NEGATIVE HYPER-RESOLUTION FOR PROVING STATEMENTS CONTAINING TRANSITIVE RELATIONS

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ABSTRACT

A modified version of PHR~- resolution comprising negative hyper-resolution and paramodulation is introduced to reduce the search of statements that contain transitive relations.

Let R be a symbol of transitive relation and let a nucleus contain the literal tRp, and let a factor of an electron be of the form C V nt,Rt_2.Moreover let us suppose that tU = t,U, where U is a simultaneous most general unifier for the corresponding clash. Then the resolvent of this clash contains the subclause (C V pRt_2)U. This rule of deduction is said to Be TPHR - rule.

It is shown that TPHR"- resolution is complete. More precisely, it is shown that the empty clause a can be deduced from a set G of clauses which contains the axiom of transitivity Tr for the relation R by using PHR""- resolution iff can be deduced from the set G-{Tr }by using TPHR"- resolution.

The efficiency of the use of TPHR~- rule is illustrated by examples.

1. INTRODUCTION

When using resolution method to prove statements with transitive relations the main difficulty is the usage of the axiom of transitivity, i.e. the clause $Tr_R \stackrel{d}{=} \neg xRy \vee \neg yRz \vee xRz$ where R is a relation symbol.

The inconvenience while working with this clause is of two kinds. First, Tr_R can be resolved with a great number of "outsider" clauses. Second, the resolvents contain a new variable y which is not in connection with any other variable of the clause to be resolved. Note that if Tr_R is applied several times then we have more such new variables. So the number of possible usage of resolution and paramodulation rules further grow resulting in an uncontrollable growth of the number of generated clauses.

The possibility for the strategy of positive hyper-resolution was shown in L"3] to reduce the search to some extent. Some further ideas on this subject can be found in Cl] . In the present paper we show how to avoid the explicit use of transitivity axiom within the strategy of negative hyper-resolution. This strategy is chosen because it can be combined with paramodulation as it is shown in C2] and because each derivation of the empty clause can be easily transformed into direct proof of the statement in question by using negative hyper-resolution (cf. [4].

Recall that under negative hyper-resolution strategy we divide the set of all clauses into two parts: clauses of nucleus (which contain one or more positive literals) and electrons (which consist of only negative literals). Resolvents are always electrons.

The idea developed below is as follows. Let some electron clause C contain literal t1 Rt2 This means that statement t1Rt2 should Be proved. If a nucleus contains a positive literal of the form t1.,Rt' and U is the most general unifier or the terms t1, and t' then the statement $(t'Rt_2)U$ might be proved instead of the original one. Now we formally develop all the constructions necessary.

2. TPHR""- RULE

First we formulate the so called PHR-rule of resolution type which comprises both negative hyper-resolution and paramodulation. As usually, negative clauses are electrons, all the rest are nucleus. Factorization is allowed only for the electrons.

Let $D \vee L_1 \vee \ldots \vee L_k \vee t_1 = p_1 \vee \ldots \vee t_m = p_m$ be a nucleus, where D consists only of negative literals. Let $C_1 \vee \neg M_1, \ldots, C_k \vee \neg M_k, C_{k+1} \vee \neg M_{k+1}(t_1'), \ldots, C_{k+m} \vee \neg M_{k+m}(t_m')$ be electron factors and let a simultaneously most general unifier exist for this clash i.e. there should exist such a substitution U that $L_1U \equiv M_1U, \ldots, L_kU \equiv M_kU$, $t_1U \equiv t_1'U, \ldots, t_kU \equiv t_k'U$. (Here $\equiv$ is the symbol of graphic identity.) Then the resolvent of the above clash is the clause $(DVC_1V \ldots VC_kVC_{k+1}V \neg M_{k+1}(p_1)V \ldots VC_{k+m}V \neg M_{k+m}(p_m))U$.

It is shown in [2] that PHR$^-$-resolution is complete for the first order calculus with equality. Let us modify PHR$^-$-rule as follows.

Definition 2.1: Let R be the symbol of a transitive relation and let the nucleus contain literal tRp and let the electron factor have the form $CV \neg t_1Rt_2$ thereby $tU=t_1U$ where U is a simultaneous most general unifier of the corresponding clash. In this case the subclause $(CV \neg pRt_2)U$ comprises the resolvent of the clash to be considered. This derivation rule is said to be TPHR$^-$- rule.

3. COMPLETENESS OF TPHR$^-$-RESOLUTION

We show that TPHR$^-$-resolution is complete for the first order logic with equality. More precisely we show that for a set G of clauses that contains the axiom of transitivity (Tr_R) for a relation R the clause $\Box$ is derivable by the use of PHR$^-$- resolution iff $\Box$ is derivable from $G \sim \{Tr_R\}$ by the use of TPHR$^-$-resolution. Note that the "only if" part of this statement is obvious.

Definition 3.1: T-rule is the derivation rule which permits to obtain the clause $(C_1VC_2V \neg p_1Rp_2)U$ from the clauses $C_1 \vee t_1Rp_1$, $C_2 \vee \neg t_2Rp_2$, where U is the most general unifier for t_1 and t_2.

Now we formulate the analogon of the lifting lemma to the T-rule.

Lemma 3.2: If w is such a substitution that $t_1w = t_2w$ and U is the most general unifier of the terms t_1 and t_2 then the clause $C_1 \vee C_2 \vee \neg p_1Rp_2$ subsumes the clause $C_1wVC_2w \vee \neg p_1wRp_2w$. In other word, the result of using T-rule to two given clauses subsume the result of using T-rule to any instances of the two given clauses.
Proof is obvious.

Note that if the starting set of the clauses contains axioms of functional reflexivity which we always suppose then lifting works for paramodulation as well. So the investigation can be restricted to the ground case.

Let G be a set of clauses and let us suppose that some clause C can be derived from G by the use of PHR$^-$- resolution, i.e. $G \vdash_{PHR^-} C$. We represent this derivation in a tree form where the root is C, the leaves are instances of starting electrons. To each node of the tree a nucleus and usage of PHR$^-$- rule correspond, i.e. to each node so many arrows go in as many the number of the electrons existing in the corresponding clash are. From each node one arrow goes out. This tree is said to be the derivation tree of clause C. It is clear that the axiom of transitivity Tr_R is a nucleus, i.e. to the resolution with Tr_R such a node corresponds in the derivation tree, where one arrow goes in (since Tr_R contains a unique positive literal).

Let leaf D of the derivation tree contain literal $\neg t_1 R t_2$, to the offsprings of which the axiom of transitivity is applied n times (i.e. the path in the derivation tree from D to the root goes through n nodes corresponding to Tr_R). Let us substitute literal $\neg t_1 R t_2$ with the clause $\neg t_1Rx_1 \vee \neg x_1Rx_2 \vee \ldots \vee \neg x_pRt_2$, where $x_1, \ldots, x_p$ are new variables, and from the path from D to the root let us eliminate n nodes corresponding to the usage of Tr_R to the offsprings of the literal t_1Rt_2.

It is obvious that the tree obtained after such reconstruction is still a derivation tree for the clause C.

Let us do the aboves for each such leaf of the tree that contains literal to the offsprings to which the axiom of transitivity has been applied. Let L_G be a family of leaves of the reconstructed tree. Some of the leaves contain clauses of the length of the transitive part. It is obvious that the reconstructed tree does not contain nodes corresponding to Tr_R. This corresponds to the fact that resolution with nucleus Tr_R can be lifted up, i.e. it can be done in the beginning. Thus we have

Lemma 3.3: Let G be a set of clauses, T be a derivation tree of the clause C from G, L be the set of leaves of T, L' that of the reconstructed tree T'. Then $L' \vdash_{PHR^-} C$ and the corresponding tree T' does not contain nodes corresponding to Tr_R.

Now we are ready to prove the main

Theorem 3.4: If $G \vdash_{PHR^-} \Box$ then $G \sim \{Tr_R\} \vdash_{TPHR^-} \Box$

Proof: Let T be a derivation tree of the clause C from G and let T' be a reconstructed ground tree of derivation of C. We show how T' can be transformed into the derivation tree by the use of TPHR$^-$-resolution so that the leaves of the new tree become instances of the starting clauses, i.e. they should not contain transitive parts.

The proof of the possibility to do such transformation is done by induction on the sum $S(T')$ of the length of transitive parts of the leaves.

Base of induction: $S(T') = 0$, i.e. tree T has not contained resolution with Tr_R. This case is trivial.

Induction step: Let $S(T')>0$ and for all trees T with $S(T) < S(T')$ let us suppose that the possibility of the transformation is proved. Since $S(T') > 0$ then there exists such a leaf D in T′ the length of the transitive part of which is greater than 0. Let this leaf be of the form $D=D_1 \vee \neg t_1Rs_1 \vee \neg s_1Rs_2 \vee \ldots \vee \neg s_nRt_2$.

Let us consider the path P from D to the root of the tree. The following two cases are possible:

Case 1. On the path P literal $\neg t_1Rs_1$ has been resolved earlier than $\neg s_1Rs_2$. Let literal $\neg t_1Rs_1$ be resolved (possibly after having used paramodulation several times for the term t_1) in the node N with the corresponding literal $t_1'Rs$ of the nucleus, where term t_1' is the result obtained after making all possible paramodulations in t_1.

Let us substitute leaf D in T′ by the clause $D_1 \vee \neg s_1Rs_2 \vee \ldots \vee \neg s_nRt_2$. We get tree T_1'. It is obvious that $L(T_1') \vdash_{PHR} C_1$, where C_1 subsumes C. Indeed, all possible paramodulations for t_1 should be done together with the resolution in the node N.

Moreover $S(T_1') < S(T')$. Thus according to the inductive hypothesis T_1' can be transformed into the derivation tree T_1 of clause C_1' with $TPHR^-$- resolution while C_1' subsumes C_1. But the starting set of clauses for T_1 contains the clause $D_1 \vee \neg s_1Rt_2$ together with $D_1 \vee \neg t_1 R t_2$. Now we substitute leaf D in T′ by the clause $D_1 \vee \neg t_1 R t_2$.

Let us execute all possible resolutions and paramodulations connected with literals from D_1 and $\neg s_1Rs_2, \ldots, \neg s_nRt_2$ in T′. In node N instead of PHR^--rule let us use $TPHR^-$-rule. Then obviously literal $\neg t_1 R t_2$ is substituted by literal $\neg s_1' R t_2$. The derivation T_2' will be a derivation of the clause which subsumes $D_1 \vee \neg s_1Rt_2$. Moreover $S(T_2')<S(T')$. Thus T_2' can be transformed into tree T_2 of the derivation with $TPHR^-$ - resolution of the clause which subsumes $D_1 \vee \neg s_1Rt_1$. Let us substitute all starting occurrences of $D_1 \vee \neg s_1Rt_2$ in T_1 by the tree T_2. Obviously the tree thus obtained is the required one.

Case 2. On the considered path literal $\neg s_1Rs_2$ has been resolved earlier than $\neg t_1Rs_1$. Let $\neg s_1Rs_2$ be resolved with s_1Rs_1' in some node N. Then literal $\neg t_1'Rs_1$ was resolved with literal $t_1''Rs_1'$ in the node N′. On the path from N to N′ let the paramodulations be executed from literals $p_1 = q_1, \ldots, p_k = q_k$ into the term s_1 such that s_1' is the result of these paramodulations. Let the paramodulations be executed in the nodes $N_1, \ldots, N_k$ in the respective order from left to right, i.e. the occurence of the left part of the equivalence is substituted by the right one.

Now let us substitute leaf D in T′ by the clause $D_1 \vee \neg s_1'Rs_2 \vee \ldots \vee \neg s_nRt_2$.

Let us execute all possible paramodulations in T′ connected with term t_1 and do the resolution in the nodes N and N′. Then on the path from N to N′ let us inverse the order of paramodulations, i.e. the order of nodes become as follows $N_k, \ldots, N_1$. Simultaneously the "direction" of each paramodulation is also changed, i.e. now they are executed from right to left. Let us use these paramodulations to the term s_1' of literal $\neg s_1'Rs_2$. Obviously we get literal $\neg s_1Rs_2$. Then in the node N′ we execute resolution with the nucleus having been earlier in N. The tree thus obtained is a derivation of the clause which subsumes clause C.

Moreover $S(T_1')<S(T)$. Thus T′ can be transformed into tree T_1, which is a derivation by the use of $TPHR^-$ - resolution of the clause which subsumes C. But the clause $D_1 \vee \neg s_1Rs_2$ while not being a starting one occurs as a leaf in T_1.

The rest is analogous with case 1.

Now we illustrate the efficiency of using $TPHR^-$-rule by examples.

4. EXAMPLES

4.1. Let us take a topological theorem with all its necessary definitions as shown below.

"Let E be a set. Let E be a metric space.

Definition 1: Let X,Y be sets. Y is a subset of X iff every element of Y is an element of X too.

Definition 2: Let X be a subset of E. X is open in E iff for any point z of X there exists an open sphere included in X, with the center in z.

Definition 3: Let z be a point of E and let A be a set. A is the neighbourhood of z iff A is a subset of E and there exists a set Y which is open in E so that z is a point of Y and Y is included in A.

Each subset of a set is a set.

<u>Theorem:</u> Let A be a subset of E. Then A is open in E iff A is a neighbourhood of every point of A."

From this text the following set of clauses can be obtained

1. ¬xRy V ¬yRz V xRz
2. SS(A,E)
3. ¬P(x,A) V P(x, f_1(x))
4. ¬P(x,A) V f_1(X) RA
5. ¬SS(A,E) V P(z_o,A)
6. ¬SS(A,E) V ¬OS(v,z_o) V ¬vRA
7. ¬P(x,A) V ¬P(z,f_1(x)) V OS(f_2(x,z), z)
8. ¬P(x,A) V ¬P(z,f_1(x)) V f_2(x,z) R f_1(x)

where v,c,y,z are variable symbols, A,E, z are constant symbols, f,, f2 are Skolem functions, R, SS, P, OS are binary relation symbols. The latters are of the following meaning

R(x,y) stands for "x is included in y"
SSIA,E) stands for "A is a subset of E"
P(x,A) stands for "x is a point of A"
OS(v,x) stands for "v is an open sphere with center in x".

Due to transitivity of relation R we use the infix notation, i.e. we write xRy instead of R(x,y).

In order to prove the theorem the strategy of negative hyper-resolution has been applied for. while this done three levels of possible resolvents have been obtained (see Table 1).

Table 1

Level number	Clause number	Parent numbers	substi-tution	result
I	9	6,4	v←f_1(x)	¬SS(A,E) V ¬OS(f_1(x),z_o) V ¬P(x,A)
	10	6,7	z ← z_o, v ← f_2(x,z_o)	¬SS(A,E) V ¬P(x,A) V ¬P(z_o,f_1(x)) V ¬f_2(x,z_o) RA
II	11	9,5	x ← z_o	¬SS(A,E) V ¬OS(f_1(z_o),z_o)
	12	10,3	x ← z_o	¬SS(A,E) V ¬P(z_o,A) V ¬f_2(z_o,z_o) RA
	13	10,5	x ← z_o	¬SS(A,E) V ¬P(z_o,f_1(z_o)) V ¬f_2(z_o,z_o) RA
III	14	12,5		¬SS(A,E) V ¬f_2(z_o,z_o) RA
	15	13,3		¬SS(A,E) V ¬P(z_o,A) V ¬f_2(z_o,z_o) RA

Note that due to unforeseen circumstances nucleus 1 did not participate in the deduction.

LiteralISS (A,E) is present in all clauses, but resolution with nucleus 2 eliminates it. So having obtained resolvent (TSS(A,E)} the proof seeking process is considered to be successfully terminated.

Note that clause 15 is subsumed by clause 14 .

Having noticed that no further possibilities for resolution are available nucleus 1 was called into the process and the deduction went on. Namely, the number of resolvents of level n+3 was of n! The empty clause D was obtained at the eighth level.

Yet if we go on with the deduction by using TPHR-rule we obtain the resolvents presented in Table la, which is the continuation of Table 1.

Table la.

IV.	16	14,8	x ← z_o, x ← z_o	¬SS(A,E) V ¬P(z_o,A) V ¬P(z_o,f_1(z_o)) V ¬f_1(z_o) RA
V.	17	16,3	x ← z_o	¬SS(A,E) V ¬P(z_oA) V ¬f_1(z_o) RA
	18	16,4	x ← z_o	¬SS(A,E) V ¬P(z_oA) V ¬P(z_o,f_1(z_o))
	19	16,5	x ← z_o	¬SS A,E V P(z_o,f_1(z_o)) V ¬P(z_o,f_1(z_o)) RA
VI.	20	17,4	x ← z_o	¬SS(A,E) V ¬P(z_o,A)
	21	17,5		¬SS(A,E) V ¬ f_1(z_o) RA
	22	18,3	x ← z_o	¬SS(A,E) V ¬P(z_oA)
	23	18,5		¬SS(A,E) V ¬P(z_o,f_1(z_o))
	24	19,3	x ← z_o	¬SS(A,E) V ¬f_1(z_o) RA
	25	19,4	x ← z_o	¬SS(A,E) V ¬P(z_o,A) V ¬P(z_o,f_1(z_o))
VII	26	20,12		¬SS(A,E)

It is evident that the hereby described deduction is much simpler than that within which nucleus 1 i.e. the transitivity axiom of relation R is used.

4.2. Let us now take a theorem from the field of commutative rings having the following text.

"Let R be a commutative ring with unit element. Let a,b€R. The ideal generated by the element a is denoted as a.R. Let xea.R iff alx (where a|x means that a divides x, i.e. azx=a.z. Here • is the binary operation of multiplying in the ring R.

Product of ideals a.R and b.R is a set denoted as (a.R)o (b.R) such that $x \in (a.R) \circ (b.R)$ iff $\exists u \exists v (x = u.v \wedge u \in a.R \wedge v \in b.R)$
Intersection of ideals $a.R \cap b.R$ has the usual set-theoretic meaning.

<u>Theorem:</u> $(a.R) \circ (b.R) \subseteq a.R \cap b.R$"

From the above text we have the following set of clauses.

1. $\neg (a.R) \circ (b.R) \subsetneq a.R \cap b.R$
2. $\neg\ x=y.z \vee y|x$
3. $\neg\ x=y.z \vee z|x$
4. $\neg\ y|x \vee x \in y.R$
5. $\neg\ x \in y.R \vee y|x$
6. $x_o \in A \vee A \subsetneq B$
7. $\neg\ x_o \in B \vee A \subsetneq B$
8. $\neg\ x \in A \vee \neg x \in B \vee x \in A \cap B$
9. $\neg\ z \in (a.R) \circ (b.R) \vee z = f_1(z).\ f_2(z)$
10. $\neg\ z \in (a.R) \circ (b.R) \vee f_1(z) \in a.R$
11. $\neg\ z \in (a.R) \circ (B.R) \vee f_2(z) \in b.R$

where a, b, R, x_o are constant symbols and f_1, f_2 are Skolem functions.

Here 1 is the unique electron while the other clauses (2-11) form the nucleus.

Note that the binary relation symbol I is transitive, though the axiom claiming this property is absent. The proof of the theorem, i.e. the deduction of □ by the use of TPHR-rule consists of 79 resolvents (including subsumation).

The derivation tree containing only the necessary resolvents is shown on Fig.l.

However if the transitivity axiom of I is attached to the above set of clauses the number of resolvents will be more than 100 (including subsumation)on the seventh level already.

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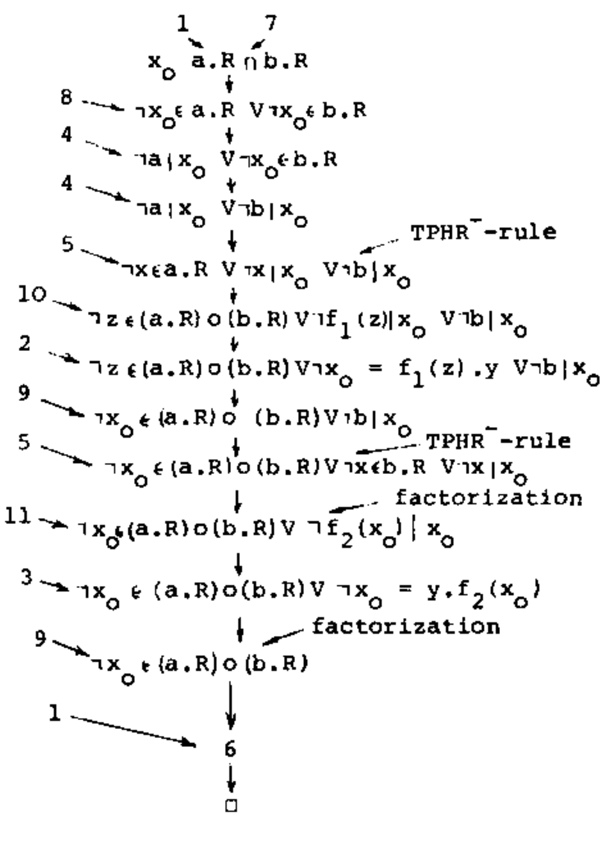

Fig.l.