STRUCTURING MATHEMATICAL KNOWLEDGE

Edwina R. Michener
Department of Mathematics
Massachusetts Institute of Technology

When a mathematician says he fully understands a theorem or a theory, he has a lot more in mind than just the deductive details. He also has command intuitively, if not self consciously, of certain special classes of examples and heuristic advice.

At least three broad categories of information are necessary to represent mathematical knowledge: *results* which contain the traditional logical aspects of mathematics, i.e., theorems and proofs; *examples* which contain illustrative material; and *concepts* which include mathematical definitions and heuristic advice.

Just as results can be structured by the relation of *logical deduction* in which A --> B means that result A is used to prove result B, examples and concepts can also be organized by relations. Examples can be ordered by the relation of *fabricational derivation* in which A -•> B means that example A is used to construct example B. For instance, the Cantor function is fabricated from the Cantor set, which in turn is fabricated from the unit interval. Concepts can be structured by the relation of *pedagogical ordering* which embodies the pedagogical judgement that concept A should be introduced before concept B.

These three fundamental categories of items together with the special relations define three representation spaces for a mathematical theory: *Re suits-space, Examples-space, Concepts-space.*

Recognizing that some theory items serve different functions than others in our understanding, we single out those that play special roles by establishing *epistemological classes*

For instance, when we learn a theory for the first time, there are certain perspicuous *start-up* examples which we can grasp immediately. *Reference examples* are examples that we refer to over and over again as we wend our way through a theory. *Model examples* are paradigm situations that suggest to us the essence of a result or concept. And of course, there are *counter-examples.*

In addition to definitions, *Concepts-space* contains the heuristic advice that we give to ourselves (and to others) while working in a theory. *Mega-principles* provide kernels of wisdom in the form of powerful suggestions such as "Let n=2". *Counter-principles* are cautions that indicate possible sources of blunders or troubles, such as the warning "Don't divide by 0".

Results-space also has many subclasses of items Among them are *basic results* which establish elementary, but important, properties of concepts, and *culminating results* which are results towards which the theory drives.

When we consider a theory item, we can fit it into its representation space by determining its predecessors and successors. We can also consider the items outside of its "home" space to which it is attached. The *dual idea* highlights relations between the three representation spaces which are said to act as *epistemological dual spaces* to each other. For instance, the *dual* of a result consists of the examples motivating it and the concepts needed to state and prove it and also the concepts and examples that are derived from it. Thus each item has two associated sets of dual items.

While the placement of an item within its graph determines one definition of closeness, the dual idea leads to additional definitions. For Instance, two results are related Or close in the *example dual sense,* if they share common examples The power of the dual idea is that it provides a good first approximation to the intuitive notion of what it means for two items to be related or close in one's understanding of a theory Two items can be considered *equivalent,* or *identified,* if their dual items are the same.

The resulting epistemology and the interconnections among its items provide a rich representation for mathematical theories which is sufficiently precise for the specification of a knowledge base for mathematics and a functional model of expert understanding. These ideas suggest powerful methods to help one understand, teach or explore mathematics

This analysis is the foundation of the design of the proposed interactive *CROKKER SYSTEM (GS)* to enable mathematicians to easily retrieve and manipulate mathematical knowledge, especially in dense, richly-developed subjects such as real analysis. It can be augmented by the *GROKKER LEARNING ADVISOR (GLA)* which is designed primarily to help neophytes understand mathematics. It forms its advice from its epistemological knowledge, its model of expert understanding, and its assessment of the user's current level of understanding.

The combined CS/GLA system could enter into partnership with a theorem prover or an analogy-generating program by guiding and advising the program's search for relevant information.

Bibliography

Michener, E.R. *Epistemology, Representation, Understanding and Interactive Exploration of Mathematical Theories.* Doctoral dissertation, Department of Mathematics, MIT, 1977.