A NOTE ON DEDUCTION RULES WITH NEGATIVE PREMISES

Ivan Kramosil
Institute of Information Theory and Automation
Czechoslovak Academy of Sciences
Prague.Czechoslovakia

The so called frame axioms problem or frame problem is well-known to anybody who is interested in automatic problem solving.Some information *concerning* this problem can be found in (1) or (2).In (2) the author proposes to eliminate the frame axioms in such a way that they are replaced by a new deduction rule denoted as UNLESS-operator.In fact,this operator is a rule enabling to derive that something concerning the environment is valid at the present situation supposing it was valid in a past situation and it is not provable that a change concerning the validity of this statement has occured.For our *reasona* only the formally logical aspect of this solution is interesting es it leais immediately to a new and very interesting modification of the notion of formalized theory.

The aspect just mentioned consists in in the fact that in (2) the author formalizes his UNL_.SS-operator in the following form of a deduction rule: If a formula A is deducible and a formula B is not deducible;then a formula C is deducible.Written in symbols

(1) $$\frac{\vdash A, \ \underline{\text{not}} \ \vdash B}{\vdash C}$$

where,of course."<u>not</u>" and " $\vdash$ " are symbols of an appropriate metatheory,not the investigated theory itself.

The use of a deduction rule of this type may involve some doubts whether we are justified to do so.The first objection arises from the fact that any deduction rule,including those of the type (1), is an integral part of the definition of the notion "deducible formula",ao this notion itself.neither in its negative form "<u>not</u> I- B",is not allowed to occur in a deduction rule.It is a matter of fact that in "usual" formalized theories the deduction rules are also written in the form using the symbol $\vdash$,e.g.

(2) $$\frac{\vdash A, \ \vdash A \supset B}{\vdash B}$$

if the modus ponens rule is considered. However,it is a well-known fact that in this case the symbol can be eliminated so that the definition of the notion "provable formula" should be correct.It is the g08l of this paper to investigate the deduction rules of the type (1) in order to see,wheher a set of deduction rules,containing at least one rule of the type (1) and considered together with a recursive set of axioms defines unambiguously and correctly e set of deducible formulas.The second objection concerning the deduction rules of the type (1) arises from the fact that when a proof is constructed we have at our disposal in every step only a finite number of formulas proved already to be theorems,so we are never justified to apply a rule of the type Cl) .First of all,we shall investigate the first objection leaving the other one till the end of this paper.

Consider a formalized language *L* based on the first-order predicate calcuTus (FOPC).Let Ax be a set of axioms.This set is supposed to be recursive and,without *any* loss *of* generality,it *may* be also supposed to be finite and its elements to be closed formulas (i.e.not containing free indeterminstes)•

Moreover,consider a finite set of deduction rules.Deduction rule (generalized 'or purposes of this paper) is a metalanguage schema

(3) $$\frac{A_1, A_2, \cdots, A_n}{\vdash B}$$

where B is a formula and every A. is either $\vdash$A- or <u>not</u> $\vdash$ A- .A-cL.Clearly, (3) includes (1) and *(z)* as special cases.Let us divide the deduction rules into two sets,R and S.

R, contains the deduction rules in whi~h every A. is of the form $\vdash$ A-${}_f$i.e. the "usual" deduction rules.Suppose,in the rest of this paper,that R contains all the deduction rules usual in the first-order theories (and-maybe,some other deduction rules derived from them).

S contains the deduction rules in which at least one A- is of the form $\vdash A_i$

Let us suppose,for a moment,that the set S is empty.In such a case the previous description coincides with the usual definition of formalized theories based on the FOPC.The set of provable formulas (theorems) is then defined as the smallest set of formulas containing the set Ax and closed with respect to the deduction rules.To make this formulation precise the following pattern must be used.

Let $\underline{P}_o$ be the system of subsets of the set $\underline{L}$ having the following properties

a) if $X \in \underline{P}_o$, then $\underline{Ax} \in X$,

b) let $A_1, A_2, \ldots, A_n$ be formulas to which a deduction rule

$$(4) \quad \frac{\vdash A_1, \vdash A_2, \ldots, \vdash A_n}{\vdash B}$$

from $\underline{R}$ is applicable, then $A_1 \in X, A_2 \in X, \ldots, A_n \in X$ implies $B \in X$,

c) $\underline{P}_o$ contains all the sets satisfying the conditions a) and b).

Then the intersection

$$\bigcap \underline{P}_o = \{x: x \in \underline{L}, (\forall X \in \underline{P}_o)(x \in X)\}$$

also satisfies a) and b) and the set of theorems, denoted here by $\underline{Cn}(\underline{Ax}/\underline{R})$, is defined

$$\underline{Cn}(\underline{Ax}/\underline{R}) = \bigcap \underline{P}_o .$$

Notice that the fact that $\bigcap \underline{P}_o \in \underline{P}_o$ is the basic point assuring the correctness of the definition of $\underline{Cn}(\underline{Ax}/\underline{R})$.

Now, suppose that $\underline{S}$ is not empty. Define the system $\underline{P}$ of subsets of $\underline{L}$ as follows:

a) if $X \in \underline{P}$, then $\underline{Ax} \subset X$,

b) if $X \in \underline{P}$, $A_1 \in X, A_2 \in X, \ldots, A_n \in X$ and the deduction rule (4) is applicable to the formulas $A_1, A_2, \ldots, A_n$, then $B \in X$.

c) if $X \in \underline{P}$, if a rule of the type (3) is in $\underline{S}$ and is applicable to formulas $A_1, A_2, \ldots, A_n$ and if $A_i \in X$ in case when $\underline{A}_i$ is $\vdash A_i$ and $A_i \in \underline{L} - X$ in case when $\underline{A}_i$ is not $\vdash A_i$, then $B \in X$,

d) $\underline{P}$ contains all the subsets of $\underline{L}$ satisfying a), b) and c).

It can be easily shown, now, that the quadruple $\langle \underline{L}, \underline{Ax}, \underline{R}, \underline{S} \rangle$ defines a formalized theory if and only if $\bigcap \underline{P} \in \underline{P}$.

In usual logical theories, where only the rules of the type (4) occur, all the deduction rules possess a property which is usually celled the truth-preserving property. This property consists in the following fact: if a rule of the type (4) is applicable to formulas $A_1, A_2, \ldots A$ and if these formulas are true in a model then also B is true in this model. In other words the truth-preserving deduction rules do not lead outside the set of true formulas. This property seems to be quite natural and necessary for any deduction rule if the formalized theory is expected to have a relation to the real world, represented by a model. In other case the formalized theory becomes just a pure play with formulas. It is why we try to generalize this notion also for the deduction rules of a more general type, namely (3).

A rule of the type (3) possesses the truth-value-translating property if for every n-tuple $A_1 A_2, \ldots, A_n$ of formulas to which this rule is applicable the following is valid : Supposing M is any model such that if A_i is $\vdash A_i$ then A_i is true in M and if A_i is not $\vdash A_i$, then A_i is not true in M, them B is true in M.

Theorem 1. Let L, Ax, R, S, P, p, and Cn (Ax/R) have the same meaning as above and let all deduction rules from R ∪ S possess the truth-value-translating property. If the set Cn(Ax/R) is closed with respect to every deduction rule from S, then

$$\bigcap \underline{P} = \underline{Cn}(\underline{Ax}/\underline{R}) ,$$

otherwise

$$\bigcap \underline{P} \notin \underline{P} .$$

Proof. Clearly, $L \in P$ and if $X \in P$, then $Cn(Ax/R) \subset X$. Denote by A_o the conjunction of all formulas from *Ax*. Consider a formula $y \in L$ such that

$$y \in \underline{L} - \underline{Cn}(\underline{Ax}/\underline{R}),$$
$$\neg y \in \underline{L} - Cn(Ax/R),$$

where $\neg y$ stands for the negation of y.

The fact that $\neg y \in \underline{L} - \underline{Cn}(\underline{Ax}/\underline{R})$ means that $A_o \supset \neg y$ is not a theorem of the FOPC. A well-known theorem of mathematical logic states that this calculus is semantically complete. This yields that a formula is a theorem of the FOPC if and only if it is true in all models. Hence, there exists a model in which $A_o \supset \neg y$ is not true, so in this model, denote it M(y), the formula $\neg(A_o \supset \neg y)$ is true. The set $\underline{Tr}(\underline{M}(y))$ of all the formulas from L which are true in the model M(y) is closed with respect to the usual deduction rules of the propositional and first-order calculi. The equivalence

$$\neg(A_o \supset \neg y) \equiv A_o \,\&\, y$$

is a propositional tautology, so the fact that $\neg(A_o \supset \neg y) \in \underline{Tr}(\underline{M}(y))$ implies that also $A_o \& y$ and y itself belong to $\underline{Tr}(\underline{M}(y))$. As the set $\underline{Tr}(\underline{M}(y))$ is a set of formulas true in a model, no pair of contradictory formulas may occur in this set, so $\neg y \in \underline{L} - \underline{Tr}(\underline{M}(y))$.

It is a well-known fact that all the

deduction rules in FOPC preserve the truth,i.e. if the premises *are* true in a model the same holds for the conclusions. However,the truth-velue-translating property assures that Tr(M(y)) is closed also with respect to (Induction rules from S.This fact should be immediately clear from the definition,let us prove it for the rule (1).

Let the rule (1) be applicable to formulas A, B c L, let A e Tr(M(y)), let B e L - Tr(M(y)T. Then,according to the truth-vaTue~preserving property,C is also true in the model M(y;,so C e Tr(M(y)), The way of generalizing this resuTt to any rule from S is quite clear.

Hence,the set Tr(M(y)) c L;contains all the axioms from Ax (since""at contains A),contains y,does not contain $\neg y$ and is closed with respect to all the deducti on rules from R u S.This gives that

$$\underline{Tr}(\underline{M}(y)) \in \underline{P}$$

and,immediately,we obtain

$$\bigcap \underline{P} = \underline{Cn}(\underline{Ax}/\underline{R}) \cup \underline{B},$$

where

$$\underline{B} \subset \{x: \neg x \in \underline{Cn}(\underline{Ax}/\underline{R})\} \cup$$
$$\cup \{\neg x: x \in \underline{Cn}(\underline{Ax}/\underline{R})\}.$$

The set Cn (Ax/R) $\cup$ B belongs to P if and only if it is closecf with respect to the rules from R U S.This is possible only if B is empty, i.e. only if

$$\bigcap \underline{P} = \underline{Cn}(\underline{Ax}/\underline{R}).$$

If this set is closed also with respect to the rules from S,then it belongs to P, in all other cases tehe set $\bigcap$ P does not belong to P.The theorem is proved.

The result of this theorem is following: the quadruple <IL,Ax,R,S> defines a formalized theory if and onTy if this theory is equivalent to that defined by the triple <L,Ax,R> . In other words, the deduction rules""from S,i.e.the rules with negrtive premises,are" either useless or meaningless.This result agrees with our intuizive feeling that the rules of the type (1) are useless as we ere never allowed to say that a formula is not derivable having at our disposal only a finite number of formulas derived until now and not having at our disposal any metatheoretical assertions concerning the investigated theory as these assertions cannot be formalized end deduced before defining the notion of provability.

In order not to limit ourselves just to a criticism of the model suggested in (2) let us propose another way how to define a formalized theory using the deductive rules with negative premises.Remember that in usual formalized theories it is possible to define,first of all,the notion of formalized proof and then define a formula to be theorem if and only if there exists a formalized proof proving this formula.Let us follow this pattern and define the set D of formalized proofs as follows,here x,y7A,B,etc.belong to L.

a) If $x \in \underline{Ax}$, then $\langle x \rangle \in \underline{D}$.
b) Let $\langle x_1, x_2, \ldots, x_m \rangle \in \underline{D}$, let $y \in \underline{Ax}$, then
$$\langle x_1, x_2, \ldots, x_m, y \rangle \in \underline{D}.$$
c) Let $\langle x_1, x_2, \ldots, x_m \rangle \in \underline{D}$, consider a rule of the type (3) from $\underline{R} \cup \underline{S}$. If for any $i \leq n$ such that $\underline{A}_i$ is $\vdash A_i$ there is a $j \leq m$ such that A_i is x_j and if, at the same time, for no $i \leq n$ such that $\underline{A}_i$ is not $\vdash A_i$ the formula A_i occurs in the sequence $\langle x_1, x_2, \ldots, x_m \rangle$, then $\langle x_1, x_2, \ldots, x_m, B \rangle \in \underline{D}$.
d) $\underline{D}$ is the smallest set satisfying a), b) and c).

Now,the set T of theorems can be defined as followsT

$$\underline{T} = \{x: x \in \underline{L}, (\exists \langle x_1, x_2, \ldots, x_m \rangle \in \underline{D})(x_m = x)\}.$$

Theorem 2. The definition of the set of theorems is logically correct.The set I) of formalized proofs is recursive, the set T of theorems is recursively enumerable."

Proof. Consider the intersection of all sets of finite sequences of formulas satisfying a),b) and c).This intersection also satisfies a),b) and c),ao D is just this intersection and the definition of D is correct.For any finite sequence < x1, , x2 , . . * ,xm> of formulas we are able to decide,in an effective way,whether it is a formalized proof or not,so the set I) is recursive.The fact that the set T of theorems is recursively enumerable follows immediately from its definition and from the Kleene's Representation Theorem (see,e.g.,(3)).The theorem is proved.

It can be easily *seen* that this definition of deducibility does not capture the intention explained in (2),namely, there may be formulas in T which should not be provable in the framework of the model developed in (2).However,the intentions explained in (2) are not correctly formalizable and we believe that certain difference between the author s suggestion in (2) and our definition of the set T is a penalty the necessity of which foTlows from Theorem 1.Certainly, it would be interesting to ask,whether Theorem 1 will be valid supposing the truth-velue-preserving property is ommited or to ask for some more properties of the set T defined above.However,both those uestTons would be beyond the scope of the main problem to which this paper hns been devoted.

References:

(1) P.J.Hayes: Frame problem and related problems in artificial intelligence. In: Artificial and Human Thinking (eitors A.Littlehorn and D.Jones),Elsevier 1973, pp.45-59.

(2) E.Sandewall: An approach to the frame problem and its implementation. In: Machine Intelligence 7 (editors B.Meltzer and D.Michie), Edinburgh University Press, 1972, pp. 195 - 204.

(3) M.Davis: Computability and unsolvability. Mc Grew - Hill Book Comp., New York - Toronto - London, 1958.