Allowed Arguments

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Abstract

Starling with a first-order modal conditional logic which allows unlimited nesting of defaults and emheddings into any context analyzable in possible-worlds theory I introduce two simple notions of default reasoning the syntactic notion of (pnorniied) Allowed Consequence and the semantic notion of (prioritized) Allowed Entailment I prove that the one is sound and complete relative to the other

1 Introduction

The better known accounts of defeasible reasoning are somewhat limited in the expressive power of their object languages This is apparent for example when it comes to formalizing Razs [1990] *prima facie* reasons to act which underlie the planning and justification *of* acts These suggest an analysis in a formalism for expressing defeasible knowledge such as Reiters [1980J Default Logic Lhat *p* is a *prima facie* reason to do *a* can be thought of as a normal' default rule with the reason *p* as its prerequisite and as its consequent a sentence expressing that *a* ought to be done Such an analysis runs into a difficulty though In some cases not discussed by Raz the very fact that there is a reason to act is itself in turn a reason lo act Thus Jones lateness is a reason for his boss lo fire him and this very fact is in turn a reason for Jones lo act it is a reason for him lo arrive on time in the future (assuming of course he doesn't want lo be fired) This second claim cannot be represented in way I ve suggested though since a default rule which represented it would have as us prerequisite a default rule expressing that Jones lateness is a *prima facie* reason for him lo be fired Such *embedded* defaults are not within the scope of Reiter s formalism since its default rules are not a part ol Us object language They are metalinguistic rules of i nference

McCarthy's [1980] Circumscription can in principle be used to represent embedded defaults, since it represents default in the object language But in order to represent the notion that an act *ought to* be done which *prima facie* reasons seem to require this formalism will have lo be generalized from classical logic lo some more expressive underlying language

It is unlikely that Default Logic can be extended lo cover embedded defaults Circumscription on the other hand can certainly be extended lo more expressive languages (Thomason [1990] extends it to Montagues Intensional Logic) But you wonder whether it is not better to start off with a more expressive language formalizing defaults as *ceteris paribus* conditionals in modal logic since modal conditional logic allows unlimited emheddings of conditionals also prepositional attitudes deontic contexts counterfactuals and other constructions are already quite well understood in that setting

This idea *was* first persued by Dclgmnde I198KJ Others including Asher and Morreau [1991] and Bouuher [19921 laler took it up bui their results so far have been disappoinling Delgrande had a difficulty which Circumscription and Default Logic do not face wilh what he called 'irrelevant information ' Asher & Morreau [1991] solved this particular problem but their nonmonotonic consequence notion Commonsensc Entailment is conceptually exotic and technically formidable Bounlier [1992] treated conditionals expressing defaults as necessarily true if true at all this squanders the resources of modal conditional logic for representing nested defaults and emheddings of defaults inio belief contexts and counterfactuals Finally these approaches have not led lo attractive syntactic characterizations of nonmonolonic reasoning Not even characterizations of the sound-though-incomplete kind which McCarthy [1980] offers

In Uns paper 1 will present an account of nonmonotonic reasoning based in modal conditional logic which does not have these shortcomings The goal is the same as dial of Asher and Morreau [1991] lo provide an underpinning for analyses requiring the resources of possible-worlds theory In regard to the inferences it allows the notion of nonmonolonic consequence which I will present here is very similar *lo* their *Common sense* Entailment (though section 7 shows how it allows defeasible inferences including *strengthening of conditional antecedents* and the *hypothetical syllogism* whjch have as far as I know nol

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been dealt with before) One important advantage of the present account over Commonsense Entailment is that it is technically and conceptually simpler For this reason it provides a better foundation for analyses which depend on Commonsense Entailment, such as Lascandes and Asher's [1993] treatment of rhetorical structure in natural language

I begin in the next section with a weak first-order modal conditional logic with which to represent defaults ' In section 3 I present a syntactic notion of nonmonotonic consequence *Allowed Consequence* and analyze several familiar patterns of nonmonotonic reasoning In section 4 1 present *Allowed Entailment* which is an analog, in possible-worlds theory of McCarthys [1980] Minimal Entailment In sections 5 and 6 I generalize these to allow a priority order on defaults and show that (prioritized) Allowed Consequence is sound and complete relative lo (prioritized) Allowed Entailment

2 Default Conditionals

Here is a formal language for expressing defaults Let L be a countable language of first-order logic with no function symbols Add a binary conditional operator $>$ The result is $L_>$, where φ and ψ are formulas of $L_>$, so is $\varphi>\psi$ Familiar monadic modal operators expressing belief (B) knowledge (K) and obligation (O) can be added loo where (p is a formula of $L_>$ so are $B\varphi$ $K\varphi$ and $O\varphi$

Where φ and ψ are sentences $\varphi>\psi$ is intended to express thai normally ψ if φ Letting j be an individual tonslanl standing for Jones for example letting Dyz express that y is a day in month z letting Lxy express that x is late tor work on day y and letting Fx express that x is fired $\forall y(D(y\ december)\rightarrow L(j,y)) > OFj$ is intended lo express that other things being equal Jones ought to be fired if he is late every day in december This formalizes ihe notion that Jones' being lale all december is *a prima facie* reason lor him lo be fired

Where φ and ψ are predicalcs or relations Vx((p>y) is intended lo express thai if something is (p then normally u is a ψ For example, letting Hxy express that y is an event in which x is thoroughly heated and Idling Fxy express lhal y is an event in which x will catch fire the monadic predicate $\forall y(Hxy>Fxy)$ expresses thai x is *flammable* it is the sort of Ihing whiLh in ihe event thai n is thoroughly heated will normally catch fire

There is something implausible about understanding flammabihty as a universally quantified conditional lhal a piece of wood is flammable wrongly entails that it normally will calch fire in the event thai it is thoroughly healed while say immersed in waler (Imagine boiling H for hours on end) There are alternative ways ol understanding flammabihty that a given piece of wood w is flammable clearly has something to do with cvenis in which w is thoroughly heated But inslcad of supposing ihis claim lo quantify over events of this kind u could, say be taken to express that two properties — the property of being an event in which w is heated and lhal of being an event in which w catches fire — stand in an appropriate relation to one another The generic *dogs bark* would on this view express thai the properly of being a dog and being a barking thing stand in this same relation, and so on

Such a relauonal accounl of defaults can be developed on the basis of a suitable theory of properties And once developed Il will be possible lo extend to it the accounl of nonmonotonic reasoning of sections 3 and 4 But that is lor another time

Unlimited nesling of > allowed Letting Ax express that x attempts to arrive at work on time (more of the resources of modal logic could be used lo analyze A further but that would be a distractions here) the following nested sentence expresses lhal the fact that Jones' lateness is a reason lo fire him is itself a *prima facie* reason for him lo try to arrive on time

$$(\forall y(D(y\ december)\rightarrow L(j\ y)) > OFj) > OAj$$

For another example of nesting letting Rx express that x ought lto he kept well away from radiators we can write $\forall x(\forall y(Hxy>Fxy) > Rx)$ lo express that flammable things normally ought to be kept well away from radiators

What is the meaning of > and what is us logic' As Delgrande Asher & Morreau and Boutilier all point out in keeping with the intended interpretation of $\varphi>\psi$ there is a standard axiom scheme of conditional logic which is nol valid *modus ponens* or $\varphi>\psi \rightarrow \varphi\rightarrow\psi$ It can happen lhal normally ψ if φ lhal φ holds and yet that ψ does nol hold These authors helped themselves lo interpretations of conditional logic within possible worlds semantics as pioneered by Stalnaker and Thomason [1970] and l^ewis 11971] making suitable adjustments so as to keep *modus ponens* from being valid I will do the same

I begin by interprcling $L_>$ in possible-worlds models (which are presented in Chellas 11980]) The basic notion is thai of a *possible worlds frame* which is a triple $<\mathcal{W},\Delta\ *>$ consisting of a nonempty sel #'of possible worlds a sel A of individuals over which ihe quantifiers of $L_>$ range and a *worlds selection function* $*\ \mathcal{W}\times\wp(\mathcal{W})\rightarrow\wp(\mathcal{W})$ This latler component which is used to inlerpret > maps each world and proposition onto a proposition {*Proposition* is the technical term for a subset of Hl a proposition *p holds* in a possible world o'jusl in case $w\in p$) Thus **(wp)* is a sel of worlds informally speaking they are the worlds where things are as they normally are il pholds Jusl which set this is dep Js on ii/,sincc how things normally arc it pcan vary fron me to time and place to place Additional structure can added to modal frames in order lo interpret B K O I other modal constructions like counterlactuals what is i eded is quite well known however and I will not discus it here See Cheilas 1980] for details

In keeping with its intended purpose * is subject lo a very basic constraint of conditional logic

FACTIC1TY A frame $<\mathcal{W},\Delta\ *>$ satisfies facticity if

for all $w\in\mathcal{W}$ and $p\subseteq\mathcal{W}$ $*(w\ p)\subseteq p$

Another familiar constraint is however quile inappropriate in this connection Say lhal $w\in\mathcal{W}$ is *(entered* in a frame $<\mathcal{W}\ *>$ if for all $p\subseteq\mathcal{W}$ if $w\in p$ then $w\in *(w\ p)$ Then we have

1 H is neither new nor is it unique to this analysis of non monotonic reasoning Delgrande Ashcr & Morreau and Boutihcr all rely on some such underlying conditional logic

CENTERING A frame $<\mathcal{W},\Delta,*>$ is centered if each of its possible worlds is centered

We cannot require that all frames be centered since that w is a p world in no way guarantees that w is a world where everything is as things normally are if p Along with Delgrande Asher and Morreau and Boutilier I therefore do not require that modal frames satisfy this constraint The following examples of frames will be used in later examples

EXAMPLE 1 The following singleton structure satisfies facticity and is centered

where $*(u,\{\}) = \{\}$ $*(u,\{u\}) = \{u\}$

EXAMPLE 2 The following structure satisfies facticity and is not centered

where for each $x \in \{u,v\}$ $*(x\ \{\}) = \{\}$ $*(x\ \{u\}) = \{u\}$ $*(x\ \{u,v\}) = *(x\ \{v\}) = \{v\}$

Together with any given domain of quantification each of these structures forms a modal frame

A modal frame becomes a model $<\mathcal{W}\Delta\ *, I>$ for $L_>$ on *adding an interpretation function* I which in any given possible world assigns elements of Δ to individual constants and which assigns appropriate intensions to the atomic predicates of $L_>$ The conditions of truth of formulas of $L_>$ in possible worlds $w \in \mathcal{W}$ of a model $\mathcal{M} = <\mathcal{W}\Delta\ *\ I>$ are stated relative to assignments a of $L_>$-variables to Δ The clauses of the truth definition which deal with atomic formulas sentential connectives apart from > and the quantifiers are all quite familiar I have not included them here The clause for > is the standard one in modal conditional logic

$\mathcal{M}\ w \models \varphi > \psi\ [a]$ if and only if $*(w\,[\![\varphi]\!]_a) \subseteq [\![\psi]\!]_a$

In this expression $[\![\varphi]\!]_a$ the *proposition expressed by* φ *in* $\mathcal{M}$ *relative to* a, is just $\{v \in \mathcal{W}\ \ \mathcal{M}\ v \models \varphi\ [a]\}$ What is required for a sentence $\varphi>\psi$ to be true then is that ψ holds in any world where things are as they normally are if φ

EXAMPLE 3 Choosing $\Delta = \{e\ w\}$ let $\mathcal{M}$ be obtained by adding to the singleton structure of EXAMPLE 1 any interpretation function I such that $I_u(e) = e$ $I_u(w) = w$ and $I_u(H) = I_u(F) = \{(w\ e)\}$ It is straightforward to verify that $\mathcal{M}\ u \models \forall y(Hwy{>}Fwy)$

OBSERVATION 4 Let u be centered in $\mathcal{M}$ (as it is in the previous example) Then for any φ and ψ $\mathcal{M}\ u \models \varphi>\psi \rightarrow \varphi\rightarrow\psi$ This follows directly from the truth-conditions for >

These models generate a notion of logical entailment in the standard way Where Γ is a set of formulas $\Gamma \models \varphi$ means that for *any model* $\mathcal{M}$ *for any possible world* w of $\mathcal{M}$ *and* for any appropriate variable assignment a, if $\mathcal{M}\ w \models \Gamma[a]$ then $\mathcal{M}, w \models \varphi[a]$ The following example concerns a model showing that *modus ponens* is not valid That is to say $\not\models \varphi>\psi \rightarrow \varphi\rightarrow\psi$

EXAMPLE 5 Choosing $\Delta = \{e,w\}$, let $\mathcal{N}$ be obtained by adding to the structure of EXAMPLE 2 any interpretation function I such that $I_u(e) = I_v(e) = e$ $I_u(w) = I_v(w) = w$ $I_u(F) = \{\}$ and $I_u(H) = I_v(H) = I_v(F) = \{(w,e)\}$ It is straightforward to verify that $\mathcal{N}, u \models \forall y(Hwy{>}Fwy)$, that $\mathcal{N}, u \models Hwe$ and that $\mathcal{N}, u \models \neg Fwe$

Logical entailment can be given a sound and complete syntactic characterization I start with a notion of derivability without premises Let $\vdash \varphi$ mean that φ can be derived using the following axioms and rules

A1 All truth-functional tautologies of $L_>$

A2 $\forall x\varphi \rightarrow (t/x)\varphi$ where t is any individual term of $L_>$

A3 $\forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$

A4 $\forall x(\varphi\rightarrow\psi) \rightarrow (\exists x\varphi\rightarrow\psi)$
where x is not a free variable of ψ

A5 $\forall x(\varphi>\psi) \rightarrow (\varphi>\forall x\psi)$
where x is not a free variable of φ

A6 $\forall x(\varphi>\varphi)$

R1 If $\vdash\varphi$ and $\vdash \varphi\rightarrow\psi$ then $\vdash\psi$

R2 If $\vdash (\psi_1\wedge\psi_2\ \ \wedge\psi_n)\rightarrow\psi$
then $\vdash(\varphi>\psi_1 \wedge \varphi>\psi_2\ \ \wedge \varphi>\psi_n)\rightarrow\varphi>\psi$

R3 If $\vdash \varphi\rightarrow(c/x)\psi$ then $\vdash \varphi\rightarrow\forall x\psi$
provided the constant c occurs neither in φ nor in ψ

R4 If $\vdash \varphi$ then $\vdash (t/x)\varphi$
provided the individual term t does not occur in φ

R5 If $\vdash \varphi\leftrightarrow\psi$ then $\vdash(\varphi/\psi)\chi \leftrightarrow \chi$
where $(\varphi/\psi)\chi$ is the result of replacing within χ an occurrence of ψ by an occurrence of φ

$\Gamma \vdash \varphi$, or φ *is derivable from premises* Γ can be defined *in terms of the above notion It means that for some finite* $\Gamma^* \subseteq \Gamma$, $\vdash \wedge\Gamma^*\rightarrow\varphi$ (This is not the standard definition but it has the advantage that compactness and the deduction theorem involving $\rightarrow$ are obvious) The following completeness theorem can be proven using a standard Henkin-style construction of a canonical model

THEOREM 6 $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$

Proof A variation on a result in Chellas [1980]

3 Allowed Consequence

In this section the logic of weak conditionals will be extended to a nonmonotonic syntactic consequence notion written $\vdash_\mu$ Informally speaking the idea is to add modus ponens wherever possible thus allowing the consequents of conditionals to be detached Technically this is done by adding to the premises of an argument as many instances of the axiom scheme $\varphi>\psi \rightarrow \varphi\rightarrow\psi$ as are consistent with them Let MP be the set of all instances of this scheme

DEFINITION 7 Ω *is maximal* Γ*-consistent within* MP iff (i) $\Omega \subseteq$ MP and (ii) $\Gamma\cup\Omega \nvdash \bot$, while for every Ω^* if $\Omega \subseteq \Omega^* \subseteq$ MP and $\Gamma\cup\Omega^* \nvdash \bot$, then $\Omega = \Omega^*$

Such sets play a role in the notion of allowed consequence which is analogous to that of extensions in Reiter's *Default Logic*

DEFINITION 8 $\Gamma \vdash_\mu \varphi$ iff for every maximal Γ-consistent set Ω within MP $\Gamma\cup\Omega \vdash \varphi$

Where $\Gamma \vdash_\mu \varphi$ I will say that Γ *allows* the consequence φ to be drawn The following two examples show that interesting instances of modus ponens across > are allowed First that e is an event in which w is thoroughly heated together with the fact that w is flammable allows the consequence w will catch fire to be drawn

EXAMPLE 9 Hwe $\forall y(Hwy>Fwy) \mid\!\sim_{\mu}$ Fwe

Proof Let the set of the premises Hwe $\forall y(Hwy>Fwy)$ be called Γ From EXAMPLE 3 and OBSERVATION 4 it follows that Γ is satisfiable together with all of MP so (with THEOREM 6) there is just a single maximal Γ-consistent subset of MP, and that is MP itself Now Γ MP $\vdash$ Fwe ■

The next example is similar but concerns the part of the language in which > is nested Intuitively speaking from the facts that w is flammable and that flammable things normally ought to be kept well away from radiators follows that w ought to be kept well away from radiators

EXAMPLE 10

$\forall y(Hwy>Fwy), \forall x(\forall y(Hxy>Fxy) > Rx) \mid\!\sim_{\mu}$ Rw

Proof Let the set of the premises $\forall y(Hwy>Fwy)$ $\forall x(\forall y(Hxy>Fxy) > Rx)$ be called Γ Adapt EXAMPLE 3 requiring in addition of I that $I_{\mu}(R) = \{w\}$ It is a matter of routine to show that Γ is satisfied in the resulting model From OBSERVATION 4 it again follows that Γ is satisfiable together with all of MP Now Γ MP $\vdash$ Rw ■

From the above demonstrations which require only that the premises are satisfiable in the frame of EXAMPLE 1 it will be plain that "irrelevant" additional information can be added to the premises without losing these conclusions For example the previous examples can be repeated on adding say an extra premise Pw expressing that w is a piece of pine Other analyses of such reasoning which depart from weak conditional logics such as the previously cited suggestions of Delgrande and Boutilier have had to go to great lengths to allow such irrelevant additional premises to be added

EXAMPLE 11 Hwe $\neg$Fwe,$\forall y(Hwy>Fwy) \mid\!\not\sim_{\mu}$ Fwe

Proof Let the set of the premises be called Γ EXAMPLE 5 shows that Γ is satisfiable It follows with the deduction theorem for $\rightarrow$ and the compactness of $\vdash$ that there is a maximal Γ-consistent subset — call it Ω — of MP Now Γ Ω $\not\vdash$ Fwe ■

The following example is similar to the well-worn *Nixon Diamond* which could have been analyzed in its place Let Wx express that x is wet so that $\forall x \forall y(Wx>\neg Fxy)$ expresses that wet things normally don't burn The following example shows that neither the inference that w will catch fire in e nor that w will not catch fire, is allowed from premises expressing that w is flammable that e is an event in which w is thoroughly heated that w is wet and that wet things normally do not catch fire

EXAMPLE 12 Let Γ be $\{$Hwe Ww $\forall y(Hwy>Fwy)$ $\forall x\forall y(Wx>\neg Fxy)\}$ Then (i) $\Gamma \mid\!\not\sim_{\mu}$ Fwe and

(ii) $\Gamma \mid\!\not\sim_{\mu} \neg$Fwe

Proof (i) With a model construction based on the frame of EXAMPLE 2 it is straightforward to show that Γ is satisfiable together with $(Ww>\neg Fwe) \rightarrow (Ww\rightarrow\neg Fwe)$ It follows that there is a maximal Γ-consistent $\Omega \subseteq$ MP such that $(Ww > \neg Fwe) \rightarrow (Ww \rightarrow \neg Fwe) \in \Omega$ Now Γ $\Omega \not\vdash$ Fwe

(ii) Analogous ■

4 and Allowed Entailment

In this section I will generalize the syntactic notion of Allowed Consequence of the previous section I will then define the nonmonotonic semantic notion of *Allowed Entailment* Later I will prove that Allowed Consequence is sound and complete relative to Allowed Entailment

Let $\mathcal{F}$ be any set of sentences called the reasoner's *focus of attention* (For the meantime $\mathcal{F}$ will always be some or other subset of MP) Now redefine Allowed Consequence relative to $\mathcal{F}$ by substituting $\mathcal{F}$ for MP in definition 6 The resulting notion $\mid\!\sim_{\mathcal{F}}$ of Allowed Consequence relative to a focus of attention clearly is a generalization of $\mid\!\sim_{\mu}$ Trivially choosing $\mathcal{F}$ as MP $\mid\!\sim_{\mathcal{F}}$ and $\mid\!\sim_{\mu}$ coincide To go to the other extreme choosing $\mathcal{F}$ as the empty set $\mid\!\sim_{\mathcal{F}}$ reduces to logical consequence Intuitively speaking the focus of attention is a parameter in nonmonotonic reasoning which reflects some of the reasoner's relative concerns to include a conditional $\varphi>\psi \rightarrow \varphi\rightarrow\psi$ in $\mathcal{F}$ is in a sense to let it matter that normally ψ if φ It is a crude way of dividing defaults into two priority classes defaults appearing in $\mathcal{F}$ are alike in having top priority defaults excluded from $\mathcal{F}$ are alike in having no effect whatsoever on the nonmonotonic consequences of a theory This part played by the parameter of focus will be greatly extended in the next section where it will be filled by a vector of sets of defaults The following notion is central in defining a preference relation on possible worlds which can be used to characterize Allowed Consequence

DEFINITION 14 $\mathcal{L}\, v \leq_{\mathcal{F}} \mathcal{M}\, w$ means

$Th(\mathcal{L}\, v) \cap \mathcal{F} \subseteq Th(\mathcal{M}\, w)$

Here $Th(\mathcal{M}\, w)$ is the *theory* $\{\varphi \;\; \mathcal{M}\, w \models \varphi\}$ of w in $\mathcal{M}$ One world lies underneath another in the sense of this relation if it gets more of $\mathcal{F}$ right" With $\mathcal{F}$ chosen within MP though more can be said about $\leq_{\mathcal{F}}$ Consider any given sentence $\varphi>\psi \rightarrow \varphi\rightarrow\psi$ in $\mathcal{F}$ We can say that w is *irregular in regard to* $\varphi>\psi$ *in* $\mathcal{M}$ if $\mathcal{M}\, w \models \varphi$ and $\mathcal{M}\, w \models \varphi>\psi$ but $\mathcal{M}\, w \not\models \psi$ Equivalently of course w is irregular in regard to $\varphi>\psi$ iff $\mathcal{M}\, w \not\models \varphi>\psi \rightarrow \varphi\rightarrow\psi$ So $\mathcal{L}\, v \leq_{\mathcal{F}} \mathcal{M}\, w$ just means that v is no more irregular than is w not at least as far as the defaults we are focusing on are concerned [2]

[2] Notice that I allow possible worlds *of different* models L and M to be compared by $\leq_{\mathcal{F}}$ not just possible worlds of one and the same model This has raised the eyebrows of some who are used to seeing worlds compared — say in regard to their similarity to a third world — but only within models In standard developments of modal conditional logic such as Lewis [1971] such similarity relations are of course what make conditionals true or false as the case may be I don't know why my comparing the worlds of different models is thought odd maybe people confuse the *metalinguistic* task of defining a nonmonotonic consequence notion which is what concerns me here with the task of giving the truth conditions of an object language conditional Indeed one of the differences between defaults and other conditionals on the one hand and consequence notions on the other namely that whereas the former can be embedded the latter cannot has been hid from view by the

This relation will be used to define a semantic notion of nonmonotonic consequence analogous to John McCarthy's [1980] notion of Minimal Entailment First there is the notion of a set of premises Γ being minimally satisfied, relative to $\mathcal{F}$ at a possible world w of a model $\mathcal{M}$

DEFINITION 15 $\mathcal{M}, w \models_{\mathcal{F}} \Gamma$ just in case

(i) $\mathcal{M}\ w \models \Gamma$ and

(ii) if $\mathcal{L}, v \models \Gamma$ and $\mathcal{L}, v \leq_{\mathcal{F}} \mathcal{M}\ w$ then $\mathcal{M}\ w \leq_{\mathcal{F}} \mathcal{L}, v$

Second a sentence φ is an *Allowed Entailment* of premises Γ relative to $\mathcal{F}$ (in symbols, $\Gamma \models_{\mathcal{F}} \varphi$) if φ holds in every world where Γ is minimally satisfied relative to $\mathcal{F}$

DEFINITION 16 $\Gamma \models_{\mathcal{F}} \varphi$ just in case for every $\mathcal{M}$ and w,

if $\mathcal{M}, w \models_{\mathcal{F}} \Gamma$ *then* $\mathcal{M}\ w \models \varphi$

Now I can state the completeness theorem for Allowed Consequence relative to Allowed Entailment

COMPLETENESS THEOREM 17 $\Gamma \vdash_{\mathcal{F}} \varphi \Leftrightarrow \Gamma \models_{\mathcal{F}} \varphi$

This *theorem provides* Allowed Consequence with a semantic underpinning the consequences which it is allowed to draw from a given set of premises are just the things which you can be sure are true in all least irregular worlds where the premises are true It is a special case of the corresponding completeness of Prioritized Allowed Consequence with respect to Prioritized Allowed Entailment which I will prove in section VI so the proof will have to wait until then

One consequence of either theorem which it is interesting to note at this point is that if Γ is consistent so are its allowed consequences and entailments This follows from THEOREM 17 together with DEFINITION 7 since with the deduction theorem for $\rightarrow$ and the compactness of $\vdash$ for any consistent theory Γ there is at least one set Ω which is maximal Γ–consistent within $\mathcal{F}$ As Davis [1980] points out the corresponding thing does not hold in general for Circumscription there are examples of theories which are classically consistent but from which contradictions follow using Circumscription

5 Prioritized Allowed Consequence

Nonmonotonic reasoning needs to be sensitive to the fact lhal some defaults take precedence over others Thus to lake the threadbare example when it comes to drawing conclusions about the ability or otherwise of any given penguin to fly the default that penguins normally cannot fly takes precedence over the default that birds normally can fly Laws, interpreted following Ra7 [1990] *as prima facie* reasons to act provide a less familiar example The example is of added interest in that in the case of defaults which express *prima facie* reasons to act which denvc from laws precedence orders do not simply mirror the taxonomic order of the kinds which the defaults are about (as it does in the case of the birds where the default about the subkind penguin takes precedence over the default about the kind birds) Instead precedence orders are in the legal case often simply declared in legal codes So for example, in virtue of the law which requires this the fact that a traffic light shows red is a *prima facie* reason to slop at an intersection Additionally, however the law requires drivers to follow the directions of traffic officers, so the fact that an officer directing traffic is waving you on is a reason to pass through an intersection And supposing you are waved on through a red light? The traffic code determines explicitly that the directions of traffic officers take precedence over traffic signs which include lights so in that case you ought to ignore the light and follow the directions

In this section I will generalize the notion of Allowed Consequence to take into account priority orders between default conditionals In the next section the notion of Allowed Entailment will be similarly generalized and I will prove the completeness theorem according 10 which these two notions coincide The first step is to generalize the notion of a focus of attention from a single set of defaults to a linearly ordered (finite) set of such sets or a *focal vector*

DEFINITION 18 $\mathcal{G} = <\mathcal{G}_1\ \mathcal{G}_2\ \ \mathcal{G}_k>$ is a *focal vector* just in case each $\mathcal{G}_i$ is a focus of attention in the earlier sense

The idea is that defaults in $\mathcal{G}_1$ will take precedence in conflicts with any defaults from $\mathcal{G}_2\ \mathcal{G}_3$ defaults in $\mathcal{G}_2$ will take precedence in conflicts with any defaults from $\mathcal{G}_3$, $\mathcal{G}_4$ and so on Letting $\mathcal{G} = <\mathcal{G}_1\ \mathcal{G}_2\ \ \mathcal{G}_k>$ be a focal vector and $\mathcal{H}$ be a set of sentences $\mathcal{G}|\mathcal{H}$ is a useful notation for the focal vector $<\mathcal{G}_1\ \mathcal{G}_2\ \ \mathcal{G}_k\ \mathcal{H}>$ Using this notation definitions and proofs can proceed by induction on the length of focal vectors For a trivial start $\mathcal{G}$ can be flattened into a set $\cup\mathcal{G}$ of sentences as follows $\cup<> = \{\}$ and $\cup\mathcal{G}|\mathcal{H} = \cup\mathcal{G} \cup \mathcal{H}$ The following notion generalizes to the case of focal vectors the earlier notion of maximal Γ-consistency within $\mathcal{F}$

DEFINITION 19 Ω *is maximal* Γ *consistent within* $\mathcal{F}$ is defined by induction on the length of $\mathcal{F}$

Base step $\{\}$ is maximal Γ–consistent within $<>$

Induction step Ω is maximal Γ consistent within $\mathcal{G}|\mathcal{H}$ if (i) $\Omega \subseteq \cup\mathcal{G}|\mathcal{H}$

(ii) $\Omega \cap \cup\mathcal{G}$ is maximal Γ–consistent within $\mathcal{G}$ and

(iii) $\Gamma \cup \Omega \nvdash \bot$ And if $\Omega \subseteq \Omega^* \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\Gamma \cup \Omega^* \nvdash \bot$ then $\Omega = \Omega^*$

OBSERVATION 20 Let $\mathcal{M}, w \models \Gamma \cup \Omega$, where Ω is maximal Γ–consistent within $\mathcal{F}$ Then $Th(\mathcal{M}\ w) \cap \cup\mathcal{F} = \Omega$ This follows directly from (iii) in the induction step of DEFINITION 19

EXAMPLE 21 Let it be agreed that in reasoning about the properties of different kinds of birds precedence orders on defaults must parallel the taxonomic order of the kinds which they are about Thus where K_1 is a subkind of K_2 any defaults about members of K_1 which are in focus must have higher priority than any defaults about members of K_2 which are in focus And let '$\varphi > \psi \rightarrow \varphi \rightarrow \psi$ be abbreviated $\varphi \mathrel{>\!\!\!\rightarrow} \psi$ Finally let an open formula in a focal vector stand for the set of its ground instantiations Then for example the focal vector

$<PENGUIN(x) \mathrel{>\!\!\!\rightarrow} \neg FLIES(x)\ \ BIRD(x) \mathrel{>\!\!\!\rightarrow} FLIES(x)>$

will be acceptable, but

$<BIRD(x) \mathrel{>\!\!\!\rightarrow} FLIES(x), PENGUIN(x) \mathrel{>\!\!\!\rightarrow} \neg FLIES(x)>$

practice of concentraung on unncsted defaults For more on the differences between consequence notions and conditionals see for example Quine [1950] will not Call the first of these vectors $\mathcal{I}$ And let our premises Γ include just

$$\forall x(\text{BIRD}(x) > \text{FLIES}(x))$$
$$\forall x(\text{PENGUIN}(x) > \neg\text{FLIES}(x))$$
$$\text{BIRD}(\text{tweety})\ \ \text{PENGUIN}(\text{tweety})$$

Using the frame of EXAMPLE 2 it is not hard to show that Γ is satisfiable along with ∀x(PENGUIN(x) → ¬FLIES(x)) So Γ is satisfiable along with all of $\mathcal{F}_1$ which is just the set of all sentences of the form PENGUIN(c) $>/_{\to}$ ¬FLIES(c) So there is a unique maximal Γ-consistent set within $<\mathcal{F}_1>$, and that is $\mathcal{F}_1$ itself $\mathcal{F}_2$ contains only sentences of the form BIRD(c) $>/_{\to}$ FLIES(c), none of which are consistent together with $\Gamma\cup\mathcal{F}_1$ It follows from DEFINITION 19 that $\mathcal{F}_1$ is the unique maximal Γ-consistent set within $\mathcal{I}$

DEFINITION 22 *(Prioritized Allowed Consequence)* $\Gamma \mid\!\!\sim_{\mathcal{I}} \varphi$ iff for every Ω which is maximal Γ-consistent within $\mathcal{I}$ $\Gamma\Omega \mid\!\!\sim \varphi$

CONTINUATION OF EXAMPLE 21 With Γ and $\mathcal{I}$ as above $\Gamma \mid\!\!\sim_{\mathcal{I}}$ ¬FLIES(tweety) but (since $\Gamma\cup\mathcal{F}_1$ is consistent) $\Gamma \not\mid\!\!\sim_{\mathcal{I}}$ FLIES(tweety)

It is not hard to see that here as in the previous examples a range of "irrelevant" background premises can be added to Γ without changing this outcome Also once this example has been understood it will be clear how others involving prioritization of defaults can be analyzed such as the traffic-law case discussed informally above

In the following section I go on to provide Prioritized Allowed Consequence with its semantic counterpart *Prioritized Allowed Entailment* All that needs to be done is to generalize DEFINITION 14 to an order $\leq_{\mathcal{I}}$ on possible worlds relative to a focal vector $\mathcal{I}$ Substitution of $\mathcal{I}$ for $\mathcal{F}$ in the earlier definitions then results in the right notions I also prove that Prioritized Allowed Consequence is sound and complete with respect to Prioritized Allowed Entailment

6 and Prioritized Allowed Entailment

Intuitively it is clear what needs to be done in order to define $\leq_{\mathcal{I}}$, for any given focal vector $\mathcal{I} = <\mathcal{F}_1\ \mathcal{F}_2\ \ \mathcal{F}_k>$ It is necessary to combine the different orders $\leq_{\mathcal{F}_1}$ $\leq_{\mathcal{F}_2}$ $\leq_{\mathcal{F}_k}$ appropriately allowing $\leq_{\mathcal{F}_1}$ to order worlds first then allowing $\leq_{\mathcal{F}_2}$ to further discriminate between worlds which as far as $\leq_{\mathcal{F}_1}$ is concerned are not discriminable and so on Such combinations of partial orders are the subject of the following definition in which v is a finite vector of partial linear orders each defined on the same domain $\mathcal{D}$

DEFINITION 23 The *alphabetic composition* $\leq_v$ *of* v is defined by induction on the length of v

base step $\leq_{<>} = \mathcal{D}^2$

induction step $\leq_{v|\leq} = \{(d_1\ d_2)\ \ d_1 \leq_v d_2$ and if $d_1 =_v d_2$ then $d_1 \leq d_2\}$

A straightforward induction on the length of v shows that $\leq_v$ is a partial linear order on $\mathcal{D}$ The familiar alphabetic order of words in a dictionary can be construed as such an alphabetic composition

OBSERVATION 24 Let $v = <\leq_1 \leq_2\ \leq_k>$ For all $d_1\ d_2 \in \mathcal{D}$ $d_1 \approx_v d_2$ just in case for each $i \leq k$ $d_1 \approx_i d_2$

This observation follows with a simple induction on the length of v It and the following observations will be needed later

OBSERVATION 25 If $(\mathit{Th}(\mathcal{M},w) \cap \cup\mathcal{I}) \subseteq \mathit{Th}(\mathcal{L},v)$ then $\mathcal{L}\ v \leq_{\mathcal{I}} \mathcal{M}\ w$ This can be seen with a simple induction on the length of $\mathcal{I}$ The base clause is trivial $\leq_{<>}$ is the universal relation on possible worlds of different models For the inductive step if $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\ \mathcal{G}|\mathcal{H}) \subseteq \mathit{Th}(\mathcal{L}\ v)$ then $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\mathcal{G}) \subseteq \mathit{Th}(\mathcal{L},v)$ and $(\mathit{Th}(\mathcal{M}\ w) \cap \mathcal{H}) \subseteq \mathit{Th}(\mathcal{L}\ v)$ By induction hypothesis $\mathcal{L}\ v \leq_{\mathcal{G}} \mathcal{M}\ w$ By definition 14 also $\mathcal{L}\ v \leq_{\mathcal{H}} \mathcal{M}\ w$ so $\mathcal{L}\ v \leq_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$

OBSERVATION 26 $\mathcal{L}\ v =_{\mathcal{I}} \mathcal{M}\ w$ if and only if $(\mathit{Th}(\mathcal{L},v) \cap \cup\mathcal{I}) = (\mathit{Th}(\mathcal{M},w) \cap \cup\mathcal{I})$

This is a direct consequence of DEFINITION 14 OBSERVATION 24 and OBSERVATION 25

Now the following notions of minimal satisfaction and Prioritized Allowed Entailment are completely analogous to the earlier definitions relative to a simple focus

DEFINITION 27 Let $\mathcal{I}$ be a focal vector Then

(a) $\mathcal{M},w \models_{\mathcal{I}} \Gamma$ means

(i) $\mathcal{M}\ w \models \Gamma$ and

(ii) for all $\mathcal{L},v$ if $\mathcal{L},v \models \Gamma$ and $\mathcal{L}\ v \leq_{\mathcal{I}} \mathcal{M}\ w$ then $\mathcal{L}\ v =_{\mathcal{I}} \mathcal{M},w$

(b) $\Gamma \models_{\mathcal{I}} \varphi$ means that for every $\mathcal{M}\ w \models_{\mathcal{I}} \Gamma$ $\mathcal{M}\ w \models \varphi$

Now we have

THEOREM 28 $\Gamma \mid\!\!\sim_{\mathcal{I}} \varphi$ just in case $\Gamma \models_{\mathcal{I}} \varphi$

Putting the focal vector as $<\mathcal{F}>$ THEOREM 17 is a special case of this theorem THEOREM 28 follows immediately from the following lemma which characterizes the possible worlds in which any given set of premises is minimally satisfied

LEMMA 29 $\mathcal{M}\ w \models_{\mathcal{I}} \Gamma$ just in case for some Ω which is Γ-consistent within $\mathcal{I}$, $\mathcal{M},w \models \Gamma\ \Omega$

The proof of this lemma makes use of the following

FACT 30 $\mathcal{M}\ w \models_{\mathcal{G}|\mathcal{H}} \Gamma$ just in case

(i) $\mathcal{M},w \models_{\mathcal{G}} \Gamma$ and

(ii) for any Ω^* if $(\mathit{Th}(\mathcal{M},w) \cap \cup\mathcal{G}|\mathcal{H}) \subseteq \Omega^* \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\Gamma\cup\Omega^* \not\models \bot$ then $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\mathcal{G}|\mathcal{H}) = \Omega^*$

Proof of **FACT 30** ⇒ Suppose $\mathcal{M}\ w \models_{\mathcal{G}|\mathcal{H}} \Gamma$

Then (i) By definition of $\models_{\mathcal{G}|\mathcal{H}}$ $\mathcal{M}\ w \models \Gamma$ Now let $\mathcal{L}\ v \models \Gamma$ and $\mathcal{L},v \leq_{\mathcal{G}} \mathcal{M}\ w$ and suppose for the contradiction that not $\mathcal{L}\ v =_{\mathcal{G}} \mathcal{M}\ w$ Then $\mathcal{L},v \leq_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$ so $\mathcal{L}\ v =_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$ From this follows by OBSERVATION 24 that $\mathcal{L}\ v =_{\mathcal{G}} \mathcal{M}\ w$

Also (ii) Suppose $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\mathcal{G}|\mathcal{H}) \subseteq \Omega^* \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\mathcal{L}\ v \models \Gamma\cup\Omega^*$ Then $\Omega^* \subseteq (\mathit{Th}(\mathcal{L}\ v) \cap \cup\ \mathcal{G}|\mathcal{H})$ so by OBSERVATION 25 $\mathcal{L},v \leq_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$ Now by assumption $\mathcal{M}\ w \models_{\mathcal{G}|\mathcal{H}} \Gamma$ Since $\mathcal{L}\ v \models \Gamma$ it follows that $\mathcal{L},v =_{\mathcal{G}|\mathcal{H}} \mathcal{M},w$ By OBSERVATION 26 then $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\mathcal{G}|\mathcal{H}) = (\mathit{Th}(\mathcal{L}\ v) \cap \cup\mathcal{G}|\mathcal{H})$ so $(\mathit{Th}(\mathcal{M}\ w) \cap \cup\mathcal{G}|\mathcal{H}) = \Omega^*$

$\Leftarrow$ Assume (i) and (ii) Clearly $\mathcal{M}\ w \models \Gamma$ now suppose $\mathcal{L},v \models \Gamma$ and $\mathcal{L},v \leq_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$ Then (by definition of $\leq_{\mathcal{G}|\mathcal{H}}$) $\mathcal{L},v \leq_{\mathcal{G}} \mathcal{M},w$ so (by i) $\mathcal{L}\ v =_{\mathcal{G}} \mathcal{M},w$ By OBSERVATION 26 $(Th(\mathcal{M},w) \cap \cup\mathcal{G}) \subseteq Th(\mathcal{L}\ v)$ and (by definition of $\leq_{\mathcal{G}|\mathcal{H}}$) $\mathcal{L},v \leq_{\mathcal{H}} \mathcal{M}\ w$ so (by DEFINITION 14) also $(Th(\mathcal{M}\ w) \cap \mathcal{H}) \subseteq Th(\mathcal{L},v)$ Thus $(Th(\mathcal{M},w) \cap \cup\mathcal{G}|\mathcal{H}) \subseteq (Th(\mathcal{L},v) \cap \cup\mathcal{G}|\mathcal{H}) \subseteq \cup\mathcal{G}|\mathcal{H}$ By ii then $(Th(\mathcal{M}\ w) \cap \cup\mathcal{G}|\mathcal{H}) = (Th(\mathcal{L},v) \cap \cup\mathcal{G}|\mathcal{H})$ Now by OBSERVATION 26 $\mathcal{L},v =_{\mathcal{G}|\mathcal{H}} \mathcal{M}\ w$ This completes the proof of fact 30 ■

Proof of **LEMMA 29** The proof is by induction on the length of $\mathcal{I}$ $\mathcal{I} = \langle\rangle$ Trivial since $\leq_{\langle\rangle}$ is universal and $\{\}$ is the only maximal Γ-consistent set within $\langle\rangle$ So let $\mathcal{I} = \mathcal{G}|\mathcal{H}$ For convenience write Σ instead of $Th(\mathcal{M},w) \cap \cup\mathcal{G}|\mathcal{H}$ (So very trivially $\Sigma \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\Sigma \not\models \bot$) We have the following equivalencies

$\mathcal{M},w \models_{\mathcal{I}} \Gamma$ $\Leftrightarrow$ FACT 30

$\mathcal{M},w \models_{\mathcal{G}} \Gamma$, and

if $\Sigma \subseteq \Omega^* \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\Gamma \cup \Omega^* \not\models \bot$, then $\Sigma = \Omega^*$

$\Leftrightarrow$ INDUCTION HYPOTHESIS OBSERVATION 20

$\Sigma \cap \cup\mathcal{G}$ is maximal Γ-consistent within $\mathcal{G}$, and

if $\Sigma \subseteq \Omega^* \subseteq \cup\mathcal{G}|\mathcal{H}$ and $\Gamma \cup \Omega^* \not\models \bot$, then $\Sigma = \Omega^*$

$\Leftrightarrow$ DEFINITION 19

Σ is maximal Γ-consistent within $\mathcal{G}|\mathcal{H}$

$\Leftrightarrow$ OBSERVATION 20

For some Ω which is maximal Γ-consistent within $\mathcal{I}$ $\mathcal{M},w \models \Gamma, \Omega$ This completes the proof of lemma 29 ■

7 Strengthening Conditional Antecedents

Allowed Consequence and Entailment solve a problem aboul reasoning with conditionals which I will now very briefly sketch

If Shoemaker Levy 9 had hit Earth then there would have been an explosion equivalent 10 at least 500 megatons ol TNT It seems reasonable to infer that there would have been an explosion of this magnitude if Shoemaker-Levy 9 had hit Earth right after the world-cup final but the logic of conditionals does not support this inference Lewis [1973] argues convincingly that the logic of counterfactuals cannot validate the strengthening of conditional antecedents

There is also the hypothetical syllogism or *transitivity* If Shoemaker Levy 9 had hit Earth then there would have been a 500 megaton explosion And if there had been a 500 megaton explosion then life as we know n would have come to an end So, you might ihink if Shoemaker-Levy 9 had hit Earth then life as we know it would have come to an end But again the logic of counlerfactual conditionals offers no support It cannot do since as can easily be shown under minimal assumptions validity of transitivity entails thai counterfactual antecedents can always be strengthened

Let TR be the axiom scheme $(\varphi{>}\psi \wedge \psi{>}\chi) \rightarrow \varphi{>}\chi$ The logic of counterfactuals cannot be allowed to validate TR, but the notion of allowed consequence suggests another way to account for the plausibility of inferences like the examples just mentioned Just as in this paper the logic of defeasible conditionals was augmented by adding to any given premises as many instances of MP as one consistently can the logic of counterfactual or for that matter any other conditionals can be augmented by adding instances of TR This is achieved by including instances of TR in E Then the defeasible notion of allowed consequence can account for reasonable examples of strengthening of the antecedent and the hypothetical syllogism, while unreasonable examples such as inferring *the coffee would have lasted good if there had been sugar and diesel oil in it* from *the coffee would have tasted good if there had been sugar in it* arc 'blocked" by background knowledge Working through this and other examples in detail is something for another time

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