The Canonical Decomposition of a Weighted Belief.

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Abstract.

Any belief function can be decomposed into a confidence and a diffidence components. Each components is uniquely decomposable into simple support functions that represent the impact of the simplest form of evidence, the one that only partially supports a given subset of the frame of discernment. The nature of the inverse of Dempster's rule of combination is detailed. The confidence component translates the impact of 'good reasons to believe'. It is the component classically considered when constructing a belief. The diffidence component translates the impact of 'good reasons not to believe'.

Keywords: uncertainty, belief functions, simple support functions, Dempster's rule of combination and decombination.

1. Introduction.

In this paper, we take it for granted that the transferable belief model (TBM) is appropriate to represent quantified beliefs. Hence beliefs are quantified by belief functions and belief functions are combined by the unnormalized Dempster's rule of combination, denoted the ©-combination, when the sources of evidence that induce them are distinct (Smets, 1990, Smets and Kennes, 1994).

In section 2, we study a canonical decomposition of a belief function into elementary and distinct components such that their ©-combination restores the original belief function. In section 3 and 4, we analyze Dempster's rule of combination and define the decombination process. In section 5, we discuss the meaning of these elementary components. We present the concepts of 'absorbing belief state', of 'debt of belief, of 'latent belief structure', and of confidence and diffidence components of a belief state. In section 6 we present a mathematical generalization of this decomposition. In section 7 we compare our canonical decomposition with Shafer's original proposal. In section 8, we solve the decomposition problem for the dogmatic belief state.

Note: In the TBM, belief functions are unnormalized. It means that we do not require m(0) = 0 and bel is defined from its related basic belief assignment (bba) $m : 2^{\Omega} \rightarrow [0,1]$ by:

$$bel(A) = \sum_{B:\emptyset\neq B\subseteq A} m(B) \qquad \text{for every } A\subseteq\Omega$$

The normalization requirement $m(\emptyset) = 0$ that Shafer assumes is not essential in the development of the theory, except if one assumes that a belief function has to be viewed as the lower probability encountered in a Dempsterian model (1967). This view is irrelevant to the TBM.

2. The canonical decomposition.

The canonical decomposition is based on a generalization of the simple support functions introduced by Shafer (1976). A simple support function (SSF) on a frame of discernment Ω is a belief function which related basic belief assignment $m : 2^{\Omega} \rightarrow [0,1]$ satisfies:

$$m(X) = \begin{cases} w & \text{if } X = \Omega \\ 1-w & \text{for some } A\subseteq\Omega, \\ 0 & \text{otherwise} \end{cases}$$

where $w\in[0,1]$, and A is called the focal element.

We denote such a SSF by A^w where the index w of the focal element A is the basic belief mass (bbm) $m(\Omega)$ given to the frame of discernment Ω and the complement of w to 1 is the bbm allocated to A.

Shafer defines a belief function as separable if it can be represented by the $\oplus$-combination of SSF . For every separable belief function bel defined on Ω, one has:

$$bel = \bigoplus_{A\subseteq\Omega} A^{w_A}$$

The representation is unique if all focal elements are different and bel is **non-dogmatic**, i.e., $m(\Omega) > 0$. The case of dogmatic belief functions is analyzed in section 8.

Such a decomposition can be extended to any belief functions, non-separable and/or dogmatic, if one uses some generalization of the SSF. By abuse of language, we define a generalized simple support function on Ω (GSSF), also denoted A^w, as the function $\omega : 2^{\Omega} \rightarrow (-\infty,\infty)$ characterized by a weight $w\in(-\infty,\infty)$ and a focal element $A\subseteq\Omega$, such that:

$$\omega(X) = \begin{cases} w & \text{if } X = \Omega \\ 1-w & \text{if } X = A \\ 0 & \text{otherwise.} \end{cases}$$

When $w \in [0,1]$, A^w is a SSF. Those GSSF with $w \in (1,\infty)$ are called Inverse Simple Support Functions (ISSF) for reasons explained in section 4.

We prove that for any non-dogmatic belief function bel, one has the unique canonical decomposition:

$$\text{bel} = \bigoplus_{A \subseteq \Omega} A^{w_A} \tag{1}$$

where $w_A \in [0,\infty)$. The proof is immediate once the relation (1) is translated in the commonality function language. The commonality function $q : 2^\Omega \to [0,1]$ related to a belief function bel and its bba m is defined by:

$$q(A) = \sum_{B: A \subseteq B \subseteq \Omega} m(B) \quad \text{for every } A \subseteq \Omega$$

Theorem 1: For any non-dogmatic belief function bel : $2^\Omega \to [0,1]$, there exists a unique set of GSSF defined on Ω such that: $\text{bel} = \bigoplus_{A \subseteq \Omega} A^{w_A}$

where $w_A \in [0,\infty)$ for every $A \subseteq \Omega$ and

$$w_A = \prod_{X: A \subseteq X \subseteq \Omega} q(X)^{(-1)^{|X|-|A|+1}} \tag{2}$$

Proof: Let bel be a non-dogmatic belief function and q be its related commonality function. The relation to prove becomes:

$$q(C) = \prod_{A \subseteq \Omega} q_A(C) \quad \text{for all } C \subseteq A, \tag{3}$$

where q_A is the commonality function defined on Ω and build from such A^{w_A}, i.e., such that:

$$q_A(B) = 1 \quad \text{if } B \subseteq A$$
$$= w_A \quad \text{if } B \not\subseteq A$$

As bel is non-dogmatic, $q(\Omega) > 0$ and therefore $q(C) > 0$ for every $C \subseteq \Omega$. Then:

$$w_A = \prod_{X: A \subseteq X \subseteq \Omega} q(X)^{(-1)^{|X|-|A|+1}}$$

The *proof is identical to the one given in Shafer (1976).* As all $q(X)$ are positive, w_A is well-defined and positive. □

The unicity of the representation results from the fact the $\oplus$-combination operator is such that every A must be different (in which case Shafer' canonical decomposition is also unique). Theorem 1 generalizes Shafer'results in that it concerns every non-dogmatic belief functions wihtout inroducing the necessity of some underlying refinement of Ω (see section 7).

Example 1 : Let $\Omega = \{a, b, c\}$. Let q be a commonality function on Ω. Let $\alpha = w_{\{a\}}$, $\beta = w_{\{b\}}$, $\gamma = w_{\{c\}}$, $x = w_{\{a,b\}}$, $y = w_{\{a,c\}}$, $z = w_{\{b,c\}}$, $t = w_\emptyset$. Table 1 presents the values of q, of each q_A and of w_A for every $A \subseteq \Omega$.

The constraint of relation (3) means that q(A) is the product of the $q_X(A)$ taken over $X \subseteq \Omega$ (those on the same line in table 1). Note that $\emptyset$ is the focal element of one of the GSSF of the canonical decomposition of bel. Ω is not a focal element as it would always correspond to the vacuous belief function Ω^1, i.e., a bba with $m(\Omega) = 1$, and Ω^1 is the unit element of Dempster's rule of combination:

$$\text{bel} \oplus \Omega^1 = \text{bel} \quad \text{for every bel defined on } \Omega.$$

Example 2: Let $\Omega = \{a,b,c\}$. Let a bba $m : 2^\Omega \to [0,1]$ be such that $m(\{a,b\}) = m(\{a,c\}) = m(\{a,b,c\}) = 1/3$ then $w_{\{a,b\}} = w_{\{a,c\}} = 1/2$ and $w_{\{a\}} = 4/3$.

$A \subseteq \Omega$	q(A)	$q_\emptyset$	$q_{\{a\}}$	$q_{\{b\}}$	$q_{\{c\}}$	$q_{\{a,b\}}$	$q_{\{a,c\}}$	$q_{\{b,c\}}$
$\emptyset$	1	1	1	1	1	1	1	1
{a}	$t\beta\gamma z$	t	1	β	γ	1	1	z
{a,b}	$t\alpha\beta\gamma yz$	t	α	β	γ	1	y	z
{a,b,c}	$t\alpha\beta\gamma xyz$	t	α	β	γ	x	y	z

$A \subseteq \Omega$	w_A
$\emptyset$	$t = \frac{q(\{a\}).q(\{b\}).q(\{c\}).q(\{a,b,c\})}{q(\{a,b\}).q(\{a,c\}).q(\{b,c\})}$
{a}	$\alpha = \frac{q(\{a,b\}).q(\{a,c\})}{q(\{a\}).q(\{a,b,c\})}$
{a,b}	$x = \frac{q(\{a,b,c\})}{q(\{a,b\})}$

Table 1: Subsets of Ω, values of q expressed in function of its canonical components, and values of the weights w characterizing each elementary component.

The practical computation of w_A is obtained by applying the logarithmic transformation to the weights and the commonalities in (2) and applying the Fast Möbius Transform (Kennes, 1992) appropriately adapted.

3. The $\oplus$-combination of two belief functions.

Let bel_1 and bel_2 be two belief functions on Ω and let bel_{12} be the result of their $\oplus$-combination. Let $\{w^1_A: A \subseteq \Omega\}$ and $\{w^2_A: A \subseteq \Omega\}$ be the sets of weights of the canonical decomposition of bel_1 and bel_2:

$$\text{bel}_1 = \bigoplus_{A \subseteq \Omega} A^{w^1_A} \quad \text{and} \quad \text{bel}_2 = \bigoplus_{A \subseteq \Omega} A^{w^2_A}$$

Then:

$$\text{bel}_{12} = \text{bel}_1 \oplus \text{bel}_2 = \bigoplus_{A \subseteq \Omega} A^{w^1_A . w^2_A}$$

This results from the associativity and commutativity of the $\oplus$ operator, and the fact that $A^x \oplus A^y = A^{x.y}$, what is proved by the direct application of Dempster's rule of combination.

Note: the reason we use the weight given to Ω as an index of the GSSF resides in the last relation. A natural alternative to represent a GSSF would have been to use the weight given to the focal element as an index, but the index obtained after applying Dempster's rule of combination would not have been as simple.

The normalization of bel_{12} is achieved by multiplying every term of bel_{12} (and identically for m_{12} and q_{12}) by $k = \frac{1}{1 - m_{12}(\emptyset)}$. This is identically achieved by the $\oplus$-combination of bel_{12} with $\emptyset^k$.

4. The Θ-decombination operator.

We define the Θ-decombination operator, i.e. the inverse of the ⊕-combination operator. It is that operator that would restore bel_1 from $bel_1 \oplus bel_2$ and bel_2:

$$(bel_1 \oplus bel_2) \ominus bel_2 = bel_1 \quad (4)$$

Inparticular, $bel_1 \ominus bel_1 = \Omega^1$.

The meaning of the Θ-decombination operator is clear: it eliminates the impact of a belief that has been included in bel thogh Dempster's rule of combination. Shenoy (1994) called that operator the removal operator.

The decombination operation is well-defined for non-dogmatic belief functions as it translates into a division once commonality functions are introduced. Indeed let q_1 and q_2 be the commonality functions related to bel_1 and bel_2. The commonality function q_{12} related to $bel_1 \oplus bel_2$ satisfies $q_{12}(X) = q_1(X).q_2(X)$ for every $X \subseteq \Omega$. Therefore (4) becomes $\frac{q_{12}(X)}{q_2(X)} = q_1(X)$.

So given any two belief functions bel_1 and bel_2 on Ω and their related commonality functions q_1 and q_2, let $q_1 \ominus q_2$:

$$q_1 \ominus q_2(X) = \frac{q_1(X)}{q_2(X)} \quad \text{for all } X \subseteq \Omega.$$

We define $bel_1 \ominus bel_2$ on Ω as the pseudo belief function that is induced from $q_1 \ominus q_2$ by the same relation as the one that links belief functions and commonality functions.

Note: $bel_1 \ominus bel_2$ is not necessarily a belief function, hence the 'pseudo' qualification.

Let q_1 and q_2 be the commonality functions related to the SSF A^x and the ISSF A^y, respectively. Let $q_{12} = q_1 . q_2$ be the commonality function relates to $A^x \oplus A^y = A^{x \cdot y}$. Define q_2' as the commonality function related to $A^{1/y}$. By construction,

$$\begin{array}{llll} q_1(X) = 1 & q_2(X) = 1 & q_{12}(X) = 1 & \\ q_2'(X) = 1 & \text{if } X \subseteq A, & & \\ q_1(X) = x & q_2(X) = y & q_{12}(X) = xy & \\ q_2'(X) = 1/y & \text{if } X \not\subseteq A. & & \end{array}$$

So $q_{12} = q_1 . q_2 = q_1 / q_2'$ is the commonality function related to both $A^x \oplus A^y$ and $A^x \ominus A^{1/y}$. This development just shows that $A^x \oplus A^y = A^x \ominus A^{1/y}$. This can be generalized into:

$$bel \oplus A^y = bel \ominus A^{1/y}.$$

Let A^y be a GSSF with $y \in (1,\infty)$, then $A^{1/y}$ is a SSF as $1/y \in [0,1]$. This explains why we call A^y an ISSF.

5. The latent belief structure.

5.1. Meaning of A^w when w>l.

The idea underlying the canonical decomposition of a separable belief function bel into SSF is that the state of belief represented by bel could be understood as the result of the combination of distinct elementary states of belief, each one represented by a SSF. Each SSF characterizes an elementary state of belief in which only one proposition (the proposition denoted by the focal element) is somehow supported (somehow meaning 'with weight 1-w'). This simple interpretation collapses once bel is not separable, in which case some of the GSSF of the canonical decomposition are not SSF, but ISSF. The meaning of the decomposition becomes clearer once the concept of absorbing belief is introduced.

5.2. Absorbing Beliefs.

The SSF A^x, xE [0,1], represents a state of belief that translates the idea that "You have some reason to believe that the actual world is in A (and nothing more)" (You is the agent who holds beliefs). The 1-x is the weight corresponding to "some reasons". Suppose the other state of beliefs that would translate the idea that "You have some reason *not* to believe that the actual world is in A". This cannot be represented by a belief function over Ω. and it seems there is no way to represent it by a discounting or by a meta-belief over the set of belief functions over Ω. Suppose that You are in a situation where You have simultaneous some reason to believe A and some reason *not* to believe A. It might occur that the weights of both 'some reasons' are exactly counter-balancing each other. In that case, You end up in a state of total ignorance, hence Your belief over Ω is represented by a vacuous belief function. The first state of belief is represented by a simple support function A^x. So the second state of belief must be represented by 'something' which combination with A^x leads to a vacuous belief function. But there arc no belief functions which combination with another belief function by Dempster's rule of combination would result in a vacuous belief function. The state of belief encountered when there are some reasons *not* to believe A is called a state of absorbing belief as it is a state of belief that will absorb A^x. It looks like a state of belief where You have a 'debt of belief as the accumulation of new pieces of evidence could lead You to a classical state of belief. The representation of such a state of absorbing belief cannot be achieved by a single belief function.

Example 2, continuation. Suppose a frame Ω = [a,b,c] and Your belief state is characterized by a belief function given in example 2. We have: bel = $\{a,b\}^{1/2} \oplus \{a,c\}^{1/2} \oplus \{a\}^{4/3} = (\{a,b\}^{1/2} \oplus \{a,c\}^{1/2}) \ominus \{a\}^{3/4}$. How to interpret the term $\{a\}^{4/3}$ that is taken away by a Θ-decombination of $\{a\}^{3/4}$? The function $\{a,b\}^{1/2} \oplus \{a,b\}^{1/2}$ is a belief function which bbm are: m({a}) = m({a,b}) = m({a,c}) = m({a,b,c}) = 1/4. Hence {a} is somehow supported (degree 1/4). The impact of $\ominus\{a\}^{3/4}$ is to erase the support given to {a}. It seems that bel was the result of the combination of three pieces of evidence.

Evidence 1: believe {a,b}.
Evidence 2: believe {a,c}.
Evidence 3: do not believe {a}.

The sources of evidence 1 and evidence 2 receive a weight 1/2 and the source of evidence 3 receives a weight 3/4.

The way you treat the first piece of evidence consists in accepting to believe what the sources say (i.e., {a,b}) with a strength 1/2. So Evidence 1 induces in You the belief state represented by the SSF $\{a,b\}^{1/2}$: You believe at level 1/2 what the source says and the source says : "believe {a,b}", a shortcut for "believe that the actual state of affair is one of a or b". So You believe at level 1/2 that You should believe {a,b}, what reduces into 'You believe at 1/2 that {a,b}'.

(This reduction remembers the positive introspection described in epistemic logic). The same holds for Evidence 2.

With Evidence 3, You believe at level 3/4 what the source says and the source says "Do not believe {a}". The reduction cannot be achieved as in the previous cases as one has "You believe at 3/4 that You should not believe {a}". It only means that if You had some belief in {a}, You should delete it. It is exactly what is achieved by the $0\{a\}^{\wedge 4}$. You had a belief 1/4 given to {a} and it is removed. So the ISSF $\{a\}^{4/3}$ corresponds to "the support given to the fact that You should not believe the focal element {a}". You have "some good reason not to believe something", where the strength of good reason is equal to the belief / reliability / support You gave to the source.

Example 3: The Pravda Bias. You are in 1980, away from home, and read in a copy of an article published in a journal that the economic situation in Ukalvia is good. You do not know which journal the paper was copied from and You never heard about Ukalvia. So You had no a priori whatsoever about the economic status in Ukalvia, and now after having read the document, You might have some reasons to believe that the economic status is good. The 'some reasons[1] reflects the strength of the trust You put in the information published in a journal. Then a friend in which You have full confidence mention to You that Ukalvia is a region of the USSR and that the document was published in the Pravda. By experience, You have some reasons *not* to believe what the Pravda says when it describes the good economic status of Ukalvia; it might just be propaganda.

The reasons to believe (called the confidence) that the economic status in Ukalvia is good result from the information presented in the initial document and Your general belief about journal information. The reasons *not* to believe it (called the diffidence) result from what You know about the Pravda. If both 'reasons' counter-balance each other, You end up in a state of total ignorance about the economic status in Ukalvia.

It might be that the confidence component is stronger than the diffidence component. Then You will end up with a slight belief that the economic status in Ukalvia is good (but the belief is not as strong as if You had not heard that the journal was the Pravda and Ukalvia was in USSR).

If the diffidence component is stronger than the confidence component, then You are still in a state of 'debt of belief, in the sense that You will need further confidence component (some extra information that support that the economic status in Ukalvia is good) in order to balance the remaining diffidence component. In such a case, if You are asked to express Your opinion about the economic status in Ukalvia, You might express it under the form: 'So far, I have no reason to believe that the economic status is good, and I need some extra reasons before I start to believe it'.

5.3. Latent beliefs.

A way to represent belief states where both confidence and diffidence are involved consists in creating a structure of 'latent' beliefs and a structure of 'apparent' beliefs. A latent belief structure is represented by a pair of belief functions (X, Y) where X.YE B and B is the set of belief functions over Ω. X and Y are respectively quantifying the confidence and the diffidence component of the latent belief structure. Let Ω^1 represent the vacuous belief function. (X,Ω^1) describes a state of belief where You have only a confidence component. (Ω^1,Y) describes a pure state of absorbing belief where there is only a diffidence component. For example, the state of belief induced by "You have some reasons to believe A" is represented by (A^x,Ω^1) and the state "You have some reasons *not* to believe A" is represented by (Ω^1,A^y) (for $x,y\in[0,1]$, and where x and y are the complements of the weights corresponding to the 'some reasons').

Two latent belief structures are combined by Dempster's rule of combination (denoted by ⊕) applied to both the confidence and diffidence components.

$$(X,Y) \oplus (U,V) = (X\oplus U, Y\oplus V).$$

In particular,

$$(X,\Omega^1) \oplus (\Omega^1,X) = (X,X)$$

A latent belief structure can induce an apparent belief structure represented by an element of $\mathscr{B}$. Let Λ be the operator that transforms a latent belief structure into an apparent belief structure: $\Lambda:\mathscr{B}\mathrm{x}\mathscr{B}\rightarrow\mathscr{B}$. If there is only a non vacuous confidence component then we assume that the apparent belief structure is equal to the confidence component: $\Lambda(X,\Omega^1) = X$. We also want that if the confidence and the diffidence components are equal, they counter-balance each other and the resulting apparent belief structure is vacuous: $\Lambda(X,X) = \Omega^1$ and in particular $\Lambda(A^x,A^x) = \Omega^1$.

Consider now the equalities:

$$(X\oplus Y,X) = ((X,X)\oplus(Y,\Omega^1)) = ((\Omega^1,\Omega^1)\oplus(Y,\Omega^1)) = (Y,\Omega^1)$$

Thus that for every $X,Y,Z\in\mathscr{B}$, $(X,Y) = (X\oplus Z, Y\oplus Z)$.

Introducing the Θ-decombination operator, we can write:

If $X\ominus Y\in\mathscr{B}$, then $(X,Y) = (X\ominus Y,\Omega^1)$ and $\Lambda(X,Y) = X\ominus Y$

If $X\ominus Y\notin\mathscr{B}$, then $\Lambda(X,Y)$ is undefined.

So Λ is not defined on the whole space $\mathscr{B}\mathrm{x}\mathscr{B}$, but only on those elements (X,Y) where $X\ominus Y$ is a belief function in $\mathscr{B}$. We could have hoped that such a state of belief would not occur. Unfortunately we already encountered a counter example when we introduced the latent belief structure (Ω^1,A^x) that characterizes the case where all You know is that You have good reasons not to believe A. This means that the apparent belief structures are not rich enough to characterize every belief state. Some state of belief can only be represented by their latent belief structure.

What should be an appropriate apparent belief structure when $X\ominus Y\notin\mathscr{B}$ is not clear. What is the apparent belief structure in the case (Ω^1,A^x)? We could claim that $\Lambda(\Omega^1,A^x) = \Omega^1$, but then the apparent vacuous belief structure Ω^1 could correspond to many non equivalent latent belief structures. How to solve the general case? We could propose that $\Lambda(X,Y)$ is the belief function 'closest' from $X\ominus Y$. Unfortunately such a concept of 'closeness' is not available. The specialization concept can be used to create a partial order on the set of belief functions. Pointwise measures of the information contained in a belief function have been proposed (Pal and Bezdek, 1992), but none seems really convincing as THE appropriate measure to define 'closeness'.

Nevertheless, all hope is not lost as the problem does not appear if we start with a belief function that represents Your beliefs. It can only appear when beliefs are described directly from some latent belief structure. We show now how to build the latent belief structure from a given belief function.

5.4. The dissection of the latent beliefs.

Let $\{w_A: A\subseteq\Omega, w_A\in[0,\infty)\}$ be the set of weights associated with a given belief function bel defined on Ω, so:

$$bel = \bigoplus_{A\subseteq\Omega} A^{w_A}.$$

Create the partition of 2^Ω into the two subsets:

$$\mathcal{A}_C = \{A: A\subseteq\Omega, w_A\in[0,1]\}$$

$$\mathcal{A}_D = \{A: A\subseteq\Omega, w_A\in(1,\infty)\}.$$

The two subsets collect the SSF and the ISSF that belong to the canonical decomposition of bel, respectively (the C index is for confidence, the D for diffidence).

Then:
$$bel = \bigoplus_{A\subseteq\Omega} A^{w_A} = \bigoplus_{A\in A_C} A^{w_A} \oplus \bigoplus_{A\in A_D} A^{w_A}$$

$$= \bigoplus_{A\in A_C} A^{w_A} \ominus \bigoplus_{A\in A_D} A^{1/w_A} \quad (5)$$

The term $\bigoplus_{A\in A_C} A^{w_A}$ is a belief function, denoted bel_C, as each component is a SSF. The term $\bigoplus_{A\in A_D} A^{1/w_A}$ is also a belief function, denoted bel_D, as each component is also a SSF. Therefore bel is the $\ominus$-decombination of the belief function bel_C be the belief function bel_D:

$$bel = bel_C \ominus bel_D.$$

Hence (bel_C, bel_D) is the latent belief structure that underlies the apparent belief structure bel.

We have thus been able to prove that any non-dogmatic belief function bel can be uniquely decomposed into two belief functions bel_C and bel_D which are separable belief functions and the sets of focal elements of the SSF in bel_C and bel_D are disjoint.

6. Further generalization.

Up to here we have considered only GSSF with w>0. The case w = 0 corresponds to a conditioning process ($bel\oplus A^0$ is the conditional belief function obtained by conditioning bel on A by Dempster's rule of conditioning).

We focus on the generalization with $w<0$. Let A^w be GSSF with $w\in[-1,0)$. Then $A^w\oplus A^w = A^{w^2}$ and $w^2\in(0,1]$. Hence A^w is such that its $\oplus$-combination with itself produces a SSF. The relation of A^w with the SSF A^{w^2} is analogous to a square root. Hence we call A^{w^2} a square root SSF, denoted RSSF. Analogously A^w with $w\in(-\infty,-1)$ is a GSSF such that A^{w^2} is an ISSF, as $w^2\in(1,\infty)$. So A^w corresponds to a square root of an inverse simple support function, denoted RISSF.

A function $b : 2^\Omega\rightarrow[0,1]$ which canonical decomposition will include square root terms is not a belief function. Can we encounter states of belief which representation includes some RSSF and IRSSF? If yes, it would mean that the TBM is inadequate (and every models of uncertainty based on belief functions and probability functions). It is not clear that such functions can be involved in the representation of a state of belief. So far, they seem nothing more than mathematical objects without relation with the representation of belief.

7. Comparison of our canonical decomposition with Shafer's solution.

Shafer (1976) has considered the set of support functions as being the set of belief functions that can be obtained by a coarsening of the separable belief functions. Hence a support function is a belief function bel that admits the following representation:

$$bel = Coars(\bigoplus_{A\subseteq\Omega} A^{w_A})$$

where $bel : 2^\Theta\rightarrow[0,1]$, A^{w_A} is a SSF on Ω, Θ is a coarsening of Ω and Coars is the operator that transforms a belief function defined on Ω into a belief function defined on Θ, i.e., for every $B\subseteq\Theta$,

$$bel(B) = Coars(\bigoplus_{A\subseteq\Omega} A^{w_A})(B) = (\bigoplus_{A\subseteq\Omega} A^{w_A})(B)$$

as $Coars(bel(B)) = bel(B)$ if $B\subseteq\Theta$.

In example 2, a solution would be:

$$bel = \{a_1,b\}^{1/2}\oplus\{a_2,b\}^{1/2}\oplus\emptyset^{4/3}$$

(the term $\emptyset^{4/3}$ is the normalization factor), $\Omega = \{a_1,a_2,b,c\}$ and $\Theta = \{a,b,c\}$ where a, b, c are the result of the coarsening of a_1 and a_2, of b and of c, respectively.

In the canonical decomposition, we do not consider that the belief function we deal with is only defined on a coarsening of another belief function defined on a more refined space. We feel that the introduction of the spaces Ω and Θ is not satisfactory. If it were the case, then how could we justify Dempster's rule of combination? To illustrate the problem, suppose bel_1 and bel_2 are both defined on Ω, and Θ is a coarsening of Ω. Let bel^Θ denoted the belief function induced on Θ by the belief function bel^Ω defined on Ω, then it is well-known that usually

$$bel^\Theta{}_1\oplus bel^\Theta{}_2 \neq (bel^\Omega{}_1\oplus bel^\Omega{}_2)^\Theta.$$

Coarsening and Dempster's rule of combination do not commute.

If one accept, as done in Shafer (1976), that the observed bel is only defined on some frame Θ and that it results from some underlying and unknown belief function bel defined on Ω, then the combination by Dempster's rule of combination becomes hazardous, is not unjustified. The canonical decomposition studied by Shafer would result in explicitly acknowledging that our beliefs are usually (i.e. whenever bel is not separable) the results of some coarsening... in which case Dempster's rule of combination would have to be reconsidered. We prefer our canonical decomposition as we do not have to acknowledge the two spaces. The space Ω on which bel is defined is all what is needed, Dempster's rule of combination is well-justified and the canonical decomposition is based on the two separable belief functions, one representing the confidence, the other the diffidence component of our state of belief.

8. The case of dogmatic belief functions.

If $m(\emptyset) = 0$, the canonical decomposition cannot be achieved as described as far as some w cannot be computed (because some denominators are 0). In such a case, a way out consists in creating a bba m' where $m'(\Omega) = \varepsilon$ (to be rigorous, the ε should be substracted from those positive m(X) in order to keep the sum of the bbm equal to one but this subtlety is unnecessary in practice). By construction, bel' is not dogmatic, hence admits an unique canonical decomposition.

Example 4: Let $\Omega = \{a,b,c\}$ and let bel be the belief function which bba is: m({a,b}) = m ({a,c}) = 1/2. Table 2 presents the values of m and q, of the m' and q' approximations of both m and q, and the coefficients of the canonical decomposition of m'. The latent belief structure of bel is $\lim_{\varepsilon\to 0}(\{a,b\}^{2\varepsilon} \oplus \{a,c\}^{2\varepsilon}, \{a\}^{4\varepsilon(1-\varepsilon)})$. So bel is essentially the result of a conditioning on {a,b} and on {a,c} but where {a} may not receive any support (so bel({a}) = 0).

A⊆Ω	m	q	m'	q'(A)	w'$_A$
Ø	0	1	2ε	1	1-ε
{a}	0	1	0	1-ε	1/(4ε(1-ε))
{b}	0	.5	0	.5	1
{c}	0	.5	0	.5	1
{a,b}	.5	.5	.5-ε	.5	2ε
{a,c}	.5	.5	.5-ε	.5	2ε
{b,c}	0	0	ε	ε	1
{a,b,c}	0	0	ε	ε	

Table 2: Example 3. Subsets of Ω, values of m and q, and their ε-approximations m' and q' and w'$_A$.

9. Conclusions.

Thanks to our canonical decomposition, we can represent a complex belief state as the result of the combination of elementary and distinct states of belief. Each elementary belief state is represented by a SSF. Each SSF represents either 'good reasons to believe' or 'good reasons not to believe' a given event or proposition. Each SSF can be seen as a weighted proposition and a state of belief is represented by a set of independently weighted propositions. It means we have built a weighted propositions] logic where user can write propositions and give weights independently to each of them. The result will be a complex state of belief, which latent belief structure is well defined.

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