COLLECTIVE BEHAVIOUR OF AUTOMATA AND THE PROBLEMS OF STABLE LOCAL CONTROL OF A LARGE-SCALE SYSTEM

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The local choice of individual actions in the collective with antagonistic interaction may in principle lead to unstable, burst-like regimes in this collective. In the mathematical model of such a collective considered below (which is convenient to treat in terms of a collective of radiostations interfering with each other during their normal performance) the major role in this respect is played by the Λ^n region. If a vector of required qualities λ^o satisfies $\lambda^o \in \Lambda^n$, then natural local algorithms of the local choice of actions give global stability even in case of some delays in the system. On the contrary there is no stability if $\lambda^o \notin \Lambda^n$. The region Λ^n is determined with all the coefficients characterizing the interaction in the collective. For this reason the coherently local approach to the control of that system necessarily forces one to look for local criteria of control which could always provide the stable choice of actions (powers) by the radiostations under consideration. It is a matter of control elasticity that leads him further to the elastic local criterion of power control. This criterion seems to be the most "reasonable" for such a collective, but it is rather complicated in realization. From the formal point of view the analysis of the local criteria is reduced to studying properties of Nash parties in corresponding automata games.

1. n - the number of members in the collective, the number of pairs of radiostations;
2. $\mathcal{E}_i$ - the action of i-th member, the power level of the transmitter of i-th pair;
3. t - the time;
4. γ_i - the useful "effect" received by i-th member of the collective;
5. λ_i - noise-to-signal ratio on i-th receiver;
6. λ - vector of noise-to-signal ratios;
7. λ_i^o - a priori given noise-to-signal ratio;
8. λ^o - vector of given a priori ratios of noise-to-signal;
9. Λ^n - the region of obtainable vectors λ ;
10. $a_{ij}\mathcal{E}_j$ - the power level on input of the receiver of i-th pair of radiostations due to only the transmitter of j-th pair;
11. a_{ij} - positive (non-negative for $i \neq j$) values;
12. $\|a_{ij}^{(n)}\|$ - the matrix of the quantities a_{ij} ;
13. $\mathcal{M}^n(\lambda)$ - the matrix with the diagonal elements equal to $\lambda_i a_{ii}$, non-diagonal equal to $-a_{ij}$;
14. $\mathcal{M}_i^{n-1}(\lambda)$ -(n- 1)-th order principal minor of matrix $\mathcal{M}^n(\lambda)$ corresponding to i-th diagonal element;
15. $\varkappa_i, \alpha_i, \beta_i$ - fixed positive parameters;
16. $C_i(\lambda_i, \mathcal{E}_i, \alpha_i)$ - the local criterion of power control for i-th pair of radiostations;

17. α - vector of fixed positive parameters;
18. $\emptyset$ - the subclass of radiostation collectives;

A number of important problems of control systems construction which because of their complexity we usually consider as belonging to the area of artificial intelligence, leads to the necessity of study of a system with certain character of interaction of its components. Suppose a system consisting of *n* components. Suppose that the i-th component of the system performs **the action $\mathcal{E}_i$ ($0 \leq \mathcal{E}_i < \infty$, $i = 1, \ldots, n$) at the moment t and just at the moment it knows a value ν_i, which shows what** the effectiveness of applying such an notion in that system is. This "effect" is determined by all the actions chosen by all other mem bers of the collective at the same moment of time. The increase of the $\mathcal{E}_i$ leads to an increase of the effect provided that the actions of all other mem bers of the **collective are being constant. While the increase of $\mathcal{E}_j$, $j \neq i$, provided that the $\mathcal{E}_i$ is constant, leads to its decrease.**

In other words the antagonism of aims of different elements of the system and comparatively small information concerning the actions of other elements are main characteristics of the system in question.

Similar interactions are typical of Mathematical Economics. For example one can consider the gain of the firm which is putting into sale $\mathcal{E}_i$ units of goods, supposing that other firms independently are supplying the same market with the same goods, their quantities being equal to $\mathcal{E}_1, \ldots, \mathcal{E}_n$ respectively.

The studies on creation of an artificial situation of a similar kind when identifying people suitable for living in full isolation for a long period of time are well known (the test for so-called psycho-physiological compatibility).

Professor Michail Lvovich Tsetlin was the first to draw our attention to the fact that sometimes "behavioral" aspects appear in purely technical problems, which seem to be very far from the problems of behaviour of living organisms. One of such problems is that of stable local control of the power levels in the collective of radiostations, which interfere with each other during their normal performance. In this particular problem $\mathcal{E}_i$ corresponds to the power level of i-th transmitter and ν_i corresponds to the signal-to-noise ratio at the input of i-th receiver.

It is easy to see that in all examples mentioned above under the condition that the action $\mathcal{E}_i$ is chosen independenly in each system component on the base of limited, local information related only to the value ν_i, generally speaking it is possible to have unstable, burst-like regimes. Indeed, let for example the i-th component of the system increase its $\mathcal{E}_i$ with the goal to increase the effect ν_i. In that case all other components in their turn can do nothing but increase their actions because their "effects" fall down. As a result the value of the effect in the i-th component may become even lower than the previous one and a further increase of $\mathcal{E}_i$ is required, and so on.

Thus in such complicated (large-scale) system the necessity of clearing up the possibility of obtaining a stable regime of the local choice of actions begins to play a fundamental role.

Though the majority of the paper formulations and results seems to be directly applicable to both the Mathematical Economics and the group-psychology problems, all the discussions below are related only to the power control problem in a collective of radiostations. (Note that independently such a problem is of interest for one type of communication system, namely for the asynchronous address communication systems.)

The following mathematical model is used. The collective of radiostations consists of n pairs of transmitter-receiver. Let $\mathcal{E}_i(t)$ be the power of the i-th transmitter at the moment t. We suppose that generally speaking all the pairs are interacting and write noisy input power coming from the transmitter of the j-th pair as $a_{ij}\mathcal{E}_j(t)$, where $a_{ij}\geq 0; i\neq j; i,j=1,\dots,n$.

Then, the input noise-to-signal ratio for i-th receiver $\lambda_i(t)$ is eaual to

$$\lambda_i(t)=\frac{\mathcal{N}_i+\sum_{j\neq i} a_{ij}\mathcal{E}_j(t)}{a_{ii}\mathcal{E}_i(t)} \quad (i=1,\dots,n) \tag{1}$$

Here $\mathcal{N}_i$ is additive noise power at the input of the i-th receiver; $a_{ii}\mathcal{E}_i(t)$ is the signal power coming from the transmitter of the same (i-th) pair.

It is supposed that all the information concerning the situation in the collective of radiostations at the moment t which is available for i-th transmitter consists of two quantities $\lambda_i(t), \mathcal{E}_i(t)$ (that is, in each pair there is an "ideal" feed-back).

When facing such a system one should answer the following questions. Let the positive value λ_i^0 be given for the i-th pair, $i=1,\dots,n$, and let the degree of interaction in the collective be fixed, i.e., the matrix $\|a_{ij}^{(n)}\|$ is fixed. Then do powers $\mathcal{E}_1,\dots,\mathcal{E}_n$ which provide noise-to-signal ratios to be equal to respectively $\lambda_1^0,\dots,\lambda_n^0$ exist? If the powers giving $(\lambda_1^0,\dots,\lambda_n^0)$ exist then do algorithms of local choice of power level independently in each transmitter which guarantee $\lambda_i(t)\to\lambda_i^0$, $i=1,\dots,n$, provided that i-th transmitter knows only $\lambda_i(t)$, $\mathcal{E}_i(t)$ exist?

Let Λ^n be the set of those and only those positive vectors λ (all components of λ are strictly positive) for which all the principal minors of the matrix

$$\mathcal{M}^n(\lambda)=\begin{Vmatrix} \lambda_1 a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_2 a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \cdots & \lambda_n a_{nn} \end{Vmatrix} \tag{2}$$

are (strictly) positive. It turned out that the values of powers providing an a priori given λ^0 exist (the vector λ^0 is obtainable) if and only if $\lambda^0\in\Lambda^n$. The algorithm of power choice which guarantees $\lambda_i(t)\to\lambda_i^0$, $i=1,\dots,n$, if only $\lambda^0\in\Lambda^n$, has the following analytical form

$$\lambda_i^0\frac{d\mathcal{E}_i(t)}{dt}=\varkappa_i\mathcal{E}_i(t)[\lambda_i(t)-\lambda_i^0] \quad (i=1,\dots,n) \tag{3}$$

(Here $\varkappa_i$ is fixed positive parameter of the algorithm).

From the expression (3) it follows that at each moment this algorithm uses only two values: $\lambda_i(t), \mathcal{E}_i(t)$. It is convenient to speak about the automator $\mathfrak{A}_i$ of power control, which has a continuous set of states-actions $\mathcal{E}_i$, the input λ_i and λ_i^0 as a parameter. Then, the expression (3) describes the joint behaviour of the collective of n such automata interacting in accordance with the formula (1).

For the i-th pair of radiostations it is natural from a practical point of view to try to reach an a priori given noise-to-signal ratio because it is this ratio that is responsible for the possibility of normal signal reception. But on the other hand such a criterion of choice of power level has the essential limitation that the vector a priori given values $\lambda_1^0, \ldots, \lambda_n^0$ may not be obtainable (if $\lambda^0 \notin \Lambda^n$). In the latter case in accordance with the power choice algorithm (3) the powers of all the transmitters begin to grow infinitely, i.e., the system becomes unstable.

Note that the answer of whether λ^0 is obtainable or not may be given only after an analysis of matrix (2). But the problem itself was formulated so that this information is not available for any member of the collective.

One way of allowing stability to be reached despite the locality of information in the general case (i.e., the matrix $\|a_{ij}^{(n)}\|$ is arbitrary one) is to take into account besides the input noise-to-signal ratio the power consumption. To this end we introduce the following formal definition of notion of local criterion of power control C;

Assume that the value of C_i is available to the 1-th transmitter and its task is to minimize in a definite sense the C_i by means of the proper power choice. As it was up to now, we assume that the i-th transmitter knows only the quantities $\lambda_i(t), \mathcal{E}_i(t)$ and that C_i can depend only upon them (i.e., C_i may depend upon the powers $\mathcal{E}_1, \ldots, \mathcal{E}_n$ only via $\lambda_i(t)$) and maybe upon certain parameter a priori known to the transmitter.

One of the most natural way of synthesis of the local criterion of power control which provides a stable choice of powers in an arbitrary collective of radiostations is to introduce "price" for the transmitter power.

The simplest criterion of such kind has the following form

$$C_i = \lambda_i(t) + \alpha_i \mathcal{E}_i(t)$$
$$(i = 1, \ldots, n) \tag{4}$$

where $\alpha_i > 0$ is an a priori chosen fixed positive parameter ("price").

We will study asymptotic (final) behaviour of the collective of automata of power control, C_i being the penalty function for the i-th automaton, i = 1, . . . , n. Omitting the formulation of construction of each automaton we will try to prove the existence and uniqueness theorem for certain equilibrium points (Nash parties) in corresponding games. After that there will be every reason to hope that automata of simple "gradient" tactics will provide the global stability of power choice by that automata collective. However, the description of the automata picking up the Nash party is beyond the scope of this paper.

The party of a game (i.e., the set of actions chosen by its participants) which has the property that for none of the automata is it advantageous to change its action provided that the other automata persist in their actions we will call equilibrium in the sense of Nash party or simply Nash party.

Introducing into formula (4) the expression (1) for noise-to-signal ratio $\lambda_i(t)$ in terms of transmitter powers, one can see that for criterion (4) the inequality

$$\frac{\partial^2 C_i(\mathcal{E}_1,\dots,\mathcal{E}_n)}{\partial \mathcal{E}_i^2} > 0 \qquad (i=1,\dots,n) \tag{5}$$

is always fulfilled, i.e. C_i is a concave function of the variables C_i. For such functions all Nash parties and only they satisfy the system of equations

$$\frac{\partial C_i(\mathcal{E}_1,\dots,\mathcal{E}_n)}{\partial \mathcal{E}_i} = 0 \qquad (i=1,\dots,n) \tag{6}$$

Theorem 1. For every positive vector $\alpha=(\alpha_1,\dots,\alpha_n)$ there exists a unique equilibrium in sense of Nash vector $\mathcal{E}_1,\dots,\mathcal{E}_n$ and consequently a unique vector $\lambda_1,\dots,\lambda_n$

(The proofs of all theorems in the paper will be omitted.)

It is interesting to note that an analogous criterion with "logarithmic price" on the power

$$C_i = \lambda_i + \alpha_i \ln \mathcal{E}_i \qquad (i=1,\dots,n) \tag{7}$$

already doesn't have such a property. One can see that in case of the criterion (7) the equilibrium party exists only for α from a certain region, which itself depends upon the detailed structure of interactions in the collective.

Thus criterion (4) provides the solution of the problem of stable local power control in the arbitrary collective of radiostations. However, it has a limitation which forces one to look for a better criterion.

Let all radiostations not interact at all ($a_{ij}=0$, $i \neq j$; $i,j=1,\dots,n$), but the automata of power control are ignorant of that fact (the i-th automaton knows only $\lambda_i(t)$, $\mathcal{E}_i(t)$). In that case it is principally possible to obtain any a priori given nolse-to-signal ratio in each pair of radiostations. Unfortunately, the automata performing in accordance with criterion (4) cannot provide in the equilibrium point the a priori given noise-to-signal ratio, though it is always obtainable. Indeed it is seen from expression (6) in that case for any fixed α_i the noise-to-signal ratio in a Nash party will be defined by the values N_i , a_{ii} in accordance with the formula $\lambda_i=\sqrt{\frac{\alpha_i N_i}{a_{ii}}}$. But these two values are unknown to the i-th transmitter (and it is ignorant of the fact that the collective doesn't really exist, for there is no interaction at all) and the equilibrium λ_i is a priori unknown.

However the limitation just mentioned is inherent not only for criterion (4), as a matter of fact it is a property of all criteria of a similar kind.

Consider now an arbitrary collective of power control automata, $\mathcal{E}_i(t)$, $\lambda_i(t)$ being, respectively, the action and the input of i-th automaton at the moment t . Let C_i be the penalty function of i-th automaton, provided that C_i depends upon only λ_i, $\mathcal{E}_i$ and maybe upon some parameter α_i which is a priori chosen individually in each automaton (i.e., transmitter). Then the following theorem is held.

Theorem 2. Criterion C_i which simultaneously has both properties i) and ii) doesn't exist, where the properties are the following:
i) In an arbitrary collective characterized by the matrix $\|a_{ij}^{(n)}\|$ there exists the unique equilibrium party satisfying the system of equations

$$\frac{\partial}{\partial \mathcal{E}_i} C_i(\mathcal{E}_i, \lambda_i(\mathcal{E}_1, \ldots, \mathcal{E}_n), \alpha_i) = 0 \qquad (i=1, \ldots, n) \tag{8}$$

ii) When there is no interaction at all ($a_{ij}=0$, $i \neq j$; $i,j = 1, \ldots, n$), then the equilibrium noise-to-signal ratio is equal to a certain a priori given value (independent of additive noise power on receiver input).

The reason for the "non-elasticity" lies in the fact that automata can not distinguish the additive noise from the noise coming from the partners.

Nevertheless using the same local information on the collective available to each transmitter, one can construct another automata game, the equilibrium party of which doesn't have the form (8). Participants of that new game have the ability to make a distinction between the additive noise and the noise from partners and for that reason the assertion of theorem 2 for the game doesn't hold.

Indeed, applying automaton $\mathfrak{A}_i$, providing the choice of an a priori given noise-to-signal ratio (when it is possible in principal!), one can formulate the game of automata in which the noise-to-signal ratio itself will be the action of i-th automata at the moment t. It can be done by making use of an automaton of so-called "two-levels" construction, where the first level (automaton $\mathfrak{A}_i$) is to maintain a certain noise-to-signal ratio, the value of which is chosen by the second, "higher" level.

Then Nash parties will satisfy the equations analogous to (5),(6):

$$\frac{\partial C_i(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_i} = 0 \tag{9}$$

$$\frac{\partial^2 C_i(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_i^2} > 0 \tag{10}$$

$$(i=1, \ldots, n)$$

We will study the game automata in which the penalty function of the i-th automaton will be equal to the following elastic local criterion of power control

$$C_i = \lambda_i - \frac{\delta_i^2}{\mathcal{E}_i} \frac{\partial \mathcal{E}_i(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_i} \qquad (i=1, \ldots, n) \tag{11}$$

where $\delta_i > 0$ is again a fixed numerical parameter a priori chosen in the i-th transmitter. Performing in accordance with this criterion the i-th automaton decreases the value of the noise-to-signal ratio by the way of an increase of its power $\mathcal{E}_i$ until a further negligible decrease of that quantity requires too much increase of the transmitter power-

It will be shown that this elastic criterion provide sufficient "smoothness" of the transition from the situation with weak components interaction to the one with high interaction

First of all it is desirable to formulate for this criterion a theorem analogous to theorem 2 . Before that we will give an interesting geometrical interpretation of equilibrium point (or points) which follows from (9), (10), (11).

Now suppose that the matrix $\|a_{ij}^{(n)}\|$ is irresoluble. That means that $\mathcal{M}^n(\lambda)$ is irresoluble too. Then it can be proved that the boundary of Λ^n region in space of vectors $\lambda=(\lambda_1,\dots,\lambda_n)$ may be given by the equation $\det\mathcal{M}^n(\lambda)=0$ and the condition that all other principal minors of $\mathcal{M}^n(\lambda)$ are strictly positive. This theorem follows from the positive matrix theory.

Resolving the system of equations (1) with respect to $\mathcal{E}_1,\dots,\mathcal{E}_n$ we have

$$\frac{1}{\mathcal{E}_i}\frac{\partial\mathcal{E}_i}{\partial\lambda_i}=-\frac{a_{ii}\det\mathcal{M}_i^{n-1}(\lambda)}{\det\mathcal{M}^n(\lambda)} \qquad (12)$$

where $\det\mathcal{M}_i^{n-1}(\lambda)$ is the determinant of $(n-1)$-th order principal minor of the matrix (2) corresponding to the i-th diagonal element.

Now the elastic criterion (11) may be rewritten in the form

$$C_i=\lambda_i+\delta_i^2 a_{ii}\frac{\det\mathcal{M}_i^{n-1}(\lambda)}{\det\mathcal{M}^n(\lambda)} \qquad (13)$$

$$(i=1,\dots,n)$$

Then it is easy to see that all points of equilibrium in the sense of Nash (9),(10) and only they are determined by the equation

$$\delta_i=\frac{\det\mathcal{M}^n(\lambda)}{a_{ii}\det\mathcal{M}_i^{n-1}(\lambda)} \qquad (14)$$

$$(i=1,\dots,n)$$

where $\delta_i>0$.

Expression (14) leads to a very important conclusion that in the case when the interaction among the radiostations is equal to zero (it means that the matrix $\|a_{ij}^{(n)}\|$ is an upper (lower) triangular one), then the equilibrium λ_i satisfys $\lambda_i=\delta_i$. Thus, indeed, the elastic criterion lets us in this case provide an a priori chosen value noise-to-signal ratio.

Further, with the obvious transformation the relation (14) can be rewritten in the determinant form

$$\begin{vmatrix} \lambda_1 a_{11} & -a_{12} & \cdots & -a_{1i} & \cdots & -a_{1n} \\ -a_{21} & \lambda_2 a_{22} & \cdots & -a_{2i} & \cdots & -a_{2n} \\ & & & & & \\ -a_{i1} & -a_{i2} & \cdots & (\lambda_i-\delta_i)a_{ii} & \cdots & -a_{in} \\ & & & & & \\ -a_{n1} & -a_{n2} & \cdots & -a_{ni} & & \lambda_n a_{nn} \end{vmatrix}=0 \qquad (15)$$

$$(i=1,\dots,n)$$

Here $(\lambda_1,\dots,\lambda_n)$ is some equilibrium vector. Now it becomes clear that points with coordinates

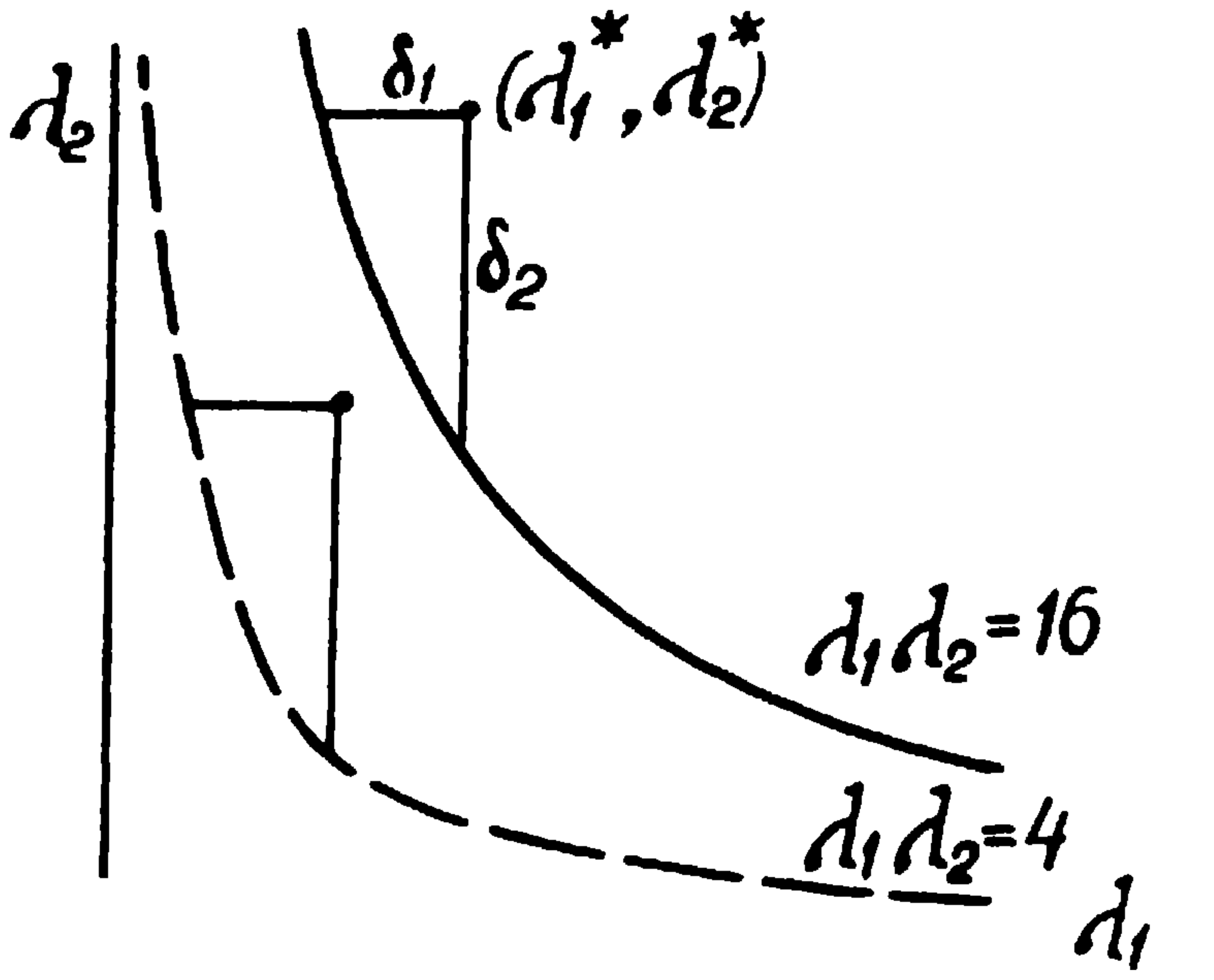
$$(\lambda_1,\dots,\lambda_{i-1},(\lambda_i-\delta_i),\lambda_{i+1},\dots,\lambda_n) \qquad (16)$$

$$(i=1,\dots,n)$$

lie on the boundary surface of Λ^n in the space of vectors λ .

In other words the equilibrium point (points) $(\lambda_1,\dots,\lambda_n)$ corresponding to a certain choice of the parameters $\delta_1,\dots,\delta_n$, which determine the criterion (11) in each automaton-transmitter lies in the Λ^n region with its distances from the boundary surface of Λ^n (measured along coordinate axes $\lambda_1,\dots,\lambda_n$, respectively) being equal to $\delta_1,\dots,\delta_n$, respectively.

The drawing gives an example for $n=2$. It is seen that exactly one equilibrium point corresponds to each pair of positive δ_1, δ_2 . (hyperbolas shown on the drawing are two examples of the location of the boundary surface of Λ^2 region, for weak and strong



interaction of radiostations, respectively).

The proof of the existence theorem for the equilibrium point is complicated enough in case n > 2 and it was done previously only for certain natural class of collective of radiostations "where the interaction is not too strong".

Consider the class Φ of all possible collectives of radiostations with each element of Φ meeting the following requirement. Any vector $(\delta_{i_1}, \ldots, \delta_{i_{n-1}})$ composed from $(n-1)$ coordinates of the vector $(\delta_1, \ldots, \delta_n)$ should be obtainable in the subcollective, the indexes of radiostations of which are respectively $i_1, \ldots, i_{n-1}$, i.e., $\det \mathcal{M}_i^{n-1}(\delta_{i_1}, \ldots, \delta_{i_{n-1}})$ and all the principal minors of this determinant are strictly positive (of course the vector $(\delta_1, \ldots, \delta_n)$ shouldn't necessarily be obtainable in the whole collective).

<u>Theorem 3»</u> There is always at least one Nash equilibrium point in the class Φ

The proof of the theorem consists of a number of more or less independent steps justifying in that case the applicability of the fixed point theorem.

Summing up, one can say that in the formulation of the problem of stable (elastic) local power control considered above, it is the introduction of a "price" on power (or on its increment) that provides global stability.
If it were not for questions of the purely technical problem, we would say that such a "split" of aim of each participant (besides the natural will to obtain the minimal value of λ_i he wants to guarantee the stability) is analogous to the necessity for an individual to stick to certain moral principles for the sake of stability of society as a whole.