ON METHODS TO DECIDE SOLVABILITY OF TRANSFORMATION PROBLEMS

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Abstract

Methods are introduced to decide the solvability of transformation problems. Such methods may be applied to save some of the problem solving effort. Supposition for applying the stated methods is that information about the structure of the problem space is available for detecting separated subsets of the situation set. Solvability of transformation problems is decided by means of systems of separated subsets of situations. For a class of problem spaces the detection of separated subsets of situations and the decision of solvability of problems are investigated in more detail.

Introduction

The solution of transformation problems can be more easily found if information about the structure of the problem space is available. Such information may be *for* instance an evaluation function describing an approximation of the goal distance or an approximative description of the applicability sets of the operators (Pohl /1/, Banerji /2/).

Structural information about the problem space may also be used to decide the solvability of any given problem. This saves fruitless effort in handling insolvable problems. Authors usually do not pay attention to this important question; one exception is the article by Tiklossy and Roach /3/.

In the present paper there will be given statements about the solvability of transformation problems in a problem space depending on structural properties of the problem space.
In section *2* we shall discuss questions of solvability of problems and give general methods for deciding solvability in a problem space.
Section 3 contains the definition of a special class of problem spaces - the so-called binary - coded problem spaces.
Finally, in section 4 we apply the general methods stated in section 2 to binary-coded problem spaces*

Come general remarks on solvability of transformation problems

Let S be the finite nonempty set of situations of a problem space and F the finite nonempty set of 1-place operators over S. That is, for each $f \in F$, $f: S_f \rightarrow S$, where $S_f \subseteq S$. S_f is the subset of situations to which f is applicable (see also Banerji /2/ and Ernst /4/).
A transformation problem in the problem space (S, F) is an ordered pair (s_1, s_2), where $s_1, s_2 \in S$ and s_1 and s_2 are the initial situation and the goal situation, respectively. A solution to the problem (s_1, s_2) is a sequence of operators from F, $\varphi = f_1 f_2 \dots f_m$ such that φ is applicable to s_1 and $\varphi(s_1) = s_2$. φ is applicable to s_1 if and only if $s_1 \in S_{f_1}$ and $f_1 \dots f_i(s_1) \in S_{f_{i+1}}$ for all $i \in \{1, \dots, m-1\}$.

v.e want to investigate two important questions on solvability of transformation problems in problem spaces (S, F). Firstly, it is intereelinc to know whether ell possible problems (s_1, s_2) in s problem space *are* solvable. Problem spaces with this property are called totally-solvable problem spaces.
econdly, we wish to formulate a method for deciding the solvability of any given problem.
For investigation of solvability of transformation problems the concept of type-k-separatec' subsets of the situation set S is introduced as follows:
Let S_1 be an arbitrary subset of S.

Definition 1:

1. S_1 is a type-1-separated subset of S if and only if for all $f \in F$ and all $s \in S_1 \cap S_f$, $f(s) \in S_1$.

2. S_1 is a type-2-separated subset of S if and only if for all $f \in F$ and all $s \in (S \setminus S_1) \cap S_f$, $f(s) \in S \setminus S_1$. *

3. S_1 is a type-3-separated subset of S if and only if S_1 is both a type-1- and a type-2-separated subset of S.

In the representation of a problem space as a directed graph a type-l-separated subset of S corresponds to a subset of nodes not being left by an arrow and a type-2-separated subset of S to a subset

* The express $S \setminus S_1$ means exactly the same as $S \cap S_1$.

of nodes not being arrived by an arrow. Therefore, for instance, in the problem space presented in Figure 1 $\{s_5, s_8\}$ is a type-1-separated subset, $\{s_1, s_2\}$ is a type-2-separated subset and $\{s_7, s_9\}$ is a type-3-separated subset of S*

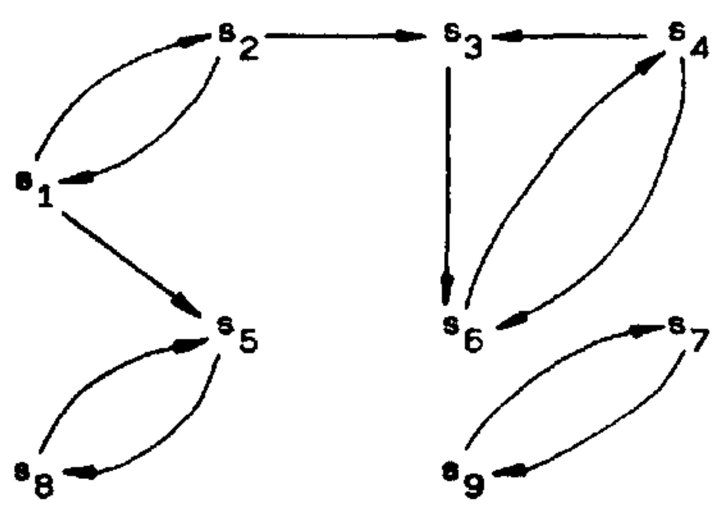


Figure 1

Using Definition 1, the following theorem may be formulated on the total solvability of a problem space.

Theorem 1; A problem space (S, F) is not totally-solvable if and only if there is a nonempty proper subset of S which is a type-1-, type-2- or type-3-separated subset of S.

Proof: ($\Longleftarrow$) Suppose there is a non-empty proper subset S_1 of S which is a separated subset of one of the three types. Obviously, in this case there also exists a nonempty proper subset S_1 of S which is a type-l-6eparated subset of S. Then the problem $(s_1, 8_2)$ with $s_1 \in S_1'$ and $s_2 \in S \setminus S_1'$ is insolvable.

($\Longrightarrow$) Suppose (S, F) is not totally-solvable and let s_1 be any situation in S. If there exists no nonempty proper subset of 5 which is a type-k-separated subset (k * 1, 2, 3) then we can show by induction that each $s_2 \in S$ can be arrived from S_1 by applying an applicable operator sequence. Hence, all possible problems in (C, F) are solvable. This contradicts the assumption and completes the proof* Q.E.D.

Theorem 1 implies that algorithms which detect type-k-separeted subsets may be used to decide the total solvability of a problem space.

Now to the second main question, the decision of the solvability of concrete problems.

Let S_1 be a nonempty proper type-1-separated subset of S. Then a problem (S_1, S_2) with $s_2 \notin S$ not solvable if $s_1 \in S_1$ and $s_2 \notin S_1$.

Analogous statements hold for type-2- and type-3-separated subsets.

Let $\gamma = \{S_1, \ldots, S_q\}$ be a set of type-1-separated subsets of 5 then, for deciding the insolvability of a problem (s_1, s_2) in (S, F) we can use the following

Decision Rule 1: (S_1, s_2) is not solvable if there is an $S_i \in \gamma$ such that $s_1 \in S_1$ and $s_2 \notin S_1$.

Similarly. Decision fule 2 and Decision Rule 3 may be formulated for type-2- and type-3-separated subsets, respectively.

On the other hand, if γ is any set of (not necessarily disjointed) subsets of S then we define so follows, assuming k - 1, 2, 3:

Definition 2:

γ is a consistent type-k-system of subsets of S if and only if for all $s_1 \in S$, $s_2 \in S$ the statement defined by Decision Rule k is true.

It may be proved in a simple way that γ is a consistent type-k-system of subsets of S if and only if for all $S_i \in \gamma$, S_i is a type-k-separated subset of S.

In the example of Figure 1 the system $\gamma = \{\{s_5, s_8\}, \{s_7, s_9\}\}$ is a consistent type-1-system of subsets of S.

Using a consistent type-k-system of subsets and the corresponding decision rule for deciding solvability of transformation problems, we commonly detect only some of the insolvable problems. But about the solvability of the other problems in the problem space we know nothing. Therefore, it seems to be necessary to introduce the concept of an admissible system of subsets.

Let γ be any set of subsets of S, then we define (k = 1, 2, 3):

Definition 3:

γ is an admissible type-k-system of subsets of S if and only if for all s_1, $s_2 \in S$ the following is true: (s_1, s_2) is solvable if and only if for all $S_i \in \gamma$ holds condition Ck, where C_1, C2 and C3 are defined as follows:

C1: $s_1 \in S_i$ **implies** $s_2 \in S_i$.

C2: $s_2 \in S_i$ **implies** $s_1 \in S_i$.

C3: $s_1 \in S_i$ **if and only if** $s_2 \in S_i$.

Then, using admissible type-k-syeteme, we decide the solvability of a transformation problem $(s_1\ s_2)$ in the following manner:

We construct the set **$E(s_1)$ of all $S_i \in \gamma$**

containing the initial situation s_1 and the set $E(s_2)$ of all $S_j \in \gamma$ containing the goal situation s_2.

Then the problem (s_1, s_2) will be decided as solvable if and only if

$E(s_1) \subseteq E(s_2)$ for type-1-systems γ;

$E(s_1) \supseteq E(s_2)$ for type-2-systems γ;

$E(s_1) = E(s_2)$ for type-3-systems γ.

An example of an admissible system of subsets v,'ill be given later in Figure 3. The concept of using systems of subsets of S to decide solvability of transformation problems mey be generalized to using systems of type-different systems of subsets of 5, for instance to using a type-1-system and a type-2-system at the same time• This comes from the possibility that sometimes both some type-1-ceprrated subsets and some type-2-separated subsets are easy to detect end to describe, but the description of an admissible type-1- or type-2-cyctem of subsets of S is more difficult.

In tht next section we will introduce a special class of problem spaces - the so-called binary-coded problem spaces.

Description of a special class of problem spaces

Let n be any positive integer, called the degree of the problem space.

Then the set S of situations of a binary-coded problem space consists of all n-dimensional vectors $x_1 \dots x_n$ with $x_i \in \{0, 1\}$ for $i = 1, \dots, n$. Any special situation $s_1 \in S$ is specified by the values of its components, $s_1(1) \dots s_1(n)$. The set F of operators is $F = \{f_1, \dots, f_n\}$, where for all $i = 1, \dots, n$ f_i negates the value of the situation component x_i if it is applicable. That is, if $f_i \in F$, $s_1 \in S$, f_1 applicable to s_1 and $f_i(s_1) = s_2$ then the following conditions are satisfied:

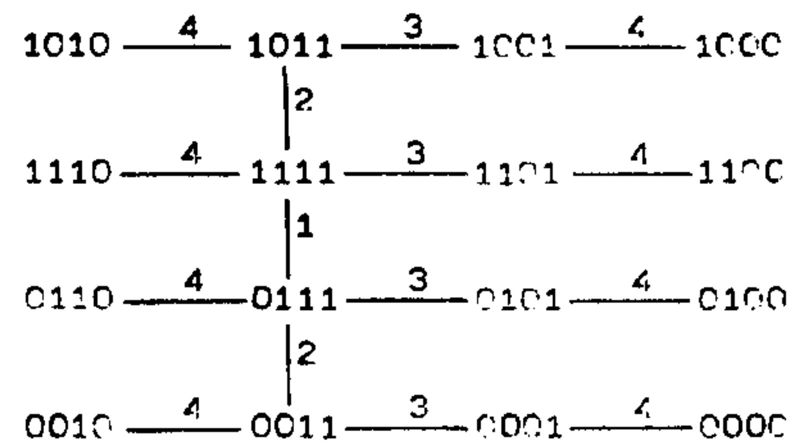
$s_2(i) = 1 - s_1(i)$;

$s_2(j) = s_1(j)$ for all $j = 1, \dots, n$ with $j \neq i$.

According to section 2 the applicability sets of the operators $f_1, \dots, f_n$ are denoted by $S_{f_1}, \dots, S_{f_n}$.

Since the situation components are binary the applicability sets may be represented by Boolean conditions (applicability conditions), $p_1, \dots P_n$, where the situation components stand for variables.

Figure 2 shows a simple binary-coded problem space of *decree A* in graph representation. The edges of the presented graph are labeled by the indices of the corresponding operators.



Applicability conditions of the operators:

$p_1 = x_2 \wedge x_3 \wedge x_4$

$p_2 = x_3 \wedge x_4$

$p_3 = x_4$

$p_4 =$ identical true

Figure 2

With respect to the form of the applicability conditions, subclasses of binery-codec! problem spaces may be defined. For instance, we define the claess of conjunctive binary-codec' problem spaces (celled briefly: conjunctive problem spaces) in the following menner:

binary-coded problem spece of degree n is a conjunctive problem space of degree n if and only if all applicability conditions p. may be represented as

$$p_i = \bigwedge_{j=1}^{n} q_{ij} \text{ where } q_{ij} \in \{x_j, \bar{x}_j, x_j \vee \bar{x}_j\}$$

Consequently, the problem space of Figure 2 is a conjunctive problem space*

For later investigations it ic convenient to define a binary relation $\succ$ over the operator set F of a conjunctive problem space as follows $(f_i, f_j \in F)$:

$f_i \succ f_j$ if and only if $q_{ij} \in \{x_j, \bar{x}_j\}$.

The class of disjunctive problem spaces may be defined analogous to the definition of conjunctive problem spaces.

solvability of transformation problems in binary-coded problem speces

In this section we will state some solvability theorems for binary-coded problem spaces•

First we give a necessary definition.

Definition 4:

A binary-coded prob.iem space of degree n, (S,F') with the applicability conditions $p_1', \dots, P_n'$ binary-coded problem space of degree n, (S, F) with the applicability conditions P_1 ,................P_n and only if for all

i = 1, ..., n the statement P_1'->P_i holds.

To decide solvability properties it may be useful to approximate problem spaces.

Let (, F') and (S , F) be two binary-coded problem spaces of the same degree. Then, (S, F') is called an undereppròximation of (S,F) if (S, F') is a subspace of (r, r). (^{r}, F') is called an overapproximation of (r, F) if (r_f F) is a subspace of (F')

Using these definitions we state the following theorem (without proof):

Theorem 2:

1* Suppose (., F') is an underapproximation of *(t,* F). Then the following is true:
   a) If (£, F') is totally-solvable then (:, F) is also totally solvable;
   b) For all s , s_2 e S:

      If (s., S_2) is solvable in (r, F') then in (^{n}, F), too.

2. Suppose (', F*) is an overapproximation of (", F), Then the following is true:
   a) If (T, F') is not totally-solvable then (^{r} , F) is also not totally-solvable ;
   b) For all s_{1f} $s_2 \in S$:

      If (s-, s_2) is not solvable in (T, F') then also not in (^{r}, F).

Suppose that a binary-coded problem space is described by applicability conditions in disjunctive normal form. Then we obtain an underapproximation if we construct a set of new applicability conditions by taking only one conjunction from each of the original conditions to the corresponding new conditions. The resulting problem space is a conjunctive problem space.

we obtain an overapproximation of the original problem space if we construct each new applicability condition as a disjunction which contains only one variable from each conjunction of the original corresponding condition. This resulting approximation is 3 disjunctive problem space. That is, conjunctive problem spaces are suitable as underapproximations of a binary-coded problem space and disjunctive problem spaces as overapproximations. Therefore, with respect to theorem *2,* solvability properties of binary-coded problem spaces can be investigated by analysing solvability properties of these two important subclasses.

In this paper we shall iljustrate this analysis for conjunctive problem spaces only.

First, we give the following definition:

Definition 5:

Within the operator set F of a conjunctive problem space there is a cycle with respect to the relation $\succ$ if and only if there are operators $f_{i_1}, \ldots, f_{i_l} \in F$ such that

$$f_{i_1} \succ f_{i_2} \succ \ldots \succ f_{i_l} \succ f_{i_1} \quad .$$

Further we state the following lemma without proof (Florath /5/):

Lemma 1: In a conjunctive problem space (S, F) there is a type-k-separated subset of S ($k \in \{1, 2, 3\}$) if and only if there is a cycle with respect to the relation $\succ$ within F.

Combining Lemma 1 and Theorem 1 we immediatly get

Theorem 3: A conjunctive problem space (S, F) is totally-solvable if and only if within the operator set F there is no cycle with respect to the relation $\succ$.

After investigating the question of total solvability we now shalJ define a method for the construction of type-k-systems of subsets for conjunctive problem spaces. For simplicity, let us assume that the graphs defined by the problem spaces are undirected. Consequently, each type-k-ceparated subset must be of type 3 and we construct type-3-systems of subsets.

It can be shown, that for each cycle with respect to the relation $\succ$ within the operator set F there exists a corresponding type-3-separated subset of S.

Let $C_1, \ldots, C_q$ be all the cycles with respect to $\succ$ within F. Let further $S_1, \ldots, S_q$ be the corresponding type-3-separated subsets. Hence, the system

$$\gamma_o = \left\{ S_1, \ldots, S_q, S \setminus S_1, \ldots, S \setminus S_q \right\}$$

is a consistent type-3-system of subsets of S. Then we construct the wanted admissible type-3-system of subsets, γ , stepwise from γ_o in the following manner:

Let γ_k be the system after k steps (k = 1, 2, ...). If for all $S_i \in \gamma_k$ and all $f_j \in F$, for which there are situations $s_1, s_2 \in S_i$ with $s_1(j) \neq s_2(j)$, there exists an $s \in S_i$ such that $s \in S_{f_j}$ then we set $\gamma = \gamma_k$,

and the process of construction terminates.

In the other case, if there are $S_i \in \gamma_k$, $f_j \in F$ such that there exist $s_1, s_2 \in S_i$ with $s_1(j) \neq s_2(j)$ and for all $s \in S_i$, $s \notin S_{f_j}$ then γ_k must be modified.

With

$$s_i^1 = \{ s \mid s \in S_i \wedge s(j) = 1 \}$$ and

$$s_i^0 = \{ s \mid s \in S_i \wedge s(j) = 0 \}$$

we construct γ_{k+1} as

$$\gamma_{k+1} = (\gamma_k \setminus \{ S_i \}) \cup \{ s_i^1, s_i^0 \} .$$

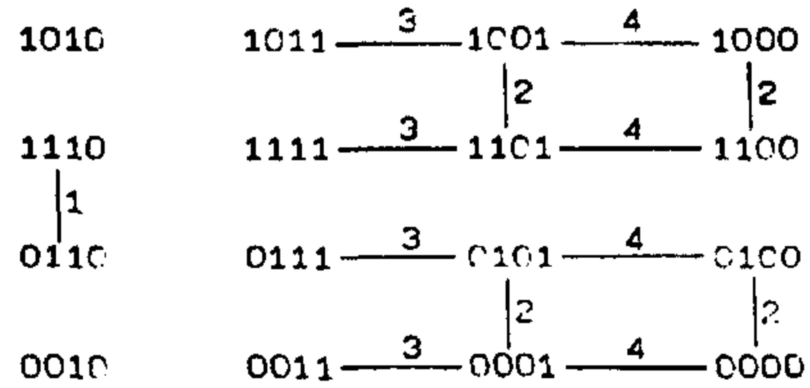
It is, possible to prove that the defined construction process yields an admissible type-2-system γ of subsets of S for all undirected conjunctive problem spaces if complete information about the applicability conditions is available. Moreover, the construction of γ can be realized very simply. Especially the identification of an element of γ_k which should be replaced by two new elements is accomplished through simple coefficient matching.

Figure 3 shows an example of a conjunctive problem space and the construction of an admissible type-3-system for this problem spece. In Figure 3 an expression like *11P means acylinder set of situations.

If information about the applicability conditions of a problem space is incomplete, then the described construction process yields a consistent but not necessarily an cdmissible type-3-system of subsets of S.

Conclusions

Insolvability of insolvable transformation problems may be decided, for instance, by means of type-k-systems of subsets of the situation set, as defined in section 2.
Such additional expense in handling problems is profitable if it is exceeded by the saved amount of problem solving effort for insolvable problems.
However, information about the structure of the problem space is necessary for detecting type-k-sepa rated subsets of situations.



Applicability conditions of the operators :

$p_1 = x_2 \wedge x_3 \wedge \bar{x}_4$

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$p_3 = x_4$

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There is the cycle $f_3 \succ f_4$ within F.
The corresponding type-3-separeted sub 6et is **1C.
Iteration of type-3-systerns of subsets yields:

$\gamma_0 = \{ **10, **** \setminus **10 \}$

$\gamma_1 = \{ *11C, *01C, **** \setminus **1C \}$

$\gamma_2 = \{ *110, 1010, 0010, **** \setminus **10 \}$

$\gamma_3 = \{ *110, 101C, 0010, 1*** \setminus **10, 0*** \setminus **10 \}$

$\gamma = \gamma_3$

Figure 3

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