Reasoning of Geometric Concepts based on Algebraic Constraint-directed Method

Hitoshi IBA*
Machine Inference Section,
Electrotechnical Laboratory
1-1-4 Umezono, Tsukuba-city,
Ibaraki, 305, Japan, iba@etl.go.jp

Abstract

We present an algebraic approach to geometric reasoning and learning. The purpose of this research is to avoid the usual difficulties in symbolic handling of geometric concepts. Our system GREW is grounded on a reasoning scheme that integrate the symbolic reasoning and algebraic reasoning of Wu's method. The basic principle of this scheme is to describe mathematical knowledge in terms of symbolic logic and to execute the subsidiary reasoning for Wu's method. The validity of our approach and GREW is shown by experiments, such as applying to learning-by-example of computer vision heuristics or solving locus problems.

1 Introduction

This paper presents a new approach for learning or reasoning of geometric concepts based on algebraic constraint-directed methods.

Geometric reasoning is available for many applications, such as robotics, CAD and computer vision. However, most previous reasoning systems, which are based on predicate logic, have difficulties in handling geometric notions. This is because the usual symbolic approach fails to grasp the essential characteristics of geometry, and cannot solve complicated problems, such as those which require auxiliary lines.

As a result, handling geometric concepts causes great trouble in many applications of reasoning. For instance, consider the heuristics called *skewed symmetry* in computer vision [Kanade81]. This is a famous geometric constraint which claims that a two-dimensional skewed symmetry is a projected image of a genuine three-dimensional symmetry (Fig.1). Because of transformation-invariant characteristics such as shear transformation, it is very difficult to represent this constraint by usual predicate logic, still more to establish the reasoning system. In order to solve these difficulties, we select Wu's method as algebraic approach, and construct a geometric reasoning system (**GREW**: Geometric REasoning based on Wu's method). This system is based on the integrated scheme of symbolic reasoning and algebraic method. We show its validity by experiments, such as learning computer-vision heuristics and solutions to locus problems.

*I wish to thank members in FAI-WG (Foundation of Artificial Intelligence) and CLP-WG (Constraint Logic Programming) of ICOT for useful comments and discussion on earlier drafts of this work.

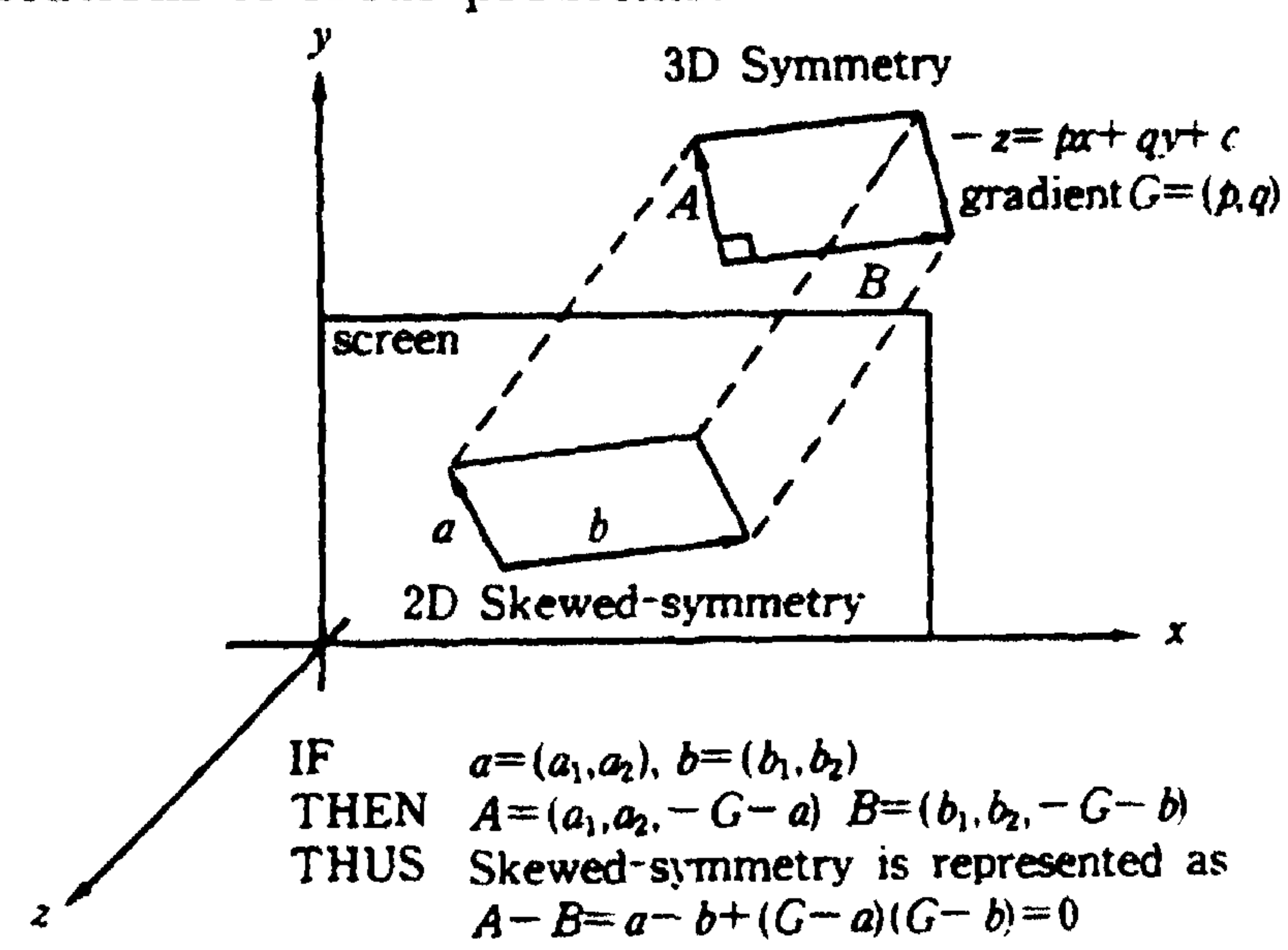

Fig.1 Skewed symmetry

2 Geometric reasoning based on algebraic method

2.1 Wu's Method

In general, the hypotheses of a geometrical theorem can be represented in triangular forms of algebraic expressions. That is,

$$tri_1(u_1, \cdots u_d, x_1, \cdots x_{r-1}, x_r) = 0$$
$$tri_2(u_1, \cdots u_d, x_1, \cdots x_{r-1}) = 0$$
$$\cdots\cdots\cdots\cdots$$
$$tri_r(u_1, \cdots u_d, x_1) = 0 \qquad (1)$$

Where $u_1, \cdots u_d$ are independent variables and $x_1, \cdots x_{r-1}, x_r$ dependent. Under these hypotheses, the conclusion is represented as follows.

$$Conc(u_1, \cdots u_d, x_1, \cdots x_{r-1}, x_r) = 0 \qquad (2)$$

With these preparations, the geometric proof is equivalent to deciding whether the expression (2) is equal to zero under the equality system of the condition (1). Wu's theorem gives a deterministic procedure for this decision [Wu78]. That is,

(2) is equal to zero under (1) ;
i.e. the conclusion of the theorem is valid
$\Longleftrightarrow$ $Rem_r = 0$

Rem_r (called (final) remainder term) is calculated as follows.

$$Rem_0 = Conc$$
$$Rem_{i+1} = (\text{ the remainder of } Rem_i \text{ divided by } tri_{i+1} \text{ under } x_{r-i})$$

The above procedure is called Wu's method [Chou84]. We have constructed an algebraic theorem proving system based on Wu's method. This system uses strategies for the efficient triangulation, such as decomposition of reducible cases, simplification of expressions, and conflict resolution of auxiliary or degenerate conditions. We have experimented in many examples to confirm that the efficient process is achieved [Iba90].

2.2 Algebraic constraint-directed method for geometric reasoning

We realized geometric reasoning based on the algebraic, method. In this section, we explain the constraint-directed principle with Wu's method.

Consider the case that the final remainder of Wu's method is not zero;

$$Rem_r \pm 0 \quad (3)$$

This expression is factorized into irreducibles as follows,

$$Rem_r = f_1^{e1} f_2^{e2} \cdots f_k^{ek}$$

If we make a new set of hypotheses such as;

$$tri_1, \cdots .tri_r \cup f_j (j = 1, \cdots k) \quad (4)$$

and retry Wu's method under this new hypotheses, then the new remainder generally equals to zero. Thus each *fj* is regarded as new algebraic constraints for validating the conclusion under the old hypotheses. These fj's are derived heuristics or candidates of geometrical descriptions. Therefore we apply Wu's method to geometric reasoning with the following fundamental principle;

CASE I: Conclusion is given beforehand.
In this case, each derived fj works like a candidate of newly-found heuristics that validates the original conclusion under the hypotheses *trij.*

CASE II: Conclusion is not given.
In this case we apply Wu's method by choosing one *trij* as Conc.The independent variables u_i, regulate the resultant remainder description. That is, the final remainder Rem_r is represented on the basis of independent variables. Therefore it is necessary to choose appropriate independent variables for fi-

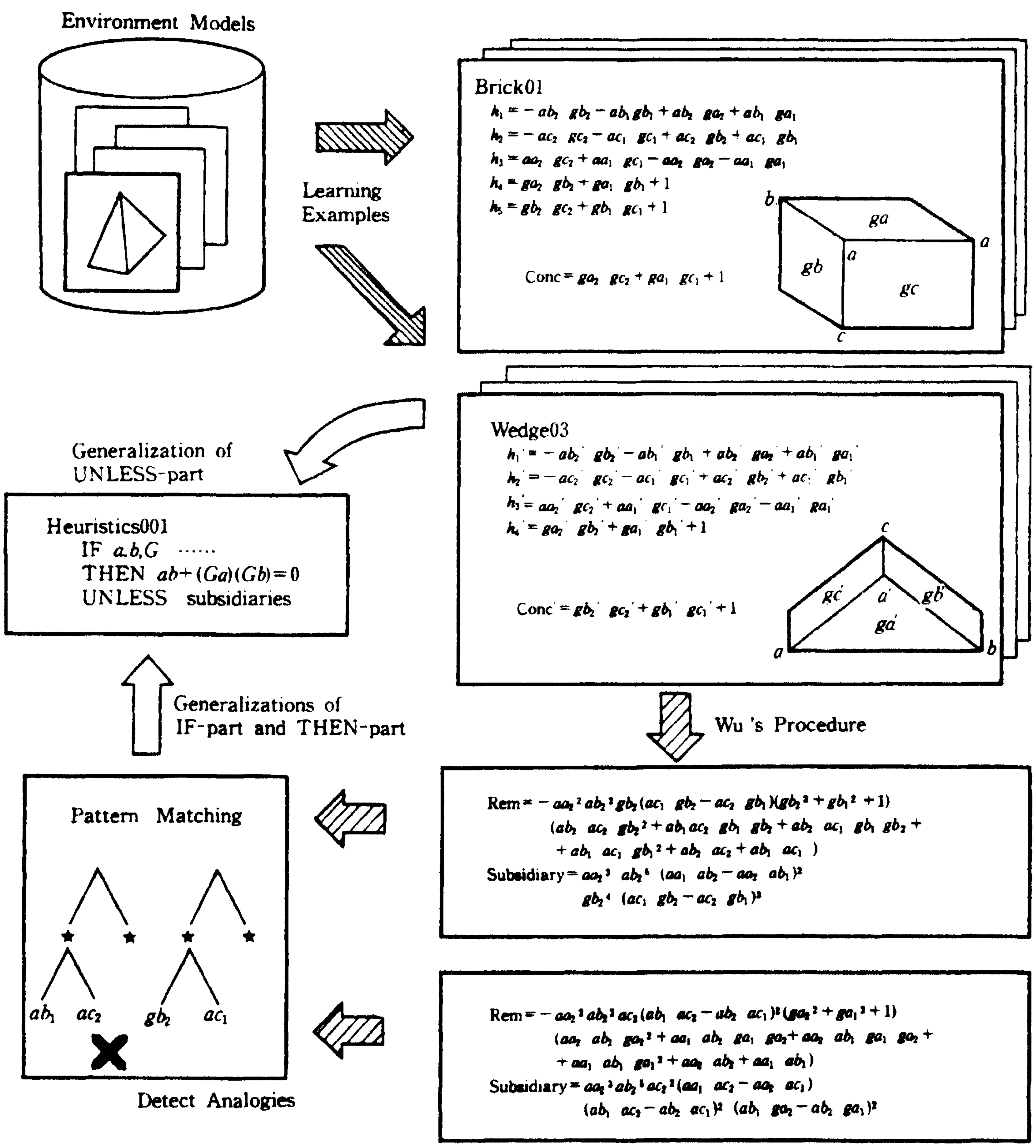

Fig.2 Acquisition of geometric heuristics

nal expressions, and *fj* is regarded as the geometric representation on the basis of ,-.

We call the above principle as the constraint-directed reasoning, which is justified alge-geometrically as follows. The final remainder (3) forms a subset of algebraic variety of hypotheses and the conclusion. Thus the desired geometric information is represented, though partially, as algebraic constraints of remainder terms.

To illustrate this algebraic constraint-directed approach, we show its application to learning-by-example of geometric heuristics in computer vision. We alge-geometrically represent heuristics or relations in computer vision as follows.

IF	<u>conditioni</u>	(constraints on variables)
THEN	<u>conclusion</u>	$f(x_1, \cdots, x_r) = 0$
UNLESS	<u>$condition_2$</u>	

<u>Conclusion</u> *f* corresponds to the part of expressions that may occur in the final remainder term of Wu's method. <u>$Condition_1$</u> describes constraints on variables in /. Learning of heuristics is realized by deriving <u>$condition_1$</u> and conclusion part from common terms in the final remainder of Wu's method. This strategy is based on the above claim in CASE I.

Fig.2 shows the learning process of Kanade's heuristics, in which the skewed-symmetry heuristic (Fig.l) is learned from exemplar bricks and wedges. In this case, hypotheses and the conclusion consist of six expressions with bricks, and five with wedges. These expressions are after [Swain86]. For instance, $h1, \cdots h_6$ represent a brick as a whole (Fig.3). Here, $h_1,\ldots h_3$ correspond to Mackworth constraints or parallel-line heuristics [Kanade81]. More precisely, h_1 shows that the vector *(GB — GA)* is vertical with the vector ab, and this is equivalent to the fact that upper and lower lines of face B (projected to parallel segments in two-dimensions) are also parallel in three-dimensions. And $h_4,\ldots\ldots h_6$ mean that faces A,B,C are vertical with each other in three-dimensions because the normal vector of face A is *{ga\,ga₂,l)* and so on (Fig.l). The whole system of $h_1,\ldots\ldots h_6$ together gives a complete description of two-dimensional visibility and three-dimensional model of bricks and their relations.

The remainder resulting from Wu's method is shown as *Rem* in Fig.2. Though omitted in the figure, learning examples are actually taken from six types of bricks and five types of wedges. The reasoning process is executed such as pattern-matching, eliminating trivial cases (aa_2), and storing analogous patterns of terms *(ab2ac2gb22-\—)* and *(aa2ab2ga22 + •••)*• The matching of *(acy <l&2-ac2<7&i)* and *(ab\ac2 — al)2dC\)* failed because the dimensions of *gb, ab, ac* are different *(gb* is gradient, and *ab, ac* are metric). After necessary generalizations, the following representation is obtained as a heuristic.

IF $\vec{a}, \vec{b}, \vec{G}$: two-dimensional vectors

THEN $(\vec{a}, \vec{b}) + (\vec{G}, \vec{a}) * (\vec{G}, \vec{b}) = 0$

UNLESS auxiliary conditions of $\vec{a},\vec{b},\vec{G}$

Where *G* is a x,y coordinate of three-dimensional gradient of the parallelogram formed by vector *a* and *b*. *(a,b)* means an inner product of vectors. UNLESS part represents a subsidiary premise which makes this heuristic applicable [Swain86]. Through the use of this UNLESS knowledge, we have established the appropriate maintenance and modification of constraint-directed models for robotics [Iba88].

The above representation is equivalent to the one in [Kanade81], which shows the validity of our approach. In the same way, other kinds of geometric heuristics in computer vision can be learned from examples; for example, parallel-line heuristics and Mackworth constraints [Iba90].

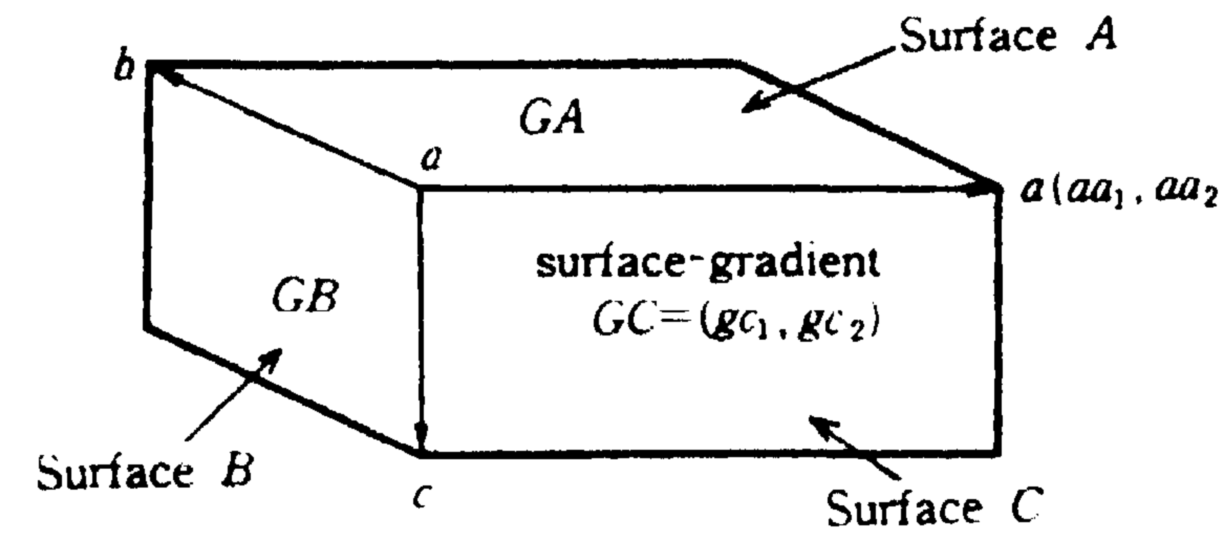

Fig.3 Line drawing of brick

3 Geometric reasoning system: GREW

Algebraic method elaborately solves reasoning problems with auxiliary lines or ad-hoc heuristics. However, at the same time, this approach is accompanied by computational problems such as the selection of independent variables or the derivation of geometric information from algebraic expressions. These kinds of problems are difficult to solve only within algebraic domains of polynomials (called syntax of expressions). Rather, reasoning with symbolic descriptions (called semantics of expressions) is required.

In order to realize an effective handling of geometric notions, we have constructed an integrated reasoning scheme; integration both of symbolic reasoning and algebraic reasoning of Wu's method (Fig.4). The basic principle of this scheme is to describe mathematical knowledge in terms of symbolic logic and to execute subsidiary reasoning for Wu's method. Thus, our system establishes appropriate handling of geometric semantics.

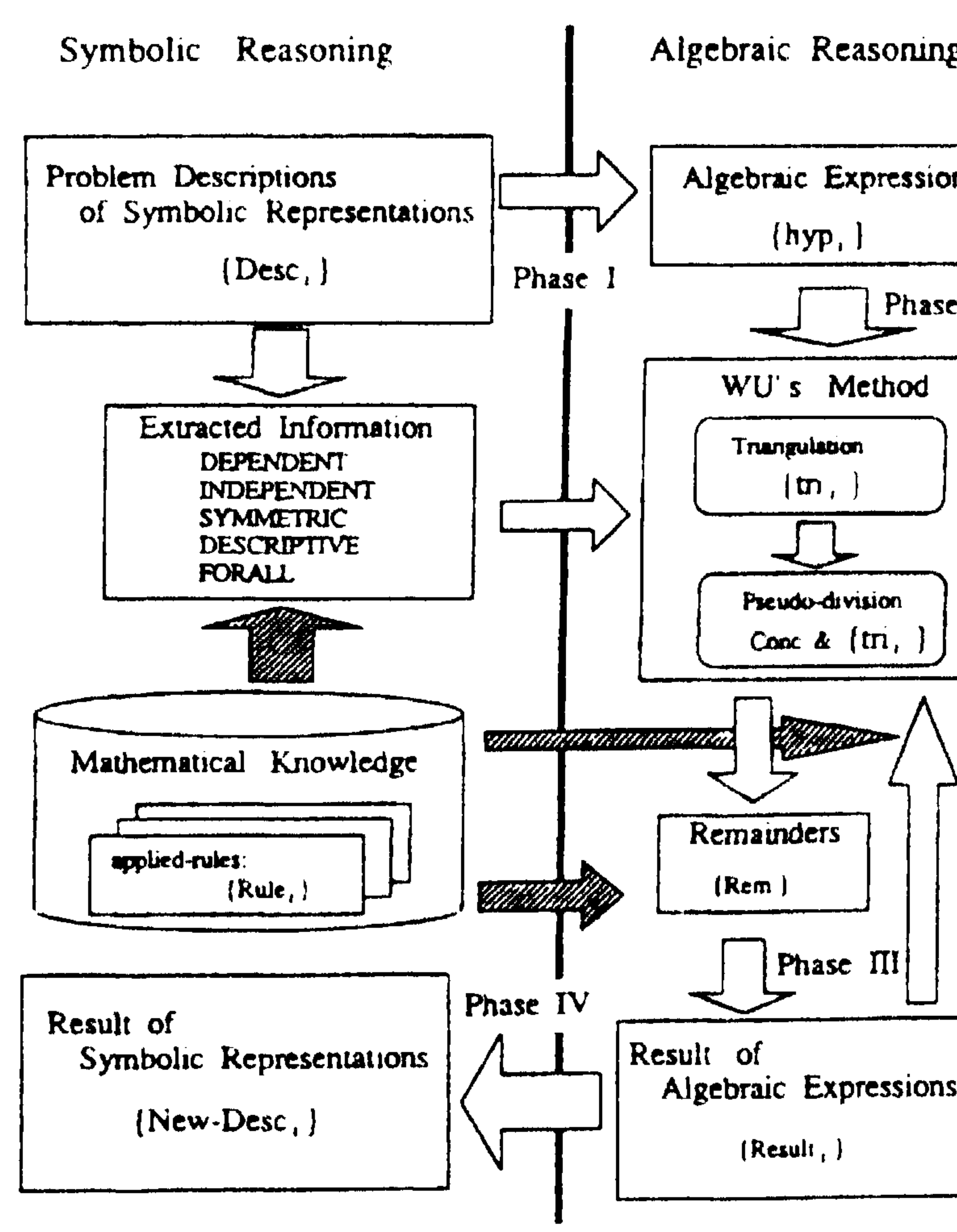

Fig.4 Symbolic and algebraic reasoning

Our system GREW consists of four fundamental phases as follows.

Phase I: Translation of symbolic representations into algebraic representations

Input expressions *Desci* for GREW are represented by symbols. In Phase I, these descriptions are translated into algebraic representations *hypi.* We realized the translation of about 20 geometric notions in two- or three-dimensional Euclidean space into algebraic representations. This translation is, in general, reversible. In Phase IV, the inverse translation is used to derive geometric descriptions in the symbolic form.

In Phase I, in addition to the above mechanical translation, the following important information is derived for the subsequent reasoning. This corresponds to the geometrical semantics in algebraic expressions.

(1) DESCRIPTIVE
The information as to which variables exist in the original problem description. This is used in Phase IV.

(2) INDEPENDENT
This maintains important information of dependent variables used in the triangular derivation (Phase II). In locus problems (Fig.5,6), variables in the FIND statement are independent because they are designated to lie in the desired locus. On the other hand, added variables are generally candidates for dependents.

(3) DEPENDENTS
This represents a set of dependent variables, which can be derived from problem descriptions by judging whether to lie on the same figure or geometric relations. It is difficult, in general, to determine INDEPENDENT and DEPENDENTS completely. Thus some sorts of heuristics are essential. These two kinds of information are used for the triangular derivation in Phase II.

(4) SYMMETRIC
The information as to the symmetry of descriptions. Symmetries are ubiquitous in geometric problems and enable an effective reasoning, as shown later.

(5) FORALL
The information as to the variable definition range for universally valid equations is maintained. This information is used in Phase III.

Fig.7 shows a part of the reasoning process of GREW for the two-circle problem. This problem is described as follows (Fig.5).

> Find the locus of the mid-point *P* of all segments QT"s, where the end points *Q* and *T* lie on 0_1 and 0_2 respectively (0_1 and O_2 are two circles outside each other and the radii are r_1 and r_2, respectively).

$Desc_9$,- - -$Desc_{15}$ and hyp_1,......hyp_8 are the results of Phase I. NIL means to be undecided slots in the original descriptions. In this case it is unnecessary to describe the center or radius of circles beforehand. These slots are appropriately filled by new symbols in Phase I. The FIND statement means that the point *P* is on the desired locus.

Phase II: Constraint-derivation based on Wu's method

This phase is the core of our reasoning system GREW, which deduces algebraic constraints with Wu's method. Here the triangular form is derived with the following principles of selecting dependents; that is, selecting as many variables that belong to the same set in DEPENDENTS as possible, and avoiding the selection of variables in INDEPENDENT and FORALL. These principles work as heuristics for Phase IV.

Phase III: Algebraic reasoning based on mathematical knowledge

The reasoning is executed on the final remainder terms. Mathematical knowledge is maintained for this execution as follows.

(1) Manipulations of algebraic expressions
Simplify the final remainder terms by factorizing or transforming. This is effective when focused variables are given beforehand, such as locus problems. In locus problems those variables are designated by FIND statements (Fig.7,8).

(2) Reasoning as to universally valid equations
In order to derive the condition for making the final remainder term zero, the reasoning about universally valid equations is executed with FORALL descriptions.

$$\text{Eg. } At + Bs = 0 \text{ (for all } t, s) \iff A = B = 0$$
$$Axy + Bx^2 + Cy^2 + D = 0 \text{ (for all } x, y)$$
$$\iff A = B = C = D = 0$$

(3) Reasoning as to trigonometric functions
Reasoning about trigonometric functions are essential for describing circles.

$$\text{Eg.} Acos(t) + Bsin(t) + C = 0 \quad (0 \le t \le 2\pi)$$
$$\Longrightarrow A^2 + B^2 \ge C^2$$

(4) Solving inequalities
Try to solve simple inequalities.

(5) Manipulations on vectors or matrices
Components of vectors or matrices are manipulated in connection with its geometric representation.

$$\text{Eg. Vectors } \vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2) \text{ are given;}$$
$$\vec{a} \text{ is vertical with } \vec{b} \iff a_1 b_1 + a_2 b_2 = 0$$
$$2 \mid a_1 b_2 - a_2 b_1 \mid = \text{the area of the parallelogram formed by } \vec{a} \text{ and } \vec{b}$$

(6) Reasoning by pattern matching between expressions
Reasoning is executed based on symmetry or analogy of derived algebraic expressions. Geometric semantics are essential for this. We illustrated the learning experiment in this reasoning (Fig.2).

Phase IV: Inverse translation into symbolic representations

Translate algebraic expressions into symbolic representations inversely. The basic strategy is to make pattern-matching with the template of deduced algebraic expressions in Phase I. DESCRIPTIVE information is used so as to derive symbolic expressions as general as possible (the least number of variables newly introduced).

4 Experimental Results

We have confirmed the validity of GREW and our algebraic constraint-directed reasoning scheme by many experiments such as locus problems (Fig.5,6) and construction problems [Iba90]

Fig.7 shows a part of the reasoning process of GREW for the two-circle problem. The constraint-directed reasoning is executed based on inequality relations of trigonometry and quadratic inequalities (type (1)(3)(4) in Phase III). *New* — Dec_5 is finally derived and shows that the point *P* is in the range which is outside a circle (the center is a mid-point of centers of circle 0_1 and circle O2, and the radius is $\frac{|r_1-r_2|}{2}$), and which is inside a circle (the center is the same, and the radius is $\frac{(r_1+r_2)}{2}$).

As another example, Fig.8 shows the solution to 3D skew-line problem . This problem is described as follows.

> Two lines XX' and YY' are given. These lines are not in the same two-dimensional plane, and are vertical with each other. Find a locus of the mid point M of all segments KL's, where the end points K and L are on lines XX' and YY' respectively, and the distance of KL is constant *P* (Fig-6)-

In phase I, new variables are generated for directional vectors of lines, and algebraic expressions are derived with these parameter variables. The reasoning as to universally valid equations is executed in Phase III, and the algebraic constraints are derived as follows.

$$(1) a_1(x_1 - y_1) + a_0(x_0 - y_0) = 0$$
$$(2) y_2 - x_2 = 0$$
$$(3) 4(\alpha - y_2)^2 + 4(\beta - x_1)^2 + 4(\gamma - x_0)^2 - p^2 = 0$$

Because of the symmetrical relation (SYMMETRIC in Phase I), the following equation is added.

$$(4) 4(\alpha - x_2)^2 + 4(\beta - y_1)^2 + 4(\gamma - y_0)^2 - p^2 = 0$$

In Phase III, from (1) and (2), it is deduced that two vectors (a_0, a_1, a_2) and $(x_0 - y_0, x_1 - y_1, x_2 - y_2)$ are vertical with each other. It is also deduced that (b_0, b_1, b_2) is vertical with $(x_0 - y_0, x_1 - y_1, x_2 - y_2)$, because (b_0, b_1, b_2) and (a_0, a_1, a_2) are vertical with each other (hyp_1,Fig.8(a)).

$$(a_0, a_1, a_2) \perp (x_0 - y_0, x_1 - y_1, x_2 - y_2)$$
$$(b_0, b_1, b_2) \perp (x_0 - y_0, x_1 - y_1, x_2 - y_2) \quad (\#)$$

Thus points of the locus are the intersection of two spheres (each center is (x_0, x_1, x_2) and (y_0, y_1, y_2) respectively, and radius is $\frac{p}{2}$, where (x_0, x_1, x_2) and (y_0, y_1, y_2) satisfy (#)). Furthermore, in checking sufficient conditions for the locus, it is also deduced that this intersection lies on the same plane, which is omitted here. As a result, $New - Desc$ descriptions are derived in Phase IV, where !- means a vector subtraction operator.

5 Future research

Our reasoning scheme has an advantage in handling of geometric notions, and we mean to make applications of this method to more practical domains; such as path-planning problems or environment model managements of intelligent robots [Iba88]. For this purpose, we now research on further extensions in the algebraic reasoning. This is to cope with the failure in Phase IV, that is, the failure to derive appropriate geometrical information from algebraic representations. This failure is caused by the lack of primitives to make inverse translations from algebraic expressions, or by the inappropriate algebraic reasoning in phase III. From the more practical viewpoint, we think it important to calculate approximated solutions based on the final remainder terms by using numeric methods or simulations. A program called TLA embodies this kind of methodology for mechanical simulations [Kramer90]. On the other hand, mathematical expert systems seem to us promising, in which formal handling of geometric semantics could be realized by the knowledge or meta-knowledge in mathematics. Thus our future research of concern is to further extend our method in this direction to realize the appropriate control of reasoning in algebraic domains, and to formalize its algorithm with both domains; symbolic and algebraic.

6 Conclusion

Reasoning about geometric notions is difficult to execute only by the usual symbolic method. This paper presents a new scheme for geometric reasoning with algebraic constraint-directed method. Although our algebraic approach avoids the complicated problem of reasoning in auxiliary lines or handling of heuristics, at the same time it encounters computational difficulty in selecting independent variables. To solve this, we have tried to derive geometric semantics of algebraic expressions from the original problem description, and to establish the constraint-directed reasoning based on these semantics. Finally, the validity of our system GREW has been shown by experiments.

References

[Chou, 1984] Chou,S. Proving elementary geometry theorems using Wu's algorithm. *Contemporary mathematics,* 29, American Mathematical Society, 1984

[Iba *et al,* 1988] Iba,H. MatsubaraJI. and Inoue,H. View and visibility for managing incomplete environment models. In *Proc. of IEEE International Workshop on intelligent robots and systems,* 1988

[Iba, 1990] Iba,H. Reasoning of geometric concepts based on algebraic method. Ph.D Thesis, University of Tokyo, 1990

[Kanade, 198l] Kanade,T. Reconstruction of the three-dimensional shape of an object, from a single view. *Artificial intelligence,* 17, 1981

[Kramer, 1990] Kramer,G.A. Solving geometric constraint systems. In *Proc. of the Eighth AAAI,* 1990

[Swain *et ai,* 1986] Swain,M.J. and Mundy,J.L. Experiments in using a theorem to prove and develop geometrical theorems in computer vision. In *Proc. of IEEE, Robotics and Automation,* 1986

[Wu, 1978] Wu,W. On the decision problem and the mechanization of theorem-proving in elementary geometry. *Scientia Sinica,* 21, 1978

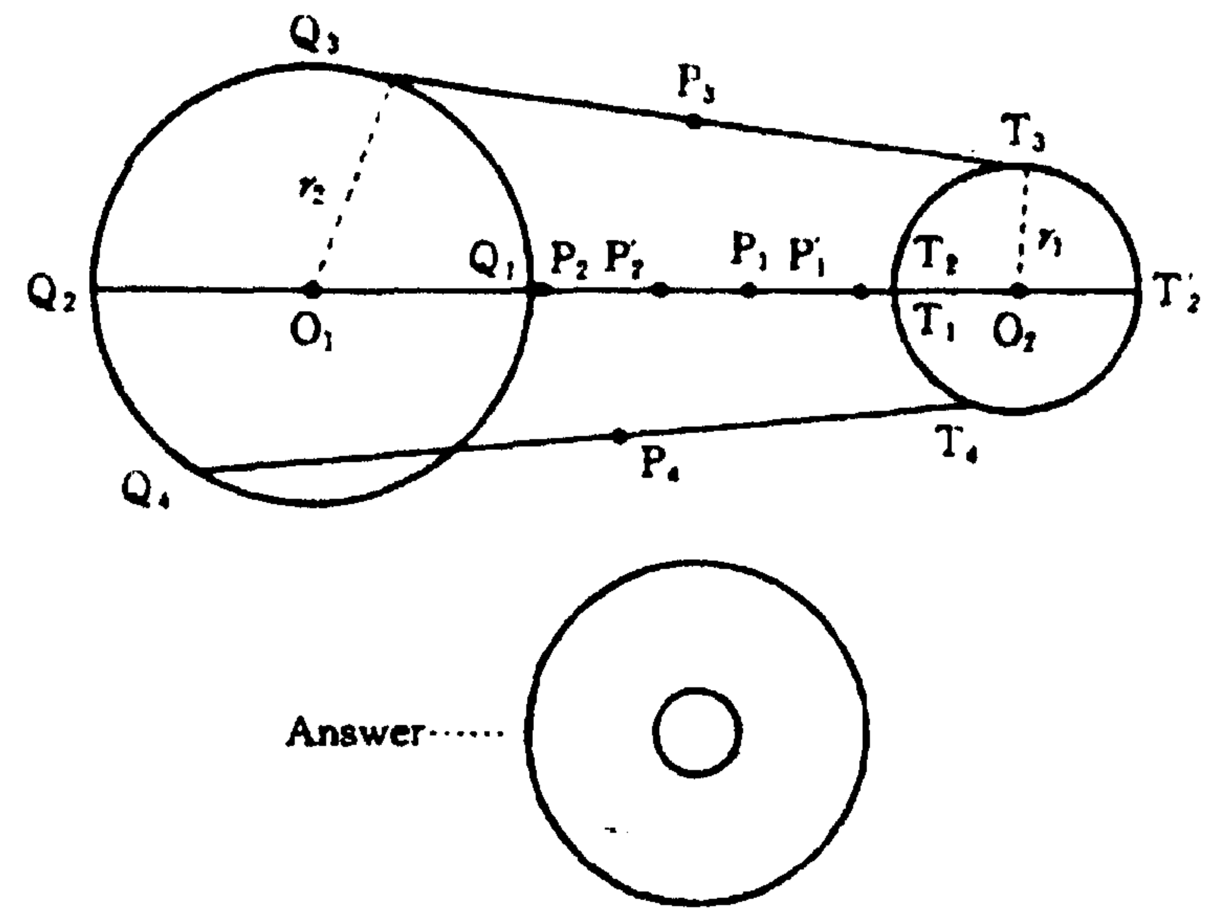

Fig.5 Two-circle problem

PHASE I : Translation into algebraic expressions

```
Desc1  : (ON $Q $O1)
Desc2  : (ON $T $O2)
Desc3  : (IS - CIRCLE $(O1 2) NIL r1)
Desc4  : (IS - CIRCLE $(O2 2) NIL r2)
Desc5  : (IS - POINT $(Q 2) NIL)
Desc6  : (IS - POINT $(T 2) NIL)
Desc7  : (IS - POINT $(P 2) (MID - POINT $Q $T))
Desc8  : (FIND $P)
```

⇓

```
Desc9  : (IS - CIRCLE $(O1 2) $A r1)
Desc10 : (IS - CIRCLE $(O2 2) $B r2)
Desc11 : (IS - POINT $(Q 2) (q1 q2))
Desc12 : (IS - POINT $(T 2) (t1 t2))
Desc13 : (IS - POINT $(A 2) (a1 a2))
Desc14 : (IS - POINT $(B 2) (b1 b2))
Desc15 : (IS - POINT $(P 2) (p1 p2))
```

$hyp_1 = -r_1 s_1 + q_1 - a_1$

$hyp_2 = -c_1 r_1 + q_2 - a_2$

$hyp_3 = s_1^2 + c_1^2 - 1$

$hyp_4 = t_1 - r_2 s_2 - b_1$

$hyp_5 = t_2 - c_2 r_2 - b_2$

$hyp_6 = -t_1 - q_1 + 2p_1$

$hyp_7 = -t_2 - q_2 + 2p_2$

$hyp_8 = s_2^2 + c_2^2 - 1$

DESCRIPTIVE

$\{O_1, O_2, r_1, r_2, Q, T, P\}$

INDEPENDENT

$\{p_1, p_2\}$

DEPENDENTS

$\{a_1, a_2, r_1, q_1, q_2\} \cup \{b_1, b_2, r_2, t_1, t_2\}$
$\cup \{p_1, p_2, t_1, t_2, q_1, q_2\} \cup \{c_1, s_1\} \cup \{c_2, s_2\}$

SYMMETRIC

$\{(O_1, O_2), ((a_1, a_2, r_1, q_1, q_2)(b_1, b_2, r_2, t_1, t_2))\}$

FORALL

(trigonometric c1 s1) ∧ (trigonometric c2 s2)

PHASE II : Wu's method

$$Rem_7 = -(4p_1 r_1 s_1 - 2b_1 r_1 s_1 - 2a_1 r_1 s_1 + r_2^2 - r_1^2 + 4c_1 p_2 r_1 - 2b_2 c_1 r_1 - 2a_2 c_1 r_1 - 4p_2^2 + 4b_2 p_2 + 4a_2 p_2 - 4p_1^2 + 4b_1 p_1 + 4a_1 p_1 - b_2^2 - 2a_2 b_2 - b_1^2 - 2a_1 b_1 - a_2^2 - a_1^2)$$

with Conc = hyp_8

$x_1 = c_1, x_2 = q_1, x_3 = q_2, x_4 = t_1,$
$x_5 = t_2, x_6 = s_2, x_7 = c_2$

PHASE III : Algebraic reasoning

Applied − rule$_1$: $\mathbf{A}cos\theta + \mathbf{B}sin\theta + \mathbf{C} \equiv 0 \ (0 \le \forall\theta \le 2\pi)$
$\Longrightarrow \mathbf{A}^2 + \mathbf{B}^2 \ge \mathbf{C}^2$
where $cos\theta = c_1 \ sin\theta = s_1$
⇓ $(FORALL - condition)$ *in* Rem_7

$Result_1$:

$$\begin{aligned}
&- r_2^4 + (2r_1^2 + 8p_2^2 + (-8b_2 - 8a_2)p_2 + 8p_1^2 + (-8b_1 - 8a_1)p_1 + 2b_2^2 \\
&+ 4a_2b_2 + 2b_1^2 + 4a_1b_1 + 2a_2^2 + 2a_1^2)r_2^2 - r_1^4 + (8p_2^2 + (-8b_2 - 8a_2)p_2 \\
&+ 8p_1^2 + (-8b_1 - 8a_1)p_1 + 2b_2^2 + 4a_2b_2 + 2b_1^2 + 4a_1b_1 + 2a_2^2 + 2a_1^2)r_1^2 \\
&- 16p_2^4 + (32b_2 + 32a_2)p_2^3 + (-32p_1^2 + (32b_1 + 32a_1)p_1 - 24b_2^2 - 48a_2b_2 \\
&- 8b_1^2 - 16a_1b_1 - 24a_2^2 - 8a_1^2)p_2^2 + ((32b_2 + 32a_2)p_1^2 + ((-32b_1 - 32a_1)b_2 \\
&- 32a_2b_1 - 32a_1a_2)p_1 + 8b_2^3 + 24a_2b_2^2 + (8b_1^2 + 16a_1b_1 + 24a_2^2 + 8a_1^2)b_2 \\
&+ 8a_2b_1^2 + 16a_1a_2b_1 + 8a_2^3 + 8a_1^2a_2)p_2 - 16p_1^4 + (32b_1 + 32a_1)p_1^3 - a_2^4 \\
&+ (-8b_2^2 - 16a_2b_2 - 24b_1^2 - 48a_1b_1 - 8a_2^2 - 24a_1^2)p_1^2 + ((8b_1 + 8a_1)b_2^2 - a_1^4 \\
&+ (16a_2b_1 + 16a_1a_2)b_2 + 8b_1^3 + 24a_1b_1^2 + (8a_2^2 + 24a_1^2)b_1 + (-4a_1a_2^2 - 4a_1^3)b_1 \\
&+ 8a_1a_2^2 + 8a_1^3)p_1 - b_2^4 - 4a_2b_2^3 + (-2b_1^2 - 4a_1b_1 - 6a_2^2 - 2a_1^2)b_2^2 - 2a_1^2a_2^2 \\
&+ (-4a_2b_1^2 - 8a_1a_2b_1 - 4a_2^3 - 4a_1^2a_2)b_2 - b_1^4 - 4a_1b_1^3 + (-2a_2^2 - 6a_1^2)b_1^2 \ge 0
\end{aligned}$$

Applied − rule$_2$: *factorize and factorsum*

⇓

$Result_2$: **LHS** $\Longrightarrow$

$$\begin{aligned}
&-(r_2^2 - 2r_1r_2 + r_1^2 - 4p_2^2 + 4b_2p_2 + 4a_2p_2 - 4p_1^2 \\
&+4b_1p_1 + 4a_1p_1 - b_2^2 - 2a_2b_2 - b_1^2 - 2a_1b_1 - a_2^2 - a_1^2) \\
&(r_2^2 + 2r_1r_2 + r_1^2 - 4p_2^2 + 4b_2p_2 + 4a_2p_2 - 4p_1^2 + 4b_1p_1 \\
&+4a_1p_1 - b_2^2 - 2a_2b_2 - b_1^2 - 2a_1b_1 - a_2^2 - a_1^2) \\
&\Longrightarrow \\
&-\{(r_2 - r_1)^2 - 4(p_2 - \tfrac{a_2+b_2}{2})^2 - 4(p_1 - \tfrac{a_1+b_1}{2})^2\} \\
&\{(r_2 + r_1)^2 - 4(p_2 - \tfrac{a_2+b_2}{2})^2 - 4(p_1 - \tfrac{a_1+b_1}{2})^2\}
\end{aligned}$$

Applied − rule$_3$: $(x - a)(x - b) \le 0 \ (a \le b) \iff a \le x \le b$
where $x = (p_1 - \frac{a_1+b_1}{2})^2 + (p_2 - \frac{a_2+b_2}{2})^2$

⇓

$Result_3$: $(p_1 - \frac{a_1+b_1}{2})^2 + (p_2 - \frac{a_2+b_2}{2})^2 - (\frac{r_1+r_2}{2})^2 \le 0$

$Result_4$: $(p_1 - \frac{a_1+b_1}{2})^2 + (p_2 - \frac{a_2+b_2}{2})^2 - (\frac{r_1-r_2}{2})^2 \ge 0$

PHASE IV : Translation into symbolic representations

```
New - Desc1 : (ON $P (DOMAIN
                       (IN - SIDE  $O3)
                       (OUT - SIDE $O4)))
New - Desc2 : (IS - CIRCLE $(O3 2)
                           $PP
                           (times 1/2 (plus r1 r2)))
New - Desc3 : (IS - CIRCLE $(O4 2)
                           $PP
                           (times 1/2 (abs (minus r1 r2))))
New - Desc4 : (IS - POINT  $(PP 2)
                           (MID - POINT $A $B))
              ⇓
New - Desc5 : (ON $P
                (DOMAIN
                 (IN - SIDE
                  (CIRCLE
                   (MID - POINT (CENTER - OF $O1)
                                (CENTER - OF $O2))
                   (times 1/2 (plus r1 r2))))
                 (OUT - SIDE
                  (CIRCLE
                   (MID - POINT (CENTER - OF $O1)
                                (CENTER - OF $O2))
                   (times 1/2 (abs (minus r1 r2)))))))
```

Fig.7 Solution to locus problem (1)

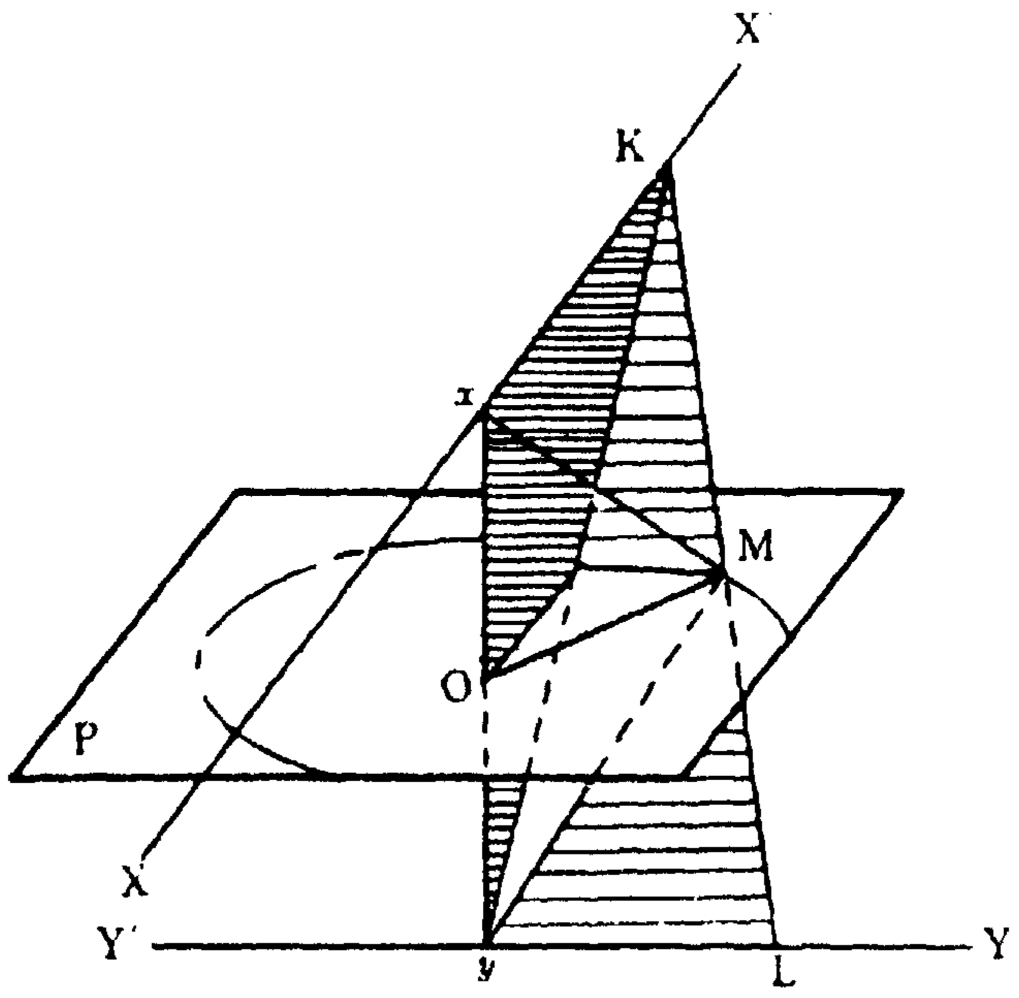

Fig.6 Skew-line problem

PHASE I : Translation into algebraic expressions

```
Desc1  :  (ON $K $XX')
Desc2  :  (ON $L $YY')
Desc3  :  (IS - LINE $(XX' 3) NIL NIL)
Desc4  :  (IS - LINE $(YY' 3) NIL NIL)
Desc5  :  (IS - POINT $(K 3) NIL)
Desc6  :  (IS - POINT $(L 3) NIL)
Desc7  :  (IS - POINT $(M 3) (MID - POINT $K $L))
Desc8  :  (IS - PERPENDICULAR $XX' $YY')
Desc9  :  (EVAL (eq (DISTANT $K $L) p))
Desc10 :  (FIND $M)
```

⇓

```
Desc11 :  (IS - LINE $(XX' 3) $X $A))
Desc12 :  (IS - LINE $(YY' 3) $Y $B))
Desc13 :  (IS - POINT $(K 3) (k0, k1, k2))
Desc14 :  (IS - POINT $(L 3) (l0, l1, l2))
Desc15 :  (IS - POINT $(M 3) (α, β, γ))
Desc16 :  (IS - POINT $(X 3) (x0, x1, x2))
Desc17 :  (IS - POINT $(Y 3) (y0, y1, y2))
Desc18 :  (IS - VECTOR $(A 3) (a0, a1, a2))
Desc19 :  (IS - VECTOR $(B 3) (b0, b1, b2))
```

$\mathbf{hyp}_1 = a_2b_2 + a_1b_1 + a_0b_0$
$\mathbf{hyp}_2 = -x_0 - a_0t_0 + k_0$
$\mathbf{hyp}_3 = -x_1 - a_1t_0 + k_1$
$\mathbf{hyp}_4 = -x_2 - a_2t_0 + k_2$
$\mathbf{hyp}_5 = -y_0 - b_0t_1 + l_0$
$\mathbf{hyp}_6 = -y_1 - b_1t_1 + l_1$
$\mathbf{hyp}_7 = -y_2 - b_2t_1 + l_2$
$\mathbf{hyp}_8 = -l_0 - k_0 + 2\alpha$
$\mathbf{hyp}_9 = -l_1 - k_1 + 2\beta$
$\mathbf{hyp}_{10} = -l_2 - k_2 + 2\gamma$
$\mathbf{hyp}_{11} = p^2 - (k_2 - l_2)^2 - (k_1 - l_1)^2 - (k_0 - l_0)^2$

DESCRIPTIVE
{XX', YY', p, K, L, M}
INDEPENDENT
$\{t_0, t_1, \alpha, \beta, \gamma\}$
DEPENDENTS
$\{k_0, k_1, k_2, l_0, l_1, l_2\} \cup \{a_2, b_2, a_1, b_1, a_0, b_0\}$
SYMMETRIC
{ (XX', YY'),
$((a_0, a1, a_2, k_0, k_1, k_2, x_0, x_1, x_2, t_0)$
$(b_0, b1, b_2, l_0, l_1, l_2, y_0, y_1, y_2, t_1))\}$
FORALL
(real − number t_0 t_1)

PHASE II : Wu's method

$$\mathbf{Rem}_{10} = -a_0^2(4y_2^2 + 4b_2t_1y_2 - 8\gamma y_2 - 4a_1t_0y_1 -4a_0t_0y_0 - 4b_2t_1x_2 + 4x_1^2 + 4a_1t_0x_1 -8\beta x_1 + 4x_0^2 + 4a_0t_0x_0 - 8\alpha x_0 -p^2 + 4\gamma^2 + 4\beta^2 + 4\alpha^2)$$

with Conc = $\mathbf{hyp}_{11}$
$x_1 = a_2, x_2 = a_1, x_3 = b_1, x_4 = b_0, x_5 = l_0,$
$x_6 = k_2, x_7 = l_1, x_9 = k_1, x_{10} = l_2, x_{11} k_0$

PHASE III : Algebraic reasoning

Applied − rule_1 : $\mathbf{A}t + \mathbf{B}s + \mathbf{C} \equiv 0 \ (\forall\ s\ t\ \in\ \Re)$
$\Longrightarrow \mathbf{A} = \mathbf{B} = \mathbf{C} \equiv 0$
where $t = t_0$ $s = t_1$
(*FORALL − condition*) *in* Rem_{10}

⇓

Result$_1$: $4\gamma^2 - 8y_2\gamma + 4y_2^2 + 4x_1^2 + 4x_0^2 - 8\alpha x_0 -p^2 + 4\beta^2 + 4\alpha^2 - 8\beta x_1 \equiv 0$
Result$_2$: $4a_1(x_1 - y_1) + 4a_0(x_0 - y_0) \equiv 0$
Result$_3$: $4b_2y_2 - 4b_2x_2 \equiv 0$

Applied − rule_2 : *Additional reasoning*
using SYMMETRIC
where $XX' \Longleftrightarrow YY'$

⇓

Result$_4$: $4\gamma^2 - 8x_2\gamma + 4x_2^2 + 4y_1^2 + 4y_0^2 - 8\alpha y_0 -p^2 + 4\beta^2 + 4\alpha^2 - 8\beta y_1 \equiv 0$
Result$_5$: $4b_1(y_1 - x_1) + 4b_0(y_0 - x_0) \equiv 0$
Result$_6$: $4a_2x_2 - 4a_2y_2 \equiv 0$

Applied − rule_3 : **Judging perpendicular** $\vec{V_1} \cdot \vec{V_2} = 0$
$\Longleftrightarrow \vec{V_1} \perp \vec{V_2}$

⇓

Result$_7$: $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \perp \begin{pmatrix} x_0 - y_0 \\ x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}$

Result$_8$: $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \perp \begin{pmatrix} x_0 - y_0 \\ x_1 - y_1 \\ x_2 - y_2 \end{pmatrix}$

PHASE IV : Translation into symbolic representations

```
New - Desc1 :  (ON $M (AND
                     (SPHERE   $X   p/2)
                     (SPHERE   $Y   p/2))
New - Desc2 :  (ON $X $XX')
New - Desc3 :  (ON $Y $YY')
New - Desc4 :  (IS - PERPENDICULAR
                     (! - $X $Y)  $XX')
New - Desc5 :  (IS - PERPENDICULAR
                     (! - $X $Y)  $YY')
```

Fig.8 Solution to locus problem (2)