A Functional Approach to Non-Monotonlc Logic

Erik Sandewall

Department of Computer end Information Science

Linkoping University

Linkoping, Sweden

Abstracts Axiom sets and their extensions are viewed as functions from the set of formulas in the language, to a set of four truth-values t,f, s for undefined, and *k* for contradiction. Such functions form a lattice with 'contains less information' as the partial order $\sqsubseteq$, and 'combination of several sources of knowledge' as the l.u.b. operation $\sqcup$. We demonstrate the relevance of this approach by giving concise proofs for some previously known results about normal default rules. For non-monotonic rules in general (not only normal default rules) we define a stronger version of the minimality requirement on consistent fixpoints, and prove that it is sufficient for the existence of a derivation of the fixpoint.

1. Introduction and overview.

Non-monotonic logic may be studied either in terms of non-monotonic inference rules (Reiter 1980, Goodwin 1984) or in terms of non-monotonic operators in the language such as the *Unless* operator (Sandewall 1972, McDermott and Doyle 1980). In this paper we pursue the former approach.

The concept of fixpoints b central to the study of non-monotonic logic: for a given set v of propositions and a given set of rules, we are looking for an extension Le. a set s' of propositions which contains v as a subset, and which is a fixpoint of the set of rules. Fixpoints are also used in the denotational semantics approach to the theory of programming languages (Scott 1970; see also e.g. Manna 1974, Stoy 1977, Blikle 1981). There, the recursive definition of a function is viewed as a functional, i.e. an operator on partial functions, and the function b viewed as the fixpoint of the same functional

In thb paper we propose that the functional approach that is taken in denotational semantics, can be adapted and serve conveniently for the study of non-monotonic logic. This b attractive since logical inference b often viewed as a high-level form of computation, and since computational inference often needs to be non-monotonic. The power of thb approach b demonstrated through simple proofs of some of Reiter's (1980) results for normal default theories. Other results in the paper apply to non-monotonic rules in general.

Although several fixpoints may exbt in the monotonie case, the criterium of being • *minimal* fixpoint (i.e. all other fixpoints are 'larger[1]) is a sufficient one, and there b only one minimal fixpoint, which b then the *lea$t* fixpoint. Other, larger fixpoints contain spurious information which b not warranted by the given facts and inference rules. In the case of non-monotonic rules, there is of course in general no single least fixpoint, and the criterium of fixpoints being minimal is not sufficient: there may be minimal fixpoints which have the given set of propositions as a subset, but which still can not be reached or approached (in the sense of a limit) by any derivation using the given set of rules. In this paper we define a concept of *approachable* fixpoint, which is stronger than the concept of minimal fixpoint, and which is proven to be a sufficient condition for the existence of a derivation that reaches or approaches the fixpoint.

The following formal machinery is used. We start from two domains, a domain L whose elements are called *formulae* and a domain J of *truth-values.* V is the domain of *valuations* i.e. continuous functions from L to J.

A set of axioms is seen as a valuation that maps some formulas (the axioms) to *t* (for true) and "all" other formulas to u (for undefined). (Exception is made for the top element of the domain L). Derivation of theorems is done by proceeding from the initial valuation to others where some formulas change value from a to *t* or / (for false). A set of inference rules corresponds therefore to a binary relation on valuations, i.e. a subset of V x V, which we shall call a *deduction.* A *derivation* using a deduction *F* is a sequence of valuations,

$$v_0, v_1, \ldots$$

where*

$$\langle v_i, v_{i+1} \rangle \in F$$

for each $i \geq 0$.

Deductions in this sense can be used for characterizing both proofs and semantics, provided that there are syntactic functions and predicates on L which characterise the abstract syntax of the language. This includes predicates which indicate whether a formula is a conjunction, a disjunction, an implication, atomic, etc., as well as functions e.g. for composing the conjunction of two other formulas. The conventions for calculating the truth-value of a propositional expression may then be seen as a dednction *F* where $F(v,v')$ e.g. in the case where:

$$v(a) = t$$
$$v(b) = t$$
$$v(a \wedge b) = u$$
$$v'(a \wedge b) = t$$
$$v'(x) = v(x) \quad \text{for all other formulas } x$$

Here a A *b* refers of course to the formula obtained by composing the formula a, the conjunction operator, and the formula 6.

The concepts and results of conventional logic can easily be re-phrased along these lines. In the case of non-monotonic logic,

This research wee eponoered by the Swedish Boerd of Technical Development.

there is however a particular advantage with doing so: a non-monotonic rule

$a \wedge Unless(b) \rightarrow c$

can now be seen as a deduction *F* which a l l o $F(v,v')$ g . in the case where

$v(a) = t$
$v(b) = u$
$v(c) = u$
$v'(b) = f$
$v'(c) = t$
$v'(x) = v(x)$ otherwise.

In other words, one derivation step using *F* will change the truth-values of the two formulas *b* and *e* at the same time. This is different from the viewpoint in ordinary logic, where the intuition is that each formula or proposition has 'its' truth-value, so that rules of inference may contribute additional information about 'the' truth of a proposition. In non-monotonic logic, we must be prepared to recognise multiple extensions of the given axioms, or multiple fixpoints of what is here called a deduction. It therefore makes sense to correlate assignments of truth-values in the way just described.

Suppose now, in the above example, that *b* has already been assigned the value *t* (maybe by an ordinary rule that does not use *Unless),* and therefore we have *v(b) - t.* In order to deal with such cases it is convenient to consider two deductions *F* and *G* as follows. For *G* there is no v'such that $G(v,v')$. This expresses the intended meaning of *Unless.* For *F* we take the view that information has now accumulated both that *b* is *t* and that it is f. A new truth-value *k* is introduced for this purpose, and there is a v'such that *v'(b) = k* and $F(v,v')$.

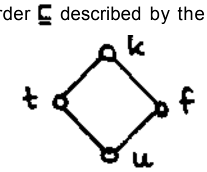
We therefore obtain a four-valued logic, with the truth-values t, f, u, k. This logic has previously been studied by Belnap (1977). One can think of these truth-values as the ones assigned by committees (cf. Borgida et al, 1984): if some members of the committee assign the value *t* to a proposition and the others assign the value «, then the committee assigns t, but if some members assign *t* and others assign */*, then the committee assigns the value *k* to the the proposition. 'Committees' are a convenient metaphor whenever there are several parallel sources of knowledge, such as when several inference rules are being used.

The deductions *G* and *F* are in general related so that $G \subseteq F$, and the difference occurs in cases like the one just discussed, where *G* 'refuses' to use a rule if it means that the non-monotonic antecedent (the formula given as argument to Unless*)* then obtains the truth-value *k*

This has been an outline of the key ideas of the paper. We now proceed to the systematic treatment.

2. Monotonlelty and fixpoints for relations.

The domain J contains the elements *{u,t,f,k}* as already discussed, with the partial order $\sqsubseteq$ described by the following figure:



Thus it is a flat lattice with ■ as the bottom element and *k* as the top element.

The flat domain L is called a *language* and its elements are called *formulas,* with *lb* as the bottom element and *lt* as the top element.

A *valuation* is a function *v* from **L** to **J** which satisfies

$v(lb) = u$
$v(lt) = k$

which in particular guarantees that valuations are continuous (and therefore monotone) functions. A valuation is *consistent* iff no formula other than *lt* is mapped to *k*. Valuations form a lattice with the partial order $\sqsubseteq$ defined in the following way:

$v \sqsubseteq v'$ iff $(\forall x) v(x) \sqsubseteq v'(x)$

We say then that *v'* is an *extension* of *v*, and that *v'* is *above* *v* (with respect to $\sqsubseteq$). A valuation *v* is *finite* iff $v(l) = u$ except for a finite number of formulas *l*.

A *deduction* is a binary relation on **V**, i.e. a subset of $V \times V$. The operation $\cup$ and the relation $\subseteq$ is therefore defined on deductions. The partial order $\sqsubseteq$ could also be extended to deductions, but that will not be needed in this paper.

A deduction *F* is *conservative* iff

$F(v,v') \rightarrow v \sqsubseteq v'$

which can now be written

$F \subseteq \sqsubseteq$

A *chain* is a sequence of valuations where

$v_0 \sqsubseteq v_1 \sqsubseteq \ldots$

It is well known that each such chain has a l.u.b. in a lattice. A *derivation* from *v* to *v'* using a conservative deduction *F* is a chain where

$v = v_0$
$F(v_i, v_{i+1})$ for all $i \geq 0$
v' is the l.u.b. of the chain.

Notice that we say a derivation — to v' — even in the case of an infinite sequence where *v'* is never reached, just approached as a limit.

A deduction *F* is *monotonic* iff

$v \sqsubseteq v' \wedge F(v,y) \rightarrow (\exists y')(F(v',y') \wedge y \sqsubseteq y')$

It is linear iff

$F(v,y) \rightarrow F(v \sqcup z, y \sqcup z)$

Clearly every linear deduction is also monotonic.

A valuation *v'* is a *fixpoint* of a deduction *F* iff $F(v',v'') \rightarrow v' = v''$

Thus in particular v' is a fixpoint if *F* does not allow any 'successor' v''. For a given valuation v and deduction F, we shall be interested in fixpoints of *F* above v, i.e. fixpoints of *F* which are $\sqsupseteq$ v.

A fixpoint of *F* above *v* is minimal iff no 'smaller' ($\sqsubseteq$) fixpoint exists for the same *F* and *v.*

What has been described so far uses some of the tools of denotational semantics, but in a diffferent fashion than usual. The differences are dictated by our desire to deal with logic using these tools. The reason for that, again being the wish to consider non-monotonic deduction. Anyway, the obvious properties of the monotonic case follow easily, in particular

Proportion 1. a) A conservative, monotonic deduction has a unique least fixpoint above each f.

b) The Lu.b. of any derivation from v is $\sqsubseteq$ the least fixpoint.

Proof, a) Consider a set $\{v_i\}$ of fixpoints of F above v. We wish to prove that $v' = \sqcap v_i$ is also a fixpoint of F. Suppose

$F(v',z)$

By monotonicity, for each v_i there exists a valuation z_i such that

$F(v_i, z_i)$

$z \sqsubseteq z_i$

Since each v_i is a fixpoint, we have

$v_i = z_i$

and therefore

$z \sqsubseteq v' (= \sqcap z_i)$

and since F is conservative,

$v' = z$ []

b) Let $v_0, v_1, \ldots$ be a derivation using F, and let v^* be the least fixpoint of F above v_0. By induction, it is easily proven that

$v_i \sqsubseteq v^*$

for each i. []

The following concept is also of interest in the non–monotonic case:

A valuation v is ***maximally consistent*** w.r.t. a deduction F iff
$F(v,v') \rightarrow v = v' \vee$ [v' is inconsistent]

3. Linearity and compactness.

There are two significant properties for deductions formed using a number of rules: linearity and compactness. This section studies consequences of those properties. Linearity was defined in the previous section

Proposition 2. If F is a conservative, linear deduction, and there exists a derivation from v to v' using F, then for every y such that

$v \sqsubseteq y \sqsubseteq v'$

there exists a derivation from y to v' using F.

Proof. Let the derivation from v to v' be

$v_0, v_1, \ldots$

where $v_0 = v$. The sequence

$v_0 \sqcup y, v_1 \sqcup y, \ldots$

is a derivation using F, and by the assumptions we have

$v_0 \sqcup y = y$

and the l.u.b. of the sequence is v'. []

Proposition 3. If F is a conservative, linear deduction, and there are derivations from v to v' and from v to v'' using F, then there is a derivation from v to $v' \sqcup v''$ using F.

Proof. Let the derivations of v' and v'' be:

$v'_0, v'_1, \ldots$

$v''_0, v''_1, \ldots$

where $v'_0 = v''_0 = v$. Since F is linear, the sequence

$v'_0, v'_1,$

$v'_1 \sqcup v''_1, v'_2 \sqcup v''_1,$

$v'_2 \sqcup v''_2, v'_2 \sqcup v''_2$ ...

is a derivation from v using F, and it has $v' \sqcup v''$ as its l.u.b. []

The proof of proposition 3 generalizes easily:

Proposition 4. If F is a conservative, linear deduction, and there are derivations using F from v to each member of a non–empty (possibly infinite) set W of valuations, then there is a derivation from v to the l.u.b. of W.

Proof (outline): use the same technique as in the previous proof, but select some ordering of the members of W (presumed denumerable), and construct the new derivation in a triangular fashion, so that it may use the first j terms of the derivation for the first member of W, the first $j - 1$ terms of the derivation of the second member, etc. []

The other important property is compactness. A deduction F is ***compact*** iff whenever $F(v,v')$ there exist some finite valuations y, y' such that

$y \sqsubseteq v$ (which of course means $v = v \sqcup y$)

$v' = v \sqcup y'$

We shall see in a moment that the deductions obtained from 'ordinary' rules of inference are compact, but first let us identify a consequence of compactness:

Proposition 5. If F is a conservative, linear and compact deduction, and there are derivations using F from v to v' and from v' to v'', then there is also one from v to v''.

Proof. Let the derivation from v to v' be

$v_0, v_1, \ldots$

and let the derivation from v' to v'' be

$v'_0, v'_1, \ldots$

where for each $i \geq 0$ there are some finite y_i, z_i such that

$F(y_i, z_i)$

$y_i \sqsubseteq v'_i$

$v'_{i+1} = v'_i \sqcup z_i$

Construct now a sequence of valuations as follows:

$v_0, v_1, \ldots v_{n_0},$

$v_{n_0} \sqcup z_0, v_{n_0+1} \sqcup z_0, \ldots v_{n_1} \sqcup z_0,$

$v_{n_1} \sqcup z_0 \sqcup z_1, \ldots$

...

where the n_i are selected so that

$n_i \leq n_{i+1}$

$y_i \sqsubseteq v_{n_i}$

for $i \geq 0$. By compactness such a sequence must exist, and by the linearity the sequence is a derivation using *F,* and its lu.b. is clearly v^*. []

We can now proceed to introducing the counterparts in our system of inference rules.

A *kernel* is a pair $\langle v,v' \rangle$ where v and v' are finite valuations, and t $\sqsubseteq v'$.

Kernels may be used for expressing how the truth-value of a composite expressions follows from the truth-value of its component(s), or vice versa. One such example was given in the introductory section. For another example, the rule that if a is true then $\neg a$ is false, is expressed by the kernel $\langle v,v' \rangle$ where

$v(a) = v'(a) = t$
$v(\neg a) = u$
$v'(\neg a) = f$

The *direct realization* of a kernel <v,v> is the deduction formed as

$\{\langle v \sqcup y,\ v' \sqcup y \rangle \mid y \in V\}$

In other words, the direct realisation of a kernel $\langle v, v' \rangle$ is the set of all possible pairs $\langle y, y' \rangle$ such that $v \sqsubseteq y$ and $y' = y \sqcup v'$. Each such pair characterizes a derivation that is allowed by the kernel, i.e. if the preconditions v are satisfied in *y* then the conclusions v' may be accumulated to *y* giving *y'*.

The direct realization of a set of kernels is defined to be the union (using U) of the direct realizations of the individual kernels. This has the effect that from each valuation v there are several successors, corresponding to the choices of which derivation step to take. From this definition it follows:

Proportion 6. The direct realization of a set of kernels is conservative, linear and compact.

4. NM-Rules.

We shall now characterize those deductions which correspond to (what we intuitively think of as) a set of non-monotonic inference rules.

Following Goodwin (1984) approximately, an *NM-rule* is a triple $\langle M,N,C \rangle$ of finite sets of formulas, where

M is the *monotonic antecedents,*
N is the *non-monotonic antecedents,*
C is the *consequents.*

The idea is that if each member of *M* is true, and each member of *N* is false or undefined, then each member of *C* can be inferred to be true. At the same time, the assumption is made that all members of *N are* false.

Each of *At, N,* and *C* may be the empty set. If *N* is empty we have a monotonic rule. If *C* is empty we have what Reiter (1080) calls a normal default rule.

The kernel that *corresponds to* an NM-rule $\langle M, N, C \rangle$ is the pair $\langle v,v' \rangle$ where

$v(m) = v'(m) = t$ for all m in *M*
$v'(n) = f$ for all n in *N*
$v'(c) = t$ for all *c* in *C*.

and all other values are a.

Thus non-monotonic rules differ from monotonic ones, partly by causing several formulas to change their truth-value as one inference step is performed. A possible objection against this way of dealing with non-monotonic antecedents, is that the resulting valuation should differentiate explicitly between that information which has been obtained as a consequent, and that which was 'merely* assumed in order to be able to apply the rule, i.e. the assignment to the non-monotonic antecedent(s). We however view that as a book-keeping issue, which need not concern the formal treatment of the deduction as such.

The direct realization of an NM-rule is the direct realization of its corresponding kernel

We let *H* be that deduction which performs the obvious deductions of conventional, propositional logic. Thus if a valuation *v* satisfies

$v(a \wedge b) = t$

then the valuation $H^*(v)$ which is the least fixpoint of *H* over *v* (proposition 1), satisfies

$H^*(v)(a) = H^*(v)(b) = t$

(unless a contradiction occurs in which case $H^*(l) = k$ for all *l*). We work presently on a more specific definition and analysis of this deduction *H*.

We want the direct realization of a set of NM-rules to be the deduction which has as subsets the realizations of each of the rules, but which is also able to do trivial derivations of truth-values. We therefore formally define the *direct realization* of a *set* of NM-rules as *H* $\cup$ (the union using $\cup$ of the direct realization of each of the rules). Clearly the direct realization of a set of NM-rules is linear, conservative, and compact.

The *restricted realization* of an NM-rule $\langle M,N,C \rangle$ is a subset of the direct realization of the same rule. It is obtained by excluding all those pairs $\langle v,v' \rangle$ where $v'(n) = k$ for some $n \in N$. Notice that pairs $\langle v, v' \rangle$ where $v'(c) = k$ for some $c \in C$ are not excluded, unless some $v'(n)$ is also *k*. The restricted realization of a set of NM-rules is obtained as the union (using $\cup$) of *H* and the restricted realization of each of the NM-rules. The restricted realization is conservative and compact but not monotonic (and therefore not linear).

An example may be useful at this point. For the examples we assume that the language consists of the formulas $\{a,b,c,...\}$. A valuation will be written as $[x,y,...]$ meaning the valuation *v* where $v(a) = x$, $v(b) = y$, etc. If either of the sets in a rule is a singleton, then the curly brackets around it will be omitted, and the empty set will be written as a dash. Thus $\langle a, -, c \rangle$ is an example of a rule, meaning the same as $\langle \{a\}, \{\}, \{c\} \rangle$.

Example 1. Suppose we have the following rules:

$\langle -, a, b \rangle$
$\langle b, -, c \rangle$

Informally, this says: *b* holds unless *a* is known to be true; if *b* then *c*. The restricted realization *G* of this set of rules satisfies:

$G([u,u,u], [f,t,u])$
(this uses the rule "*b* unless *a*")
$G([f,t,u], [f,t,t])$
(this uses the rule "if *b* then *c*")

and $[f,t,t]$ is also a fixpoint for *G* over $[u,u,u]$.

If we start instead from a valuation where *a* is known to be true, e.g. $v = [t,u,u]$, there is no valuation v' such that $G(v,v')$. The direct realization *F* of the same set of rules holds of course for the same argument pairs as *G*, but also

$F([t,u,u], [k,t,u])$

since the direct realization will proceed even if it introduces a contradiction. The valuation $[t,u,u]$ is therefore a fixpoint for *G*, but not for *F*. However, it is a maximally consistent extension of $[t,u,u]$ w.r.t. *F*.

5. Correct extensions, Approachability.

Throughout this section, we assume that v is a valuation, and *R* b a set of NM-rules whose direct realization is *F* and whose restricted realization is *G.*

A valuation v' b termed a *correct extenrion* of v w.r.t. *R* iff:

1. v' is a fixpoint of *G* above v (meaning in particular that
   $v \sqsubseteq v'$)
2. v' is consistent
3. there b some derivation from v to v' using *G.*

(Remember that the phrase $/v'$ is an extension of $v/$ means simply that $v' \sqsupseteq v$). The notion of correct extensions expresses stringently what are the desirable fixpoints for given v and *G.* If v' b not a fixpoint then some additional derivation steps remain to be performed. If v' is inconsbtent it b for either of the following reasons:
a) v is inconsistent
b) the set of NM-rules implies an inconsbtence (e.g. if $v(a) = t$ and one of the rules b $<-, -, \neg a>$)
c) some non-monotonic antecedent was assumed to be /early during the derivation, and later in the derivation we had an NM-rule with the same proposition as a consequent, which invalidated the assumption.

Finally, if there b no derivation from v to v' then v' is 'unfounded', like a non-minimal fixpoint in the monotonic case.

The third condition in the definition corresponds to Goodwin's requirement of well-founded ness. The above formulation b however problematic in that it refers to the exbtence of a derivation. By contrast, the minimality requirement on a fixpoint need not refer to derivations: it just states that no 'smaller' fixpoint exbts. We would similarly like to have a static condition, which guarantees the exbtence of a derivation, instead of having to prove its existence whenever needed. The remainder of thb section will give such a result.

Although the definition of correct extensions uses (7, the deduction *F* provides a partial characterization of them, and will be used as a tool*

Proportion 7. Each consistent fixpoint of *G* above v is a maximally consistent extension of # w.r.t. F.

Proof. Follows easily from the definition of *G* from *F,* since

$$F(v,v') \wedge \neg G(v,v')$$

implies that v 'b inconsbtent.

The converse does not hold, i.e. there are maximally consistent extensions of t w.r.t. *F* which are not fixpoints of *G* over v. Thb happens in those cases where a truth maintenance system has to shift IN nodes to OUT status. Consider

Example t. Suppose we have the following NM-rules:

$<-, a, b>$
$<-, -, a>$

Then $[f, t]$ is a maximally consbtent extension of $[u, u]$ w.r.t. F, but $G([f, t], [u, t])$.

Not every set of NM-rules has a correct extension:

Example 3. Consider the NM-rules

$<-, a, b>$
$<b, -, a>$

Then of course

$G([u,u],[f,t])$
$G([f,t],[u,t])$

and [u,u] does not have any correct extension w.r.t. these NM-rules.

Some of the propositions in earlier sections can now be extended to apply to the realizations of NM-rules. The basic idea b to first use those results for *F,* and then to transfer the result to *G* by introducing a consistency requirement.

Proportion 2A. If •' b a correct extension of v w.r.t. R, then for every y such that

$$v \sqsubseteq y \sqsubseteq v'$$

there exbts a derivation from *p* to v' uslug u.

Proof. According to proposition 2 thb holds for *F.* However, it follows from the definition of *G* that if *F(z,z')* and z' b consbtent, then *G(z,z').* Since v' is consbtent, so must all intermediate steps in the derivation from *p* be, because *G* b conservative. Therefore we have a derivation from y to v' using *G.* []

Corollarp ("minimality of extensions" - Reiter). If v' and v" are correct extensions of v w.r.t R, then

$$v' \sqsubseteq v'' \rightarrow v' = v''$$

Proportion SA. If there are derivations from rto f' and from v to v" using G, and v' U v" is consbtent, then there b a derivation from v to v' u *v"* using *G.*

Proof follows directly from proposition 3.

Corollarp. If v'and v" are dbtincl correct extensions of v w.r.t. *R,* then $v' \cup v''$ b inconsbtent.

Thb corollary subsumes Reiter's (1980) "orthogonality of extension" theorem. - Proposition 4 of course extends similarly.

Let us return now to the issue of the third criterium in the definition of a correct extension. Thb requirement can not be omitted, since that would allow fixpoints for which there is no support. For example, the valuation [t,u,u) b a consbtent fixpoint of *G* in example 1 above. In the monotonic case, such fixpoints are eliminated by the requirement to be minimal, but that requirement b not sufficient here since [f,u,u| is indeed a minimal fixpoint in the example (no 'smaller' fixpoint *exists).* However, we can substitute instead of the third requirement another one which b similar in spirit to minimality, as follows.

A fixpoint *v'* of *G* over *v* b called *approachable* from *v* iff

$$v \sqsubseteq y \prec v' \rightarrow (\exists y')(y \prec y' \sqsubseteq v' \wedge G(y,y'))$$

For lack of adequate characters in the font, $\prec$ is used for strict $\sqsubseteq$ i.e. excluding equality. Intuitively, thb says that whenever we are on the path from v to v', there b some step allowed by *G* that will take us closer to v'.

Thb concept b a strengthening of the concept of least fixpoint, since the definition directly implies:

Proportion 8. If v' b an approachable fixpoint of *G* over t, then it b minimal.

It b easily seen that the least fixpoint [t,u,u| in example 1 is not approachable. Thb approach ability condition can replace the third condition in the definition of the correct extension, since:

Proportion 9. If v' is a consbtent fixpoint of *G* over v, and v' b approachable from v, then it b a correct extension of t w.r.t. R.

Proof. *v*' immediately satisfies the first two conditions for being a correct extension. It remains to show that there b a derivation from v to v'using *G.*

Suppose this were not the case, i.e. every chain from t whose members are $\sqsubseteq v'$ (the existence of at least one such chain b guaranteed by the definition of 'approachable') has a l.u.b. $y \prec v'$. Let v'' be the l.u.b. of all such y. By proposition 4 there is a derivation from v to v''. If now $v'' \prec v'$, consider the y' whose existence is guaranteed by the approachability, such that

$$G(v'', y')$$
$$v'' \prec y' \sqsubseteq v'$$

The derivation step from v'' to y' must have been obtained using the extension of a kernel $<z, z'>$ such that $z \sqsubseteq v''$, and z' is not $\sqsubseteq v''$. By compactness, in any derivation,

$$z_0, z_1, \ldots$$

of v'' there must be some element z_n such that $z \sqsubseteq z_n$. But then there is also a derivation of $z_n \sqcup z'$, which is not $\sqsubseteq v''$. This is a contradiction. []

In this way we have obtained the desired counterpart of the fixpoint criterium of the monotonic case.

6. Normal default rules.

Reiter (1980) introduces the concept of *normal default rules*. He shows that every normal theory has an extension (in our terms: a correct extension), and proves semi-monotonicity, i.e. a larger set of normal default rules has a larger extension. His results, which are given with fairly complicated proofs, can now be obtained more easily from the material presented above.

A *normal default rule* is an NM-rule of the form $<M, \{n\}, \{\}>$.

If v is a valuation then v^* is the (obviously unique) least fixpoint of H over v. Clearly the * operation is monotonic. The valuation v is *fully consistent* iff v^* is consistent.

A valuation v is *saturated* w.r.t. a formula a iff $v(a) = v^*(a)$.

Let G be the restricted realization of a set of normal default rules. A derivation using G is *cautious* iff in each derivation step $<v,v>$ that uses a rule $<M, \{n\}, \{\}>$, v is saturated w.r.t. n. The idea is that in a cautious derivation it is *not* possible to have the following scenario. Let the formula c be $\neg a$. Start from the valuation $[u, u, f]$, i.e. $\neg a$ is false but the conclusion that a is true has not been drawn. Use the extension of the NM-rule $<\cdot, a, b>$ to derive $[f, t, f]$. After that, use the deduction H which administrates simple truth-value calculations, to derive $[k, t, f]$. The cases that we exclude by being cautious in this sense are the ones where an implementation would have to backtrack, or (in a truth-maintenance system) shift propositions from IN to OUT status, because a non-monotonic antecedent which was temporarily accepted because no proof had been bound so far, later had to be retracted when a proof was found.

Since H does derivation steps according to propositional logic, one can derive a valuation that is saturated w.r.t. n in a finite number of steps. It follows:

Proposition 10. Let v be a consistent valuation, and let G be the restricted realization of a set of normal default rules. Each step in a cautious derivation from v using G, is fully consistent.

Proof. We first prove that any step is consistent. By the definition of restricted realizations, a derivation step using a normal default rule can not introduce k into the valuation. Suppose a derivation step according to H in a cautious derivation does go from a consistent to an inconsistent valuation. Let n be the non-monotonic antecedent of the most recent derivation step, v' to v'', that uses an NM-rule. We must have $v'(n) = u$, $v''(n) = f$, and $v'(l) = v''(l)$ for all other formulas l. (By the definition of restricted realizations, we could not have had $v'(n) = t$, $v''(n) = k$). But by a familiar result of propositional logic, if there was a derivation using H from v'' to a contradiction, there must have been a derivation using H from v' to a valuation y where $y(n) = t$, which means v' was not properly saturated. Contradiction.

The full consistency follows easily, which completes the proof. []

Corollary. If there is a cautious derivation from v to v' using G, then v' is fully consistent.

Proposition 11. If G is the restricted realization of a set of normal default rules, and there is a derivation from a valuation v to a fully consistent valuation y using G, then there is a cautious derivation using G from v to some y' such that $y \sqsubseteq y' \sqsubseteq y^*$.

Proof. We prove by induction on the derivation of y, using the monotonicity of F, that there is a derivation using F from v to some such y'. (Just saturate sufficiently before applying each rule). But since y^* is consistent, so is y', and the derivation using F is also a derivation using G. []

The main result is now:

Proposition 12. If v is a fully consistent valuation, and G is the restricted realization of a set of normal default rules, then v has a correct extension w.r.t. G.

Proof. Let W be the set of l.u.b. of cautious derivations from v using G. By the corollary of proposition 10, each member of W is fully consistent. A subset $Y \subseteq W$ is called *maximally consistent* iff the l.u.b. of its members is fully consistent, but the l.u.b. of every strict superset of Y which is still a subset of W, isn't fully consistent. Consider some maximally consistent Y (it is clear that some exist, although maybe $Y = W$). Let y be the l.u.b. of Y. Clearly y is a fixpoint of H, i.e. $y = y^*$. According to proposition 4, there is a derivation from v to y, and by proposition 11 there is then a cautious derivation from v to some y' which is between y and y^*, so $y' = y$. There can however not be any derivation from y using G, except the trivial derivation consisting only of y, since otherwise Y would not be maximal. Therefore y is a fixpoint for G and a correct extension. []

Proposition 13 (semi-monotonicity): Let v be a fully consistent valuation; let $R' \subseteq R''$ be two sets of normal default rules; and let $G' \subseteq G''$ be the restricted realizations of R' and R''. If v' is a correct extension of v w.r.t. G', then there exists some correct extension v'' of v w.r.t. G'' for which $v' \sqsubseteq v''$.

Proof. let y be a correct extension of v' w.r.t. G''. Since using G'' there is a derivation from v to v', and a derivation from v' to y, there is also a derivation from v to y (proposition 5). Let W be the set of l.u.b. of cautious derivations using G'' from v to, or passing through y. Proceed as in the previous proof. [].

Reiter's proofs for the last two results are considerably more involved. The functional approach taken in this paper makes it possible to reason more abstractly, and therefore more concisely.

7. Conclusion.

The functional view, starting with the lattice of four truth-values (including k for contradiction) makes it possible to deal with non-monotonic logic on a high level of abstraction, avoiding tedious proofs, and at the same time offers the strong intuitions of partial orders, lattices, and fixpoints.

Acknowledgements

I wish to thank Jacek Leszexylowski for his constructive comments on an earlier version of this paper, and Brian Mayoh who brought Bemap's paper to my attention. The term 'conservative' was chosen to conform to the terminology in a manuscript for a paper by Allen Brown, who discussed 'belief conserving' rules of inference.

References.

Belnap, Nuel D.: "A Useful Four-Valued Logic". In: Dunn and Epstein, "Modern Uses of Multiple-Valued Logic", Dordrecht, 1077.

Blikle, A. (1981): "Notes on the Mathematical Semantics of Programming Languages". *Report LiTH-MAT-R-81-19,* Mathematics Department, Linkoping University, 1981.

Borgida, A. and T. Imielinski (1084): "Decision Making in Committees— A Framework for Dealing with Inconsistency and Non-Monotonicity", *Proceedings of AAAI Non-Monotonic Reasoning Workshop.*

Goodwin, J. (1084): "WATSON: A Dependency Directed Inference System". *Proceedings of AAAI Non-Monotonic Reasoning Workshop.*

Manna, Z. (1074): "Mathematical Theory of Computation", McGraw-Hill Book Company.

McDermott, Drew, and J. Doyle (1080): "Non-Monotonic Logic I", *Artificial Intelligence,* Vol 13:1, pp. 41-72.

Reiter, R. (1080): "A Logic for Default Reasoning", *Artificial Intelligence,* VoL 13:1, pp. 81-132.

Sandewall, E. (1073): "An Approach to the Frame Problem, and its Implementation". In: Meltser, B. and Michie, D. (Eds), *Machine Intelligence* 7 (Wiley, New York, 1072), pp. 195-204.

Scott, D. (1070): "Outline of a Mathematical Theory of Computation", *4th Annul Princeton Conf. on Information Seiences and Systems,* pp. 160-176.

Stoy, J. (1077): "Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory", The MIT Press.