2\} \prec A$  for all  $A \in \mathcal{X} \setminus \{2\}$ . Dominance implies  $\{2\} \prec \{2, 4\} \prec \{4\}$ ,  $\{3, 4\} \prec \{4\}$ ,  $\{1, 2, 4\} \prec \{2, 4\}$  and  $\{1, 4\} \prec \{4\}$ , and (strict) independence lifts the preference between  $\{2, 4\}$  and  $\{4\}$  to  $\{1, 2, 4\}$  and  $\{1, 4\}$ . Thus, dominance and independence imply  $\{1, 2, 4\} \preceq \{1, 4\}$ , and dominance and strict independence imply  $\{1, 2, 4\} \prec \{1, 4\}$ .

The standard version of the order lifting problem asks for the existence of a lifted order on  $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$  that would satisfy dominance, and independence or strict independence. As it turns out, these requirements can rarely be satisfied together. In their seminal paper, Kannai and Peleg [1984] proved that if  $|X| \geq 6$  then orders on  $\mathcal{P}(X) \setminus \{\emptyset\}$  satisfying dominance and independence are not possible. Barberà and Pattanaik [1984] showed a similar impossibility result for  $|X| \geq 3$ , when dominance and *strict* independence are required.

We show in this paper that the picture for other families of non-empty subsets of  $X$  is much more interesting. In particular, we show it to be the case for collections of subsets of  $X$  that induce connected subgraphs in some graph on  $X$ . Namely, we describe non-trivial classes of graphs defining families of sets that allow for lifted orders satisfying dominance and (strict) independence. In many cases, these lifted orders also satisfy the extension rule or its stronger version. It is important as every reasonable lifted order should satisfy the extension rule.<sup>3</sup>

### 3 Problem Statement

We are interested in families of sets that are defined in terms of connectivity of subgraphs in a graph. We consider undirected graphs only. We write  $G = (V, E)$  for a graph with the set of vertices  $V$  and the set of edges  $E$ . We write an edge between vertices  $u$  and  $v$  as  $\{u, v\}$  or  $uv$ . A *subgraph* of a graph  $G$  is a graph whose every vertex and edge are also a vertex and edge of  $G$ . If a subgraph  $H = (W, F)$  of  $G$

<sup>3</sup>The standard statement of the lifting problem does not explicitly mention the extension rule since for  $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$  the extension rule is implied by two applications of dominance via  $\{x_1\} \prec \{x_1, x_2\} \prec \{x_2\}$  for  $x_1 < x_2$ .

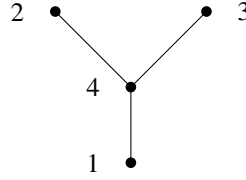


Figure 1: Graph  $G$

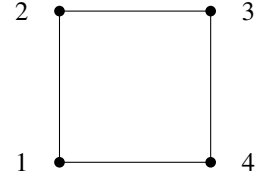


Figure 2: Graph  $G'$

contains all edges in  $G$  connecting vertices in  $W$ ,  $H$  is the subgraph induced by  $W$ . A *path* consists of a non-empty set  $S$  of vertices that can be enumerated so that every two consecutive vertices are connected with an edge; its length is given by  $|S| - 1$ . A *cycle* is a sequence of at least three different vertices, as well as the first and the last one, are connected with an edge; the length of a cycle is given by the number of vertices. A graph is *connected* if every two of its vertices are connected with a path. A *forest* is a graph with no cycles. A *tree* is a forest that is connected.

**Definition 2.** For a graph  $G$  we write  $C(G)$  for the family of sets of vertices of all connected subgraphs of  $G$ . Moreover,  $IT(G)$  denotes the family of sets of vertices  $V'$  such that the subgraphs induced by  $V'$  in  $G$  are trees.

**Example 3.** For the graph  $G$  shown in Figure 1, we get

$$C(G) = IT(G) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

On the other hand for the graph  $G'$  shown in Figure 2, we have  $C(G') = \mathcal{P}(\{1, 2, 3, 4\}) \setminus \{\{1, 3\}, \{2, 4\}\}$  and  $IT(G') = C(G') \setminus \{1, 2, 3, 4\}$ .

**Definition 4.** Let  $X$  be a set of elements and  $\mathcal{X} \subseteq (\mathcal{P}(X) \setminus \{\emptyset\})$ . We say  $\mathcal{X}$  is *strongly  $DI^{(S)}$ -orderable*, if for every linear order  $X$  there is a (linear) order on  $\mathcal{X}$  satisfying dominance and (strict) independence. We say  $\mathcal{X}$  is *weakly  $DI^{(S)}$ -orderable*, if for at least one linear order on  $X$  there is a (linear) order on  $\mathcal{X}$  satisfying dominance and (strict) independence. Analogously, we say  $\mathcal{X}$  is *strongly/weakly  $DI^{(S)}E^{(S)}$ -orderable* if for every/for one linear order on  $X$  there is a (linear) order on  $\mathcal{X}$  satisfying dominance, (strict) independence and (strong) extension. If there is no ambiguity, we say a graph  $G$  is *strongly/weakly  $DI^{(S)}$ - or  $DI^{(S)}E^{(S)}$ -orderable* if  $C(G)$  has the property.

**Example 5.** Consider the graphs  $G$  and  $G'$  from Figures 1 and 2. One can check that  $G$  is *strongly  $DI^S E^S$ -orderable* and  $G'$  is not *strongly  $DI^S$ -orderable*. If we assume the natural order on the vertices of  $G'$ , dominance and transitivity imply  $\{1\} \prec \{1, 2, 3\}$  and  $\{2, 3, 4\} \prec \{4\}$ . However, then strict independence implies  $\{1, 4\} \prec \{1, 2, 3, 4\} \prec \{1, 4\}$ , thus preventing  $\prec$  from being a strict order. This argument obviously does not work on  $IT(G')$  because  $\{1, 2, 3, 4\} \notin IT(G')$  and indeed  $IT(G')$  is *strongly  $DI^S$ -orderable*. Furthermore, if we exchange vertices 2 and 3 in  $G'$ , then there exists an order on  $C(G')$  satisfying dominance and strict independence. Hence  $G'$  is *weakly  $DI^S$ -orderable*.

It is evident that if a graph  $G$  is strongly or weakly orderable with respect to some collection of axioms selected from those discussed above, so are all its connected components. Importantly, the converse holds, too, allowing us to restrict attention to connected graphs only.

**Proposition 6.** *Let  $G$  and  $G'$  be graphs with the vertex sets  $U$  and  $V$ , respectively, such that  $U \cap V = \emptyset$ . If  $G$  and  $G'$  are strongly (weakly)  $DI^{(S)}(E^{(S)})$ -orderable, then  $G \cup G'$  is strongly (weakly)  $DI^{(S)}(E^{(S)})$ -orderable.*

*Proof.* (Sketch) Let us define  $W = U \cup V$ . Let us assume that  $\leq_U$  and  $\leq_V$  are linear orders on  $U$  and  $V$  such that some orders  $\preceq_U$  on  $C(G)$  and  $\preceq_V$  on  $C(G')$  satisfy all necessary axioms with respect to  $\leq_U$  and  $\leq_V$ . To prove the claim in all its versions, it suffices to show that for every linear order  $\leq$  on  $W$  such that  $\leq_U$  and  $\leq_V$  are restrictions of  $\leq$  to  $U$  and  $V$ , respectively, there is an order  $\preceq$  on  $C(G \cup G')$  satisfying all necessary axioms with respect to  $\leq$ . Now let  $\leq$  be such a linear order. In order to define  $\preceq$  we need the following definitions.

Let  $\{u_1, \dots, u_k\}$  be an enumeration of all elements in  $U$  such that  $\{u_i\} \preceq_U \{u_j\}$  for all  $i \leq j$ . Similarly, let  $\{v_1, \dots, v_l\}$  be an enumeration of all elements of  $V$  such that  $\{v_i\} \preceq_V \{v_j\}$  for all  $i \leq j$ . Let  $w \in W \cup \{0\}$  for dummy element  $0 \notin W$ . For  $1 \leq i \leq k-1$  and  $1 \leq j \leq l-1$ , we define

$$C_w = \begin{cases} \{A \in C(G) \mid \{u_i\} \preceq_U A \prec_U \{u_{i+1}\}\} & \text{if } w = u_i \\ \{A \in C(G') \mid \{v_j\} \preceq_V A \prec_V \{v_{j+1}\}\} & \text{if } w = v_j \\ \{A \in C(G) \mid \{u_k\} \preceq_U A\} & \text{if } w = u_k \\ \{A \in C(G') \mid \{v_l\} \preceq_V A\} & \text{if } w = v_l \\ \{A \in C(G) \mid A \prec_U \{u_1\}\} \cup \\ \{A \in C(G') \mid A \prec_V \{v_1\}\} & \text{if } w = 0 \end{cases}$$

Clearly, the sets  $C_w$ ,  $w \in \{0\} \cup W$  are pairwise disjoint. Moreover, since  $C(G \cup G') = C(G) \cup C(G')$ , it follows that  $C(G \cup G') = \bigcup_{w \in \{0\} \cup W} C_w$ .

We define an order  $\preceq$  on  $C(G \cup G')$  by setting  $A \preceq B$  for  $A, B \in C(G \cup G')$  precisely when one of the following conditions holds:

- $A, B \in C_w$ , for  $w \neq 0$ , and  $A \preceq_U B$  or  $A \preceq_V B$
- $A, B \in C_0$ , and  $A \preceq_U B$  or  $A \preceq_V B$
- $A, B \in C_0$ ,  $A \in C(G)$ ,  $B \in C(G')$
- $A \in C_w$ ,  $B \in C_{w'}$ , for  $w, w' \in \{0\} \cup W$ , and  $w < w'$ .

It can be verified that  $\preceq$  is total, reflexive and transitive. Moreover,  $\preceq$  satisfies the same axioms as  $\preceq_U$  and  $\preceq_V$ .

The case of strong extension requires a different construction. Let  $w \in U \cup V$ . We define

$$C_w = \begin{cases} \{A \in C(G) \mid \max_{\leq_U}(A) = w\} & \text{if } w \in U \\ \{A \in C(G') \mid \max_{\leq_V}(A) = w\} & \text{if } w \in V \end{cases}$$

Let us now assume that the orders  $\preceq_U$  and  $\preceq_V$  on  $C(G)$  and  $C(G')$  respectively, satisfy strong extension with respect to  $\leq_U$  and  $\leq_V$ , respectively. To define an order  $\preceq$  on  $C(G \cup G')$ , for  $A, B \in C(G \cup G')$  we set  $A \preceq B$  precisely when one of the following conditions holds:

- $A, B \in C_w$ ,  $w \in U$  and  $A \preceq_U B$
- $A, B \in C_w$ ,  $w \in V$  and  $A \preceq_V B$
- $A \in C_w$ ,  $B \in C_{w'}$  and  $w < w'$ .

Again, it is easily verified that  $\preceq$  is an order. From the definition, it follows that  $\preceq$  satisfies the strong extension property and it can be checked that it satisfies dominance and (strict) independence.  $\square$

In the forthcoming two sections we present our main results. Section 4 considers the combination of strict independence with dominance and optionally, extension or strong extension. Then in Section 5 we consider combinations of axioms containing regular independence.

## 4 Strict Independence

We start our investigations with strong orderability. Our first result concerns the family of all subsets of vertices of a graph that induce a tree.

**Proposition 7.** *For every graph  $G$ ,  $IT(G)$  is strongly  $DI^S E^S$ -orderable.*

*Proof.* (Sketch) Let  $N = |V|$  and let  $\leq$  be any linear order on  $V$ . Wlog, we assume that  $V = \{1, \dots, N\}$  and that  $\leq$  is the standard linear order on  $\{1, \dots, N\}$ . For every  $A \subseteq V$  and  $i \in A$ , we write  $deg_A(i)$  for the degree of  $i$  in the subtree of  $G$  induced by  $A$ . We associate with every set  $A$  a vector  $v_A = (a_1, \dots, a_N) \in (\mathbb{N} \cup \{\infty\})^N$ , where  $a_i = \infty$  if  $i \notin A$ , and  $a_i = deg_A(i)$  otherwise.

Let  $\leq^*$  be the reverse of the usual linear order on  $\mathbb{N} \cup \{\infty\}$ , i.e.  $k \leq^* l$  iff  $l \leq k$  and  $\infty \leq^* k$  for all  $k$ . We order  $IT(G)$  by defining  $A \preceq B$  precisely when  $v_A \leq_{lex} v_B$ , where  $\leq_{lex}$  is the lexicographic order with respect to  $\leq^*$ , with the indices considered from  $N$  to 1. That is,  $A \preceq B$  if  $a_N <^* b_N$ , or  $a_N = b_N$  and  $a_{N-1} <^* b_{N-1}$ , and so on. Obviously,  $\prec$  is a linear order, and it satisfies strong extension. It can also be checked that it satisfies dominance and strict independence.  $\square$

**Corollary 8.** *Every tree is strongly  $DI^S E^S$ -orderable.*

This result is optimal in the sense that cycles prevent a graph from being strongly  $DI^S$ -orderable.

**Proposition 9.** *A graph  $G$  is not strongly  $DI^S$ -orderable, whenever it contains a cycle.*

*Proof.* Let  $C = v_1, \dots, v_n$  be a shortest cycle in  $G$ . Then  $C(G)$  contains  $C$  and all its connected subgraphs. Let  $\leq$  be an order on  $V$  such that  $v_1 < \dots < v_n$ . Let us assume that there is an order  $\preceq$  on  $C(G)$  that satisfies dominance and strict independence with respect to  $\leq$ . Then, by dominance  $\{v_1\} \prec \{v_1, v_2\} \prec \dots \prec \{v_1, \dots, v_{n-1}\}$  and  $\{v_2, \dots, v_n\} \prec \{v_3, \dots, v_n\} \prec \dots \prec \{v_n\}$ . Therefore, by strict independence  $\{v_1, v_n\} \prec \{v_1, \dots, v_n\}$  and  $\{v_1, \dots, v_n\} \prec \{v_1, v_n\}$ , a contradiction!  $\square$

The following theorem summarizes the previous results and follows from Corollary 8, Proposition 6, Proposition 9 and the fact that any graph that is not strongly  $DI^S$ -orderable is also not strongly  $DI^S E^{(S)}$ -orderable.

**Theorem 10.** *The set of strongly  $DI^S$ -,  $DI^SE$ - or  $DI^SE^S$ -orderable graphs is exactly given by  $\mathcal{F}$ , the class of forests.*

This result states that every linear order on  $X$  can be lifted (wrt dominance, strict independence and strong extension) to every family of sets of vertices inducing a connected subgraph in a forest on  $X$ . For instance, no matter what linear order on  $\{1, 2, \dots, n\}$  we consider, it extends to a linear order on the family  $\mathcal{I} = \{[i..j] \mid 1 \leq i < j \leq n\}$  that satisfies dominance, strict independence and strong extension. It is so because every set in  $\mathcal{F}$  induces a connected subgraph in the path in which elements  $1, \dots, n$  are listed in the natural order. The same is true for the family of sets  $\mathcal{S} = \{X \subseteq \{1, \dots, n\} \mid 1 \in X\}$ . Indeed, each set in this family induces a connected subgraph in the “star” tree in which every vertex  $i \geq 2$  is connected to 1 (and there are no other edges).

We now turn to weak orderability and show in the forthcoming two results that the bipartite graphs form the crucial class for our characterization. We use the fact that a graph is bipartite if and only if it is 2-colorable.

**Proposition 11.** *Every 2-colorable graph is weakly  $DI^SE^S$ -orderable.*

*Proof.* (Sketch) Let us consider a 2-colorable graph  $G = (V, E)$ . We color  $G$  with two colors *small* and *large* and call vertices of  $G$  *small* and *large* accordingly. Let  $\leq$  be any linear order on  $V$  such that every *small* vertex is smaller than every *large* vertex.

For every  $A \in C(G)$  we define  $A_L = \{x \in A \mid x \text{ is large}\}$  and  $A_S = \{x \in A \mid x \text{ is small}\}$ . For  $A, B \in C(G)$ , we define  $A \preceq B$  if

- $A \preceq A$ ;
- $A_L \neq B_L$  and  $\max(A_L \Delta B_L) \in B_L$ ; or
- $A_L = B_L$ ,  $A_S \neq B_S$ , and  $\min(A_S \Delta B_S) \in A_S$ .

(we write  $\Delta$  for the symmetric difference of sets). It is clear that  $\preceq$  is a linear order. It can be checked that that  $\preceq$  satisfies dominance, strict independence and strong extension.  $\square$

This result shows, in particular, that if  $X$  and  $Y$  are disjoint nonempty sets, then the family of sets  $\{Z \subseteq X \cup Y \mid Z \cap X \neq \emptyset \neq Z \cap Y\}$  is weakly  $DI^SE^S$ -orderable.

Proposition 11 is tight as graphs that are not 2-colorable are not weakly  $DI^SE^S$ -orderable.

**Theorem 12.** *The set of weakly  $DI^S$ -,  $DI^SE$ - or  $DI^SE^S$ -orderable graphs is exactly given by the class of 2-colorable graphs.*

*Proof.* By Proposition 11, it remains to show that graphs that are not 2-colorable are not weakly  $DI^S$ -orderable, which obviously implies the same for  $DI^SE$  and  $DI^SE^S$ .

Let  $V$  be the vertex set of  $G$ . For every linear order  $\leq$  we say a vertex  $x$  is *large* if  $n < x$  holds for all neighbors  $n$  of  $x$  and *small* if  $x < n$  holds. We call  $x$  *intermediate* if  $x$  is neither *large* nor *small*. We claim that for every order on  $V$  there is an intermediate vertex  $x \in V$ . Indeed, assume otherwise that there is an order  $\leq$  on  $V$  without intermediate vertices. Obviously no *large* vertex can be a neighbor of

a *large* vertex and no *small* vertex can be the neighbor of a *small* vertex. Thus, we constructed a 2-coloring of  $G$ , a contradiction.

Let  $\leq$  be an order on  $G$  with a minimal number of intermediate vertices and let  $x$  be an intermediate vertex. We call a neighbor  $n$  of  $x$  *small* if  $n < x$  holds and *large* otherwise. We claim that at least one small neighbor of  $x$  is connected to at least one large neighbor of  $x$  by a path in  $G_x^-$ , the graph induced by  $V \setminus \{x\}$ . Indeed, let us assume otherwise and let  $V'$  be the set of all vertices in  $V$  reachable in  $G$  from  $x$  by paths not including any *large* neighbor of  $x$ . Let us define  $V'' = V \setminus V'$ . Clearly,  $x$  and all *small* neighbors of  $x$  belong to  $V'$  and all *large* neighbors of  $x$  belong to  $V''$ . For every element  $u \in V$ , we write  $\bar{u}$  for the dual element of  $u$ , i.e. if the rank of  $u$  in  $\leq$  is  $k + 1$ , then  $\bar{u}$  is the element with rank  $|V| - k$ . We construct a linear order  $\leq'$  by flipping  $u$  with  $\bar{u}$ , for all  $u \in V'$  in the enumeration of  $V$  with respect to  $\leq$ . It is clear that  $\leq'$  is a linear order on  $V$ . Moreover,  $x$  is no longer an intermediate vertex in  $G$  and, for all other vertices, whether they are intermediate or not does not change. Thus,  $\leq'$  is a linear order on  $V'$  with fewer intermediate vertices, a contradiction.

Let then  $n$  be a *small* neighbor of  $x$  connected in  $G_x^-$  to a *large* neighbor of  $x$  by a path, say  $n, x_1, \dots, x_k, n'$ . Let us assume there is a linear order  $\preceq$  on  $C(G)$  satisfying dominance and strict independence with respect to  $\leq$ . Then, since  $n < x$ , we have  $\{n\} \prec \{n, x\}$  by dominance. Further, by repeated application of strict independence and transitivity

$$\{n, x_1, \dots, x_k, n'\} \prec \{n, x, x_1, \dots, x_k, n'\}.$$

On the other hand, since  $x < n'$ , we have  $\{x, n'\} \prec \{n'\}$  and hence

$$\{n', x, x_1, \dots, x_k, n\} \prec \{n', x_1, \dots, x_k, n\}.$$

Thus,  $\{n, x_1, \dots, x_k, n'\} \prec \{n, x_1, \dots, x_k, n'\}$ , a contradiction.  $\square$

## 5 Regular Independence

We now exchange strict by regular independence and focus on strong  $DIE^S$ -orderability for which we give an exact characterization. The following is easy to see.

**Proposition 13.** *Let  $X$  be a set. If  $|X| \leq 3$ , then  $\mathcal{P}(X) \setminus \{\emptyset\}$  is strongly  $DIE^S$ -orderable.*

The next result shows that we cannot go much beyond 3-cycles.

**Proposition 14.** *Let  $G$  be a connected graph with four or more vertices that contains at least one cycle. Then  $G$  is not strongly  $DIE^S$ -orderable.*

*Proof.* Either  $G$  contains a cycle of length at least four or a cycle of length three connected to an additional vertex. In the first case let  $u, v \in V$  be two non-adjacent vertices contained in the cycle, and let  $u, v_1, \dots, v_n, v$  and  $u, u_1, \dots, u_m, v$  be the two paths from  $u$  to  $v$ . In the second case let  $u$  be the additional vertex and let  $u_m, v_n, v$  be the vertices in the cycle such that  $v_n$  is connected to  $u$ . Define  $\leq$  as  $u < u_1 < \dots < u_m < v_1 < \dots < v_n < v$ . Then there is no order on  $C(G)$  satisfying dominance, independence and strong extension with

respect to  $\leq$ : assume otherwise  $\preceq$  is such an order. Let  $CY$  be the set of all vertices in the cycle in the first case and  $CY = \{u, v, u_m, v_n\}$  otherwise. Then,  $\{u_m\} \prec CY \setminus \{v\}$  by strong extension, and  $\{u_m, v\} \preceq CY$  by independence. However, by dominance  $\{v_1, \dots, v_n, v\} \prec \{v\}$  and therefore, by independence,  $CY \setminus \{u, u_1, \dots, u_{m-1}\} \preceq \{u_m, v\}$  which contradicts  $CY \prec CY \setminus \{u, u_1, \dots, u_{m-1}\}$  as required by dominance.  $\square$

Therefore,  $K_3$  is the only connected graph that is strongly  $DIE^S$ -orderable but not  $DI^S E^S$ -orderable (recall Proposition 9). Also recall that a graph is strongly  $DI^S E^S$ -orderable precisely when it is a forest. The class of  $DIE^S$ -orderable graphs is thus only marginally larger, as is shown by the following result, which is immediate from Propositions 6, 13 and 14.

**Theorem 15.** *The set of strongly  $DIE^S$ -orderable graphs consists precisely of graphs whose each connected component is a tree or a cycle  $K_3$ .*

We now turn to graphs that are strongly  $DIE$ - or  $DI$ -orderable. We omit proofs due to space limits. Our first result shows that by replacing strong extension by extension or dropping extension altogether we get additional strongly orderable graphs.

**Proposition 16.** *Every cycle is strongly  $DIE$ -orderable and every unicyclic graph is strongly  $DI$ -orderable.*

On the other hand, we know that the following two classes of graphs are not strongly  $DI$ -orderable.

**Proposition 17.** *A graph  $G = (V, E)$  is not strongly  $DI$ -orderable if one of the following applies:*

- *there are distinct  $a, b \in V$  connected by three mutually disjoint paths, such that two of them have length at least two and the sum of their lengths is at least six;*
- *there is a vertex  $v \in V$  contained in two cycles  $C_1, C_2$  of  $G$  such that  $C_1 \cap C_2 = \{v\}$  and  $|C_1| \geq 4$ .*

These results allow us to outline the extent of strongly  $DI$ -orderable graphs. We recall that  $x$  is an *articulation point* in a graph  $G$  if the removal of  $G$  results in at least two connected components. Graphs without articulation points are two-connected. Propositions 16 and 17 imply that a two-connected graph that contains a cycle of length at least six and is not a cycle itself is not strongly  $DI$ -orderable. Thus, the only two-connected graphs that are strongly  $DI$ -orderable are the complete graph  $K_2$ , all cycles and, possibly, some two-connected graphs with the length of the longest cycle equal to 4 or 5. For graphs that are not two-connected, we know that unicyclic graphs are strongly  $DI$ -orderable. We also know that graphs that contain two cycles sharing one point, with at least one of these cycles having length at least four, are not strongly  $DI$ -orderable. This limits connected graphs with articulation points that might be strongly  $DI$ -orderable to tree-like graphs in which strongly orderable two-connected graphs (“meta-nodes”) are connected to each other by paths (in the case of triangle meta-nodes, they may connect via a shared point).

Finally, we provide some preliminary results on weakly  $DIE$ -orderable graphs. The result of Kannai and Peleg

[1984] implies that the complete graph  $K_N$  is not weakly  $DIE$ -orderable for  $N \geq 6$ . On the other hand, every proper subgraph of  $K_6$  is weakly  $DIE$ -orderable.

**Proposition 18.** *Every proper subgraph  $G$  of  $K_6$  is weakly  $DIE$ -orderable.*

Observe that this can not be extended to strong extension, because  $K_4$  is a proper subgraph of  $K_6$  and not weakly  $DIE^S$ -orderable.

## 6 Discussion

Lifting a preference order on elements of some universe to a preference order on *subsets* of this universe respecting certain axioms is a fundamental problem, but well-known impossibility results pose severe limits on when such liftings exist. Bossert [1995] observed that these impossibility results may be avoided by considering families of subsets of fixed cardinality. Maly and Woltran [2017] showed that, deciding whether a given linear order on a set of objects  $X$  can be lifted to an order on a given collection of subsets of  $X$  is NP-complete. Bouveret *et al.* [2017] were the first to consider graph topologies and subsets inducing connected subsets. They proposed this model for the problem of fair allocation of indivisible goods. Our work adopts their idea for an implicit representation of classes of families of non-empty subsets (in contrast to the explicit representation used in [Bossert, 1995; Maly and Woltran, 2017]). It turns out that for several interesting families of sets definable in terms of graphs the impossibility results observed for the family of all non-empty subsets of a set can be avoided!

Our main results characterize strongly and weakly  $DI^S E^S$ -orderable graphs. We also obtain a complete characterization of strongly  $DIE^S$ -orderable graphs. For strong  $DI$ -orderability we have an almost complete picture. Our results show rich classes of well-motivated families of sets that allow for lifting of linear orders in ways that combine dominance and (strict) independence. They also suggest that independence, despite being much less restrictive than strict independence, does not significantly extend that class of strongly orderable graphs. This suggests that it is strict independence that might be the axiom to focus on.

Even though in many cases our results fully resolve the order lifting problem, they also open several directions for future studies. First, we only touched on weak  $DI$ -orderability. We showed that all proper subgraphs of a complete graph  $K_6$  are weakly  $DI$ -orderable but have as of yet no general results on weakly  $DI$ -orderable graphs. Next, using graphs as implicit representations of families of sets is just one of many possibilities. Knowledge representation often uses logic formalisms towards this end. For instance, formulas can be viewed as concise representations of the families of their models. It is therefore important to study lifting of linear orders to orders on families of sets given by those representations. Further, for both graph and logical representations of families of sets it is a key challenge to establish the complexity of deciding the existence of lifted orders, and study algorithms for computing lifted orders in some concise representation. Finally, another direction for future work, is to investigate how our findings apply to the “reverse” problem

of social ranking [Moretti and Öztürk, 2017], where an order over individuals needs to be obtained from a given order over sets of individuals.

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