LARGE SYSTEMS AND THEIR REGULAR EXPRESSIONS: AN APPROACH TO PATTERN RECOGNITION

David C. Rine
Statistics and Computer Science Department
West Virginia University
Morgantown, West Virginia
U. S. A.

Abstract.

Systems theory deals with classes of identifiable parts each interacting in such a way that a given class exists together to satisfy certain specific requirements; these parts can be thought of an components of the system, some of which are permanent and some not. In analyzing a given system (see (13)) one usually offers up an m-tuple consisting of devices to be analyzed, primitives to represent any device, allowable compositions, concepts of simulation, and theorems that tell how the devices are to be analyzed.

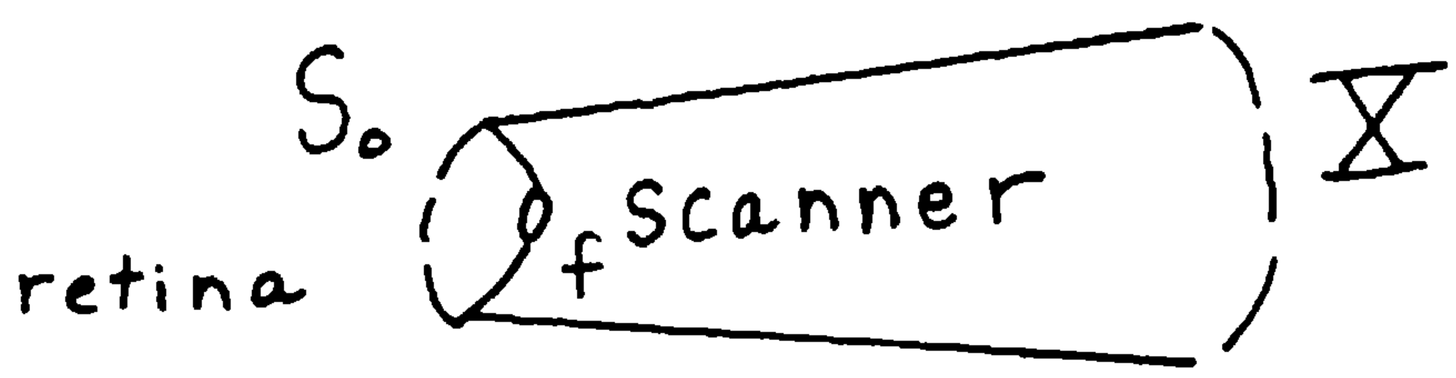
Category theory serves as an organizational tool for large systems.

L t $\mathcal{C}$ e a category with $\mathcal{S}$ the category of sets. $\mathcal{S}^{\mathcal{C}}$ is the category of functors $\mathcal{C} \longrightarrow \mathcal{S}$ whose morphisms are natural transformations between functors. We call a universe whose objects are node labels and whose morphisms are edge labels. Functor theory, having now reached a level of variety and depth in descriptive power, stands ready to help the systems theorist characterize those states which are important in the study of a particular system.

A computer with a TV camera is a tissue scanner whose job is to verify homogeneity. That is, it detects flaws or holes in a tissue. It is well know that when two topological spaces are homeomorphic, then their fundamental groups are isomorphic. The job of the functor π is to "record holes" in spaces.

Zeeman'8 (38) concept of a tolerance space is useful when dealing with visual acuity, while Wallace's (35) concept of a separation space is helpful in recording the position of patterns.

Let $(S_0, \mathcal{T}_{S_0})$ be a retina, f a lens and $(X, \mathcal{T}_X)$ an observed object. Then, (S_0, f) π-recognizes X when $\pi(S_0) \cong \pi(X)$.



The algebraic π-structure of certain spaces has already been characterized; take, for example, compact connected 2-manifolds.

A large system is determined by a subcategory of $\mathcal{S}^{\mathcal{C}}$ and is denoted $A = \langle Ob\,\mathcal{S}^{\mathcal{C}}/A, Morph\,\mathcal{S}^{\mathcal{C}}/A, \tau_A \rangle$ where $Ob\,\mathcal{S}^{\mathcal{C}}/A$ are states of A, $Morph\,\mathcal{S}^{\mathcal{C}}/A$ are inputs of A, and τ_A is the transition defined by $\tau_A(F, \eta) =$

$$\begin{cases} \eta(F) & \text{when } F = \text{domain } \eta \\ F & \text{otherwise.} \end{cases}$$

For example, $Ob\,\mathcal{S}^{\mathcal{C}}/A$ could be just those functors mapping into vector spaces (See Kelly and MacLane (17)) (groups) and $Morph\,\mathcal{S}^{\mathcal{C}}/A$ just those natural transformations between linear-functors (group functors).

If $G, F \in \mathcal{S}^{\mathcal{C}}$, then $G \xrightarrow{\eta} F$ is called an expression. An expression η is a category-regular expression if the following hold:

(1) $\exists$ large system $A = \langle Ob\,\mathcal{S}^{\mathcal{C}}/A, Morph\,\mathcal{S}^{\mathcal{C}}/A, \tau_A \rangle$ so that $\eta \in Morph\,\mathcal{S}^{\mathcal{C}}/A$;

(2) $\exists\ \mathcal{D}_A \subseteq Ob\,\mathcal{S}^{\mathcal{C}}/A$ subcollection of functors;

(3) $\exists\ S_0 \in Ob\,\mathcal{C}$, so that $[S_0, \cdot\,] \in Ob\,\mathcal{S}^{\mathcal{C}}/A$;

(4) $\exists\ F \in \mathcal{D}_A$ so that $[S_0, \cdot\,] \xrightarrow{\eta} F$ is a natural equivalence.

It is natural to define what one means by acceptance. An acceptor is denoted by $A = \langle [S_0, \cdot\,], Ob\,\mathcal{S}^{\mathcal{C}}/A, Morph\,\mathcal{S}^{\mathcal{C}}/A, \tau_A, \mathcal{D}_A \rangle$ where

$[S_0,\bullet]$ is called the initial state and $\mathcal{D}_A \subseteq \mathrm{Ob}\,\mathcal{S}^{\mathcal{C}}/A$ the final states. We assume $[S_0,\bullet] \notin \mathcal{D}_A$. Thus, $\eta \notin [[S_0,\bullet],-]$ implies $\tau_A([S_0,\bullet],\eta) = [S_0,\bullet]$, and $L(A,\mathcal{D}_A) = \{\eta \in \mathrm{Morph}\,\mathcal{S}^{\mathcal{C}}/A \mid [S_0,\bullet] \xrightarrow{\eta} F$ natural equivalence and $F \in \mathcal{D}_A\}$.

An example of an acceptor appears within the structure of Give'on's (11) study of transition systems. Let $\mathcal{S}^W$ be the category of semimodules with fixed input monoid W. Also, let $S:\mathcal{S}^W \longrightarrow \mathcal{S}$ be the forgetful functor with M_W the semimodule whose states are W and whose transition is determined by multiplication of W. Then, there is a natural equivalence $[M_W,\bullet] \xrightarrow{\rho^{-1}} S$; hence, we have an important acceptor of the form $A = \langle [M_W,], \mathrm{Ob}\,\mathcal{S}\mathcal{S}^W/A, \mathrm{Morph}\,\mathcal{S}\mathcal{S}^W/A, \tau_A, \mathcal{D}_A \rangle$ where $S \in \mathcal{D}_A$, $\rho^{-1} \in L(A,\mathcal{D}_A)$.

The following theorem of Mitchell (23) is going to see more use in systems theory.

Theorem (M). Let $\mathcal{A}$ be a cocomplete abelian category and let R be any ring (commutative with unity). Then, we have an additive, colimit preserving, covariant bifunctor

$$\otimes_R : \mathcal{G}^R \times {}^R\!\mathcal{A} \longrightarrow \mathcal{A}$$

and a natural equivalence of trifunctors

$$[M, [C,A]_{\mathcal{A}}]^R \simeq [M \otimes_R C, A]_{\mathcal{A}}$$

with $M \in \mathcal{G}^R$, $C \in {}^R\!\mathcal{A}$, and $A \in \mathcal{A}$.

Moreover, Rine(30) has coined the concept of an R-linear separated system, having some non-linear characteristics, and proved, using Mitchell's theorem, this theorem.

Theorem. Let $\mathfrak{A} = \langle U, X, \delta \rangle$ be an R-linear separated transition system. Let $W_R(Y) \xrightarrow{\alpha} V_R$ be a linear interpretation with compiler $W_R(Y) \xrightarrow{H} U \subseteq (Y \sqcup \{R,+\})^*$ in $\mathcal{G}^R$. Let $V_R \xrightarrow{\bar{h}} [X, X] \subseteq [V^* \sqcup \{0,1\}, V^* \sqcup \{0,1\}]$ be an arbitrary loading morphism. Assume that $\hat{\delta} H \neq h\alpha$ and $\mathfrak{A}$ will not execute α. Then, there exists a corrected R-linear separated transition system $\langle U, X, \bar{\delta}^{\#} \rangle$ that will execute α.

In automata theory the weakest algebraic structure permitted is usually the semigroup, and when $B \in \mathrm{Ob}\,\mathcal{C}$ one has no reason to believe that $[S_0, B]$ is a semigroup.

If $\mathcal{C}$ is a category and $[A,B]_{\mathcal{C}}$ is a semigroup, the semigroup structure may arise in several somewhat unnatural ways. This must first be clarified in the following proposition.

Proposition. Let $S_0 \in \mathrm{Ob}\,\mathcal{C}$ projective generator for $\mathcal{C}$ and $\forall_A$, $B \in \mathrm{Ob}\,\mathcal{C}$ $[A,B]$ a semigroup; then $[S_0,\bullet]$ is an embedding into the category of semigroups, preserving and reflecting monics and epics.

Hence, we shall assume the hypothesis of this proposition and call such a $\mathcal{C}$ admissible' for semigroup (category)-regularity.

Hence, we call $\eta \in \mathrm{Morph}\,\mathcal{S}^{\mathcal{C}}$ regular if

(1) η is category-regular with respect to some acceptor $A = \langle [S_0,\bullet], \mathrm{Ob}\,\mathcal{S}^{\mathcal{C}}/A, \mathrm{Morph}\,\mathcal{S}^{\mathcal{C}}/A, \tau_A, \mathcal{D}_A \rangle$;

(2) $\mathcal{C}$ is admissible;

(3) $\forall\ S_0 \xrightarrow{f} B$ in $\mathcal{C}$ isomorphism, then $F(S_0) \xrightarrow{F(f)} F(B)$ is a variables-isomorphism, where variables depends upon those things A is observing (vector spaces, top. spaces).

Proposition. $[S_0,\bullet] \xrightarrow{\eta_0} F \in L(A, \mathcal{D}_A)$,

$F_1 \xrightarrow{\eta} F_2$ natural equivalences so that $F = F_1$ and $F_2 \in \mathcal{D}_A$ implies $\eta_\circ \eta \in L(A, \mathcal{D}_A)$. Moreover, for arbitrary $\eta_\circ, \eta$ $\tau_A([S_0, \bullet],$ $\eta_\circ \eta) = \tau_A(\tau_A([S_0, \bullet], \eta_\circ), \eta)$ is defined.

Let $\mathcal{B} = I$ be a small category. Then, we will call nodes (states) $D: I \longrightarrow \mathcal{S}$ diagrams. $\eta \in \mathrm{Morph}\,\mathcal{S}^{\mathcal{B}}$ is small category-regular if (1) η is category-regular with respect to $\mathcal{B}$, and (2) $\mathcal{B}$ is a small category.

The corresponding $A = \left\langle [S_0, \bullet], \mathrm{Ob}\,\mathcal{S}^I/A, \mathrm{Morph}\,\mathcal{S}^I/A, \tau_A, \mathcal{D}_A \right\rangle$ is called a small system.

Proposition. η small category-regular implies that the coordinates class of η has cardinality.

η is small regular if (1) η is small category-regular, (2) I is admissible, and (3) $\forall\, S_0 \xrightarrow{f} B$ in I isomorphism, then $F(S_0) \xrightarrow{F(f)} F(B)$ is a variables-isomorphism.

Proposition. $\forall\, f \in [S_0, B]$ bijection $\exists$ semigroup isomorphism $[S_0, S_0]_{\circ} \xrightarrow{f\bullet} [S_0, B]_{\circ_{f^{-1}}}$, where "o" is composition and $(f\bullet)(x \circ y) = (f\bullet)(x) \circ_{f^{-1}} (f\bullet)(y)$.

In the following diagram η_{S_0}, η_B need not be semigroup isomorphisms, so that variables-isomorphism may only be bijective.

$$\begin{tikzcd} {[S_0, S_0]} \arrow[d, "f\bullet"'] \arrow[r, phantom, "\overset{\eta_{S_0}}{\cong}"] & F(S_0) \arrow[d, "F(f)"] \\ {[S_0, B]} \arrow[r, phantom, "\overset{\eta_B}{\cong}"] & F(B) \end{tikzcd}$$

Let $\mathcal{B} = I$ be a finite category where $I = \coprod_{i \in \mathrm{Ob}\,I} \{i\} \sqcup \coprod_{m \in \mathrm{Morph}\,I} \{m\}$. Then, we will call state (nodes) $D: I \xrightarrow{\text{finite}} \mathcal{S}$ graphs and $S_0 = i_0 \in \mathrm{Ob}I$ a point (points are not identically the same as nodes or states). $G_1 \xrightarrow{\eta} G_2 \in \mathcal{S}^{I\,\text{finite}}$ is finite category-regular if (1) η is small category-regular with respect to I, and (2) I is a finite category.

The corresponding $A = \left\langle [i_0, \bullet], \mathrm{Ob}\,\mathcal{S}^I/A, \mathrm{Morph}\,\mathcal{S}^I/A, \tau_A, \mathcal{D}_A \right\rangle$ is called a graph system.

η is finite regular if (1) η is finite category-regular, (2) I^{finite} is admissible, and (3) $\forall\, i_0 \xrightarrow{f} b$ in I isomorphism, then $F(i_0) \xrightarrow{F(f)} F(b)$ is a variables-isomorphism.

Remark. In applications to programming $D(i_0)$ can be thought of as a subroutine reached from $[i_0, i_0]$.

Remark. One might try to think of the automorphism functors as something that changes properties being considered. Can one think of it as being a rearrangement of subroutines, a switch in control, a new sorting level, sorting level, sorting routine, scanning, searching!

Shaw (34) has considered the parsing of graph-representable pictures and gives a picture parsing algorithm that is an n-dimensional analog of a classical top-down string parser, and an application of an implemented system to the analysis of spark chamber film.

Remark. One can show how the abstract notion of category theory, in particular taking limits of diagrams in complete (colimits in cocomplete) categories (Mitchell, 23), attacks these problems.

ACCEPTANCE

In the last section we introduced the notion of ACCEPTORS; we now consider acceptance.

As was mentioned previously, one of the important natural phenomena that cocomplete (complete) category studies handle is the notion of sorting. With this in mind, let us turn to Berthiaume's (A) definition of cofinal functor.

Let $\mathcal{A}$, $\mathcal{B}$ be small categories with $\mathcal{C}$ $\mathcal{A}$, $\mathcal{B}$ -cocomplete; then, there is a natural transformation $(\mathcal{C}^{\mathcal{B}} \xrightarrow{F^I} \mathcal{C}^{\mathcal{A}} \xrightarrow{colim} \mathcal{C})$ $\overset{\delta}{\rightsquigarrow}$ $(\mathcal{C}^{\mathcal{B}} \xrightarrow{colim} \mathcal{C})$ between functors with $F^I(D) = D \circ F$ for any $D \in \mathcal{C}^{\mathcal{B}}$. $D \circ F$ is a "sub-functor" of D.

Let $\mathcal{Y}, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be a sequence of small categories, and let $S_1 S_2 \ldots S_n$ be a composition of n functors where $S_1 \in \mathcal{Y}^{\mathcal{X}_1}$ and $S_k \in \mathcal{X}_{k-1}^{\mathcal{X}_k}$; k = 2,...,n.

$$\mathcal{X}_n \xrightarrow{S_n} \cdots \mathcal{X}_2 \xrightarrow{S_2} \mathcal{X}_1 \xrightarrow{S_1} \mathcal{Y} \longrightarrow \mathcal{C}$$

Let $\mathcal{C}$ be $\mathcal{X}_n, \ldots, \mathcal{X}_1, \mathcal{Y}$ -cocomplete; then, there exist natural transformations $\mathrm{colim}(S_1 \ldots S_i)^I \overset{\delta_i}{\rightsquigarrow} \mathrm{colim}$, $\mathrm{colim}(S_1 \ldots S_j)^I \overset{\epsilon_{j-1}}{\rightsquigarrow} \mathrm{colim}(S_1 \ldots S_{j-1})^I$ so that $\delta_i = \delta_{i-1}\, \epsilon_{i-1}$, where i = 1,...,n and j = 2,...,n+1. If each S_k is cofinal (If $S_1 S_2 \ldots S_n \in \mathcal{Y}^{\mathcal{X}_n}$ is cofinal), then, by Berthiaume's (4) proposition, each δ_k (δ_n) is a natural equivalence.

Let us define $S_0 = \mathrm{colim}(S_1 \ldots S_n)^I$ so that $S_0(D) = \mathrm{colim}(D \circ S_1 \ldots S_n) \in \mathrm{Ob}\, \mathcal{C}$. One can now give another definition of acceptance and (colimit) category-regular expression. Let $S_n(n) = S_0$ arise from a sequence $S_1, \ldots, S$ of functors as before, and let $\mathcal{C}$ satisfy the ne cessary cocompleteness properties. We admit a 5-tuple A = $\left\langle S_0(n), \mathrm{Ob}\, \mathcal{C}^{\mathcal{X}_n}/A, \mathrm{Morph}\, \mathcal{C}^{\mathcal{X}_n}/A, \tau_A, \mathcal{D}_A(n) \right\rangle$ where $\mathcal{D}_A(n) = \mathcal{D}_A \subseteq \mathrm{Ob}\, \mathcal{C}^{\mathcal{X}_n}/A$. At this point one should substitute for $\mathcal{C}$ the category of sets $\mathcal{S}$. Then, looking at some sequence of functors of length m, a natural question to ask is whether or not $S_0(m)(D)$ is a semigroup, group, or whatever. Thus, the notion of variables mentioned earlier becomes a search for properties after a certain level: semigroups, groups,...

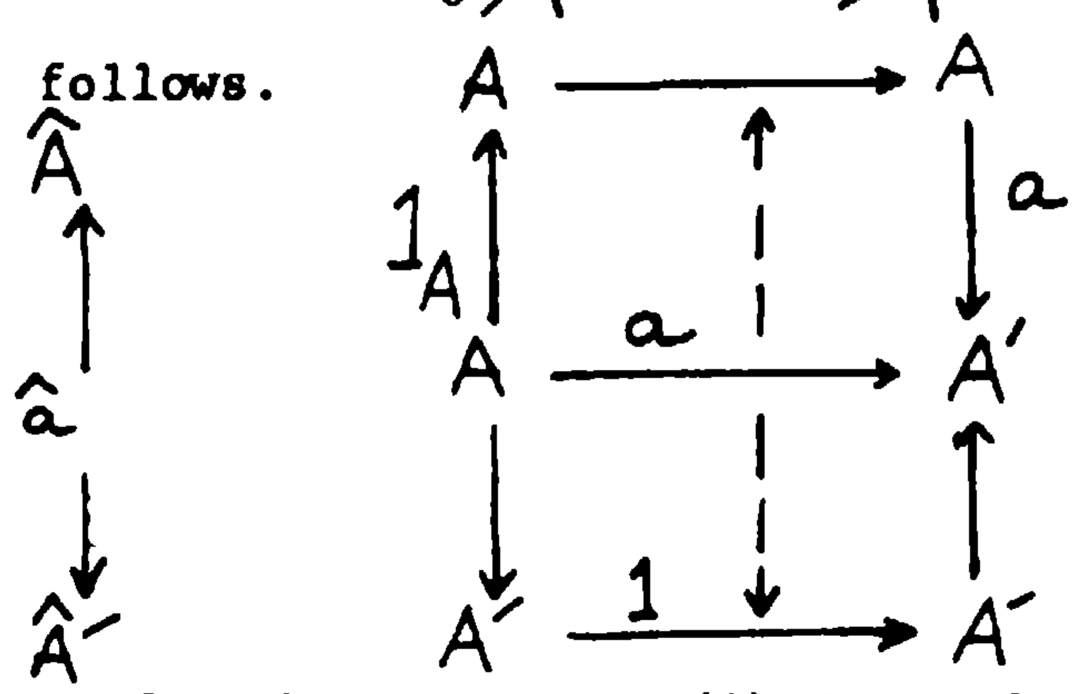
There are two ways to approach a problem (analysis and synthesis). One notion, like "sorting", is given F - colim, then find S_0; this seems to be what category theory is all about at this point, i.e. that of synthesis or putting things together. The other notion, like "listing", is given S_0, then find F « colim; this is more the notion of analysis or seeing how things are put together; for example, what pictures are described by a given picture description grammar or what languages can be written down by a given finite state acceptor (automaton, directed graph with entry point) ; language theorists call an analysis a parse. We include one more theorem about these very general acceptors.

Theorem. System B = $\left\langle [j_0, \cdot\,], \mathcal{S}^I/B, \mathcal{D}_B \subseteq \mathrm{Ob}\, \mathcal{S}^I, \tau_B \right\rangle$ can simulate system A = $\left\langle [i_0, \cdot\,], \mathcal{S}^I/A, \mathcal{D}_A \subseteq \mathrm{Ob}\, \mathcal{S}^I, \tau_A \right\rangle$ if and only if $\exists!$ functor to functor correspondence by natural transformations between $\mathcal{D}_A$ and $\mathcal{D}_B$.

The study of systems with foundations in category theory is not entirely new (see 10, 11, 12, 33); but, a general theory of abstract systems using simulation as a category morphism in order to better understand coordination problems is new (27, 28, 29).

SYSTEMS THEORY

If $\mathcal{A}$ is a category, then one can construct the Kan subdivision category, $\hat{\mathcal{A}}$, of $\mathcal{A}$ (17, 4). Also, from $\mathcal{A}$, one can construct the category $\tilde{\mathcal{A}}$ of "twisted morphism" of $\mathcal{A}$ (17, 4). Let $(\tilde{\mathcal{A}})^*$ stand for the dual of $\tilde{\mathcal{A}}$; then, there is a functor $T_0: \hat{\mathcal{A}} \longrightarrow (\tilde{\mathcal{A}})^*$ described as follows.



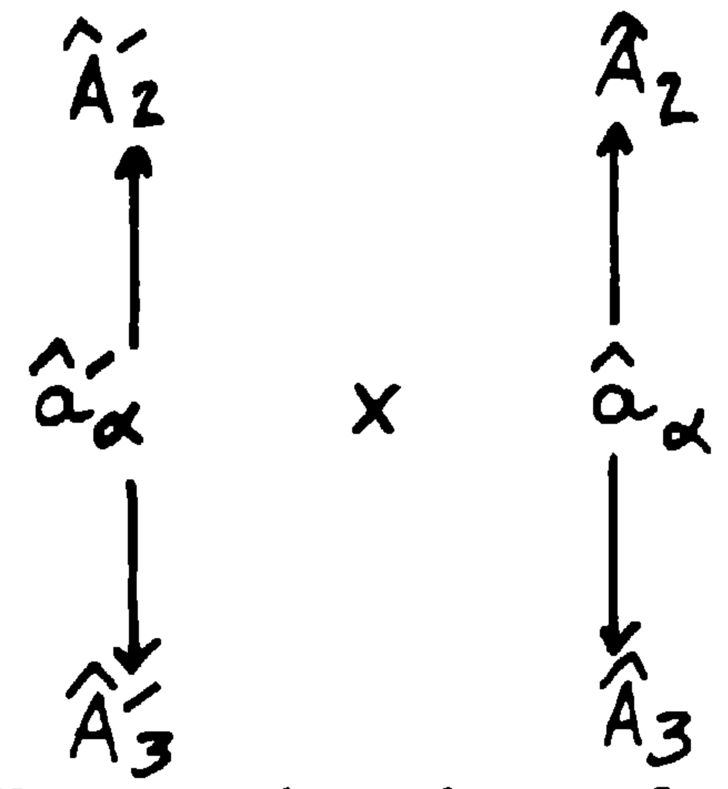
We refer the reader to (4) for definitions of connected category and cofinal functor. Berthiaume (4) proves the following proposition.

Proposition (B). If $F: \mathcal{A} \longrightarrow \mathcal{B}$ is cofinal, $D: \mathcal{B} \longrightarrow \mathcal{C}$ and colim D exists in $\mathcal{C}$, then δ (D) is an isomorphism. If δ is $\mathcal{B}$-co-complete, then δ is a natural equivalence.

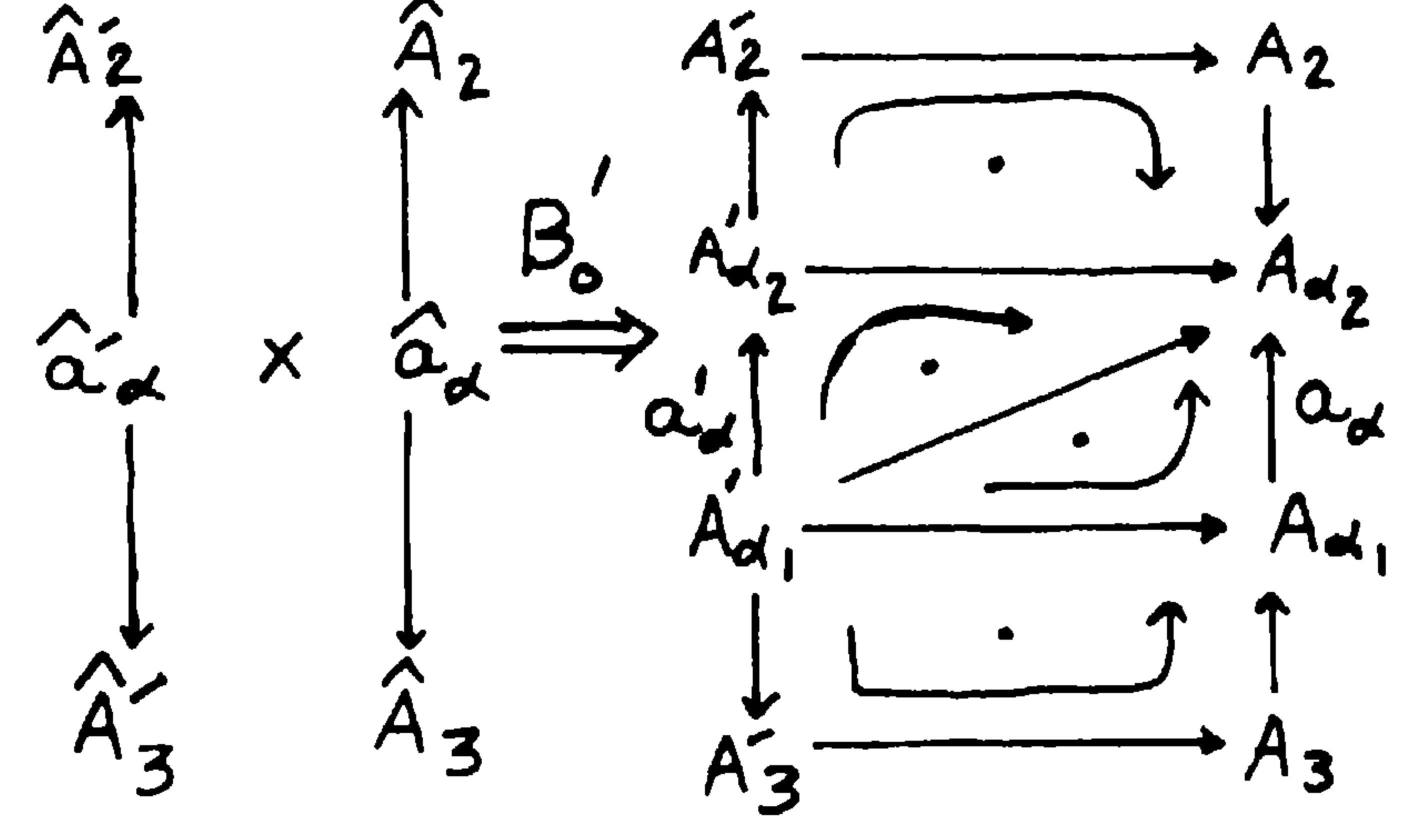
Moreover, the following theorem is due to MacLane (17).

Theorem (M). $T_0: \hat{\mathcal{A}} \longrightarrow (\tilde{\mathcal{A}})^*$ is confinal.

We now extend the Kan category to a more general form; let $\mathcal{A}' \subseteq \mathcal{A}$ be a subcategory of $\mathcal{A}$. Then, let $(\mathcal{A}', \mathcal{A})$ be the category of abstract systems with inputs from $\mathcal{A}'$ and state-outputs from $\mathcal{A}$. We define the generalized Kan category over the pair $\mathcal{A}', \mathcal{A}$ by considering pairs as follows.



Next, we introduce a functor $B_0': \hat{\mathcal{A}}' \times \hat{\mathcal{A}} \longrightarrow (\mathcal{A}', \mathcal{A})$ described below, where it is assumed that every [A', A] is non-empty.



It is obvious that the first property (4) of cofinality is satisfied, but it is not apparent that for every a $\in$ [A', A]$_{(\mathcal{A}',\mathcal{A})}$ a/B_0'-is connected. The same difficulty is encountered in the more restricted "classical" definition of cofinality mentioned in (17,4). Moreover, MacLane's proof for T_0 cofinal does not work when trying to show B_0' cofinal.

A good discussion of classical input-output systems motivated from differential equations can be found in references (36, 3, 22), especially where Zadeh and Polak discuss consistent abstract objects (36). We are making much use of the following notion. Let us consider the binary form of a general system (21); this is a relation $S \subseteq U \times Y$. The u's are in U_s and the y's in Y_s. Let Z be a set and S_ a function mapping any subset of Z x U into Y (not nec. U_S, Y_S), denoted $(Z \times U) \xrightarrow{S_Z} Y$.

Z represents the <u>initial global state objec</u> for S if $S_Z(z, u) = y \Longleftrightarrow (u,y) \in S$. $\forall\ z \in Z$ $\exists$ function $(U) = U_z \xrightarrow{S_z} Y$. $S = \bigsqcup_{z} S_z$. $U' \subseteq U$ <u>acceptable</u> by S_Z if and only if $\exists\, z \in Z \ni U' = U_Z$.

Consider the following two explicit representations of circuits.

$$y_1(t) = \alpha_0 e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\hat{t})} u_1(\hat{t})\, d\hat{t}$$

$= \alpha_0 E(t) = E * u_1(t)$, where "*" is convolution, and

$$y_2(t) = \beta_0 e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-\hat{t})} u_2(\hat{t})\, d\hat{t}$$

$= \beta_0 E(t) + E * u_2(t)$. Let $S^1_{\alpha_0}$, $S^2_{\beta_0}$ represent these two devices. We will say that $S^2_{\beta_0}$ <u>simulates</u> $S^1_{\alpha_0}$ if there is an encoder H and a decoder h so that the following diagram is commutative.

$$\begin{tikzcd} \bullet \arrow[r, "S^1_{\alpha_0}"] \arrow[d, "H"'] & \bullet \\ \bullet \arrow[r, "S^2_{\beta_0}"'] & \bullet \arrow[u, "h"'] \end{tikzcd}$$

This means that $h(y_2) = y_1 = \alpha_0 E + E * u_1$ and $y_2 = \beta_0 E + E * H(u_1) = \beta_0 E + E * u_2$.

$S = \bigsqcup_{z} S_z$ (S is a relation and S_z a function) forms the class of objects of a category, where S_{z_1} is mapped to S_{z_2} if there exists a pair (H, h) such that the following diagram commutes.

$$\begin{tikzcd} X_{z_1} \arrow[r, "S_{z_1}"] \arrow[d, "H"'] \arrow[dr, phantom, "\bullet"] & Y_{z_1} \\ X_{z_2} \arrow[r, "S_{z_2}"'] & Y_{z_2} \arrow[u, "h"'] \end{tikzcd}$$

Now $\bigsqcup X_{z_i} = (X) = \Delta_0 S$ and $\bigsqcup Y_{z_i} = (Y) = \Delta_1 S$; moreover, (X) is just a set, whereas it may turn out that the X_{z_i} have more structure. For example, each X_{z_i} might be a monoid with left cancellation; these monoids are important to automata theory (10, 11). All told, we have a class $\mathcal{S} = U \times Y$, $\mathcal{S} = \bigsqcup_{\alpha} S^{\alpha}$ (usually finite), and categories (general system) $S^{\alpha} = \bigsqcup_{z} S^{\alpha}_{z}$.

<u>Proposition</u>. Let $\mathcal{A}$ be a category with products and finite intersections, and let D be a diagram in $\mathcal{A}$ over a scheme (I, m, d). Then, a limit for D is given by the family of compositions

$$\bigcap_{m \in M} \mathrm{Equ}\,(p_k, D(m)p_j) \subset \bigtimes_{h \in I} D_h \longrightarrow D_i$$

where $d(m) = (j, k)$, and p_i represents the <u>ith</u> projection from the product.

Let $W \in \mathcal{M}$ (monoids) and $S \in \mathcal{S}$. A function $s \in [W, S]_{\mathcal{S}}$ is called an <u>abstract machine</u>. A simulation relation can be established between two abstract machines s_1, s_2 by the following commutative diagram where H is a monoid morphism and h is a function.

$$\begin{tikzcd} W_1 \arrow[r, "s_1"] \arrow[d, "H"'] \arrow[dr, phantom, "\bullet"] & S_1 \\ W_2 \arrow[r, "s_2"'] & S_2 \arrow[u, "h"'] \end{tikzcd}$$

It is obvious that two such relations (H', h'), (H, h) can be composed in a natrual way; but, it is also obvious that a single relation can have many domains and codomains. Hence, the usual procedure is to replace each (H, h) by a triple (s, (H, h),t) to establish a <u>morphism</u> from s to t; we therefore can extend the class of machines and relations to a category of machines and morphisms that we denote by $(\mathcal{M}, \mathcal{S})$.

Proposition. $\mathcal{M}$ has (equalizers, finite products, pullbacks), $\mathcal{S}$ has (co-equalizers, finite co-products, pushouts), and $(\mathcal{M}, \mathcal{S})$ has (equalizers, pullbacks).

Moreover, since $\mathcal{M}$ has limits and $\mathcal{S}$ has colimits every connected diagram of abstract machines has a best possible representation in $(\mathcal{M}, \mathcal{S})$.

Lawvere and Berthiaume (16,4) have used the notion of *comma category*: if $\mathcal{B} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{D}$ is a pair of functors, then an *object* in the comma category (F,G) is a triple (B,c,D), where $F(B) \xrightarrow{c} G(D)$, and a *morphism* (B,c,D) $\longrightarrow$ (B',c',D') in (F, G) is a pair ($B \xrightarrow{b} B'$, $D \xrightarrow{d} D'$) such that G(d)c = dF(b), i.e. (c,b,d,c').

$$\begin{array}{ccc} F(B) & \xrightarrow{\quad c \quad} & G(D) \\ {\scriptstyle F(b)}\downarrow & & \downarrow{\scriptstyle G(d)} \\ F(B') & \xrightarrow{\quad c' \quad} & G(D') \end{array}$$

Replacing G by G^o gives us a (covariant, contravariant) pair of functors. Hence, assume the pair $\mathcal{B} \xrightarrow{F} \mathcal{C} \xleftarrow{G^o} \mathcal{D}$ where $\mathcal{A}'$ has limits and $\mathcal{A}$ has colimits. Moreover, let F be limit preserving and G^o colimit preserving where $(B,c,D) \xrightarrow{(F(b),G^o(d))} (B',c',D')$. Thus, we have a "twisted comma category", denoted (F, G^o); and, we have the following theorem.

Theorem.

Hypothesis: (1) Let $\mathcal{A}' \subseteq \mathcal{A}$ be a subcategory of $\mathcal{A}$, and let $\mathcal{B}, \mathcal{D}$ be connected, small categories with infinite products;

(2) (F, G^o) is a twisted comma "category;

(3) $F: \mathcal{B} \longrightarrow \mathcal{A}'$ has a left root $\underleftarrow{\lim}\, F$;

(4) $G^o: \mathcal{D} \longrightarrow \mathcal{A}$ has a right root $\underrightarrow{\lim}\, G^o$.

Conclusion: (1') (F, G^o) can be extended to a category;

(2') $(\mathcal{B}, \mathcal{D}) \xrightarrow{(F,G^o)} (\mathcal{A}', \mathcal{A})$ has a left root $\underleftarrow{\lim}$ (F, G^o);

(3') in $(\mathcal{A}', \mathcal{A})$ $\underleftarrow{\lim}\, F \overset{\lim}{\Longleftarrow} \xrightarrow{(F,G^o)} \underrightarrow{\lim}\, G^o$.

Assertion. Let I be a small category with infinite products (a scheme). Let I^o be the dual of I; $I \times I^o$ is the scheme whose objects are $i \in I$; if $i \xrightarrow{m} j$ is a morphism, then a morphism in $I \times I^o$ is denoted $i \xrightarrow{(m,m^o)} j$. Hence, $i = m^o jm$ and I is *equivalent* to $I \times I^o = (I, I^o)$.

Corollary. Let $(\mathcal{M}'; S', T'; \mathcal{A}', \mathcal{B}')$ be an adjoint situation and let co-$(\mathcal{M}; S, T; \mathcal{A}, \mathcal{B})$ be a co-adjoint situation where $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq \mathcal{B}$, $S' = S/\mathcal{A}'$, and $T' = T/\mathcal{B}'$, into $\mathcal{B}'$ and $\mathcal{A}'$ respectively. Then, there exists $((\mathcal{M}', \mathcal{M}); \bar{S}, \bar{T}; (\mathcal{A}', \mathcal{A}), (\mathcal{B}', \mathcal{B}))$ adjoint situation.

Freyd (9) has shown that $\mathcal{A}' \subseteq \mathcal{A}$ subcategory is coreflective (relfective) if and only if its inclusion functor has a left-adjoint (right-adjoint).

Proposition. Let $\mathcal{A}_o \subseteq \mathcal{A}_1' \subseteq \mathcal{A}$. Assume that $I: \mathcal{A}_o \longrightarrow \mathcal{A}$ inclusion functor has a left-adjoint $R: \mathcal{A} \longrightarrow \mathcal{A}_o$ and $J: \mathcal{A}_o \longrightarrow \mathcal{A}_1'$ inclusion functor has a right-adjoint $\bar{R}: \mathcal{A}_1' \longrightarrow \mathcal{A}_o$. Then, $\underset{\sim}{R}: (\mathcal{A}_1', \mathcal{A}) \longrightarrow$

$(\mathcal{A}_0 \cdot \mathcal{A}_0)$ is a reflector if and only if $\underset{\sim}{I} =$ (J, I): $(\mathcal{A}_0, \mathcal{A}_0) \longrightarrow (\mathcal{A}_1', \mathcal{A})$ inclusion functor has a right-adjoint $\underset{\sim}{R}$.

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