The Rationality and Decidability of Fuzzy Implications

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Abstract

It is well known that knowledge-based systems would be more robust and smarter if they can deal with the inconsistent, incomplete or imprecise knowledge, which has been referred to as common sense knowledge. In this paper, we discuss fuzzy implications in the sense of common sense reasoning. Firstly, we analyse the rationality of some existing fuzzy implications based on the discussion of implicational paradoxes. Secondly, we present a new fuzzy preferential implication that is nonmonotonic, paraconsistent and without the general implicational paradoxes. Finally, we propose sound and complete decision tableaux of such implications, which can be used as the inference engines of adaptive expert systems *or* frameworks for the fuzzy Prolog.

1 Introduction

Implication is the heart of logic. The truth value of the material implication A->B in classical logic is determined by the truth values of the antecedent A and the consequent B, i.e. A-*B is true iff A is false or B is true. The truth value of the implication should be determined by the conditional relation between the antecedent and the consequent. Because the truth values of the antecedent A and the consequent B cannot determine the causal link between A and B, there are implicational paradoxes in classical logic, i. e. if one regards the material implication as the entailment and each logical theorem in the logic as a valid reasoning form, then some logical axioms or theorems in the logic, such as $A\rightarrow(\sim B\vee B)$, $(\sim A\wedge A)\rightarrow B$, $A\rightarrow(B\rightarrow A)$, $\sim A\rightarrow(A\rightarrow B)$, $(A\rightarrow B)\vee(B\rightarrow A)$, $(A\rightarrow B)\vee(\sim A\rightarrow B)$, $(A\rightarrow B)\vee(A\rightarrow\sim B)$ and so on, present some paradoxical properties and therefore they have been referred to as 'implicational paradoxes'[Anderson & Belnap, 1975]. For example, '$(\sim A\wedge A)$->B" means 'a contradiction implies anything', "A-*$(\sim B\vee B)$' means 'a tautology is implied by anything'. In 1912, Lewis established modal logic in order to avoid implicational paradoxes.In 1955, Sugihara pointed out that a logic system is paradoxical if it has either a weakest formula (WF for short) or a strongest formula (SF for short). Where A is a WF iff B entails A for any formula B.Where A is a SF iff A entails B for any formula B.We call a logic system general-implicational- paradox-free (GIPF for short) if it contains neither WF nor SF. In 1966, Ackermann pointed out: ' Rigorous implication, which we write as A-»B, should express the fact that a logical connection holds between A and B, that the content of B is part of that of A. ..That has nothing to do with the truth or falsity of A or B. ' During the period from the 1950s to the 1970s, Anderson and Belnap [1975] extended the work of Ackermann and proposed variable-sharing as a necessary but not sufficient formal conditon for the relevance between the antecedent and consequent of a valid entailment, i. e. if A entails B then A and B share a variable. They introduced the concept of generic implication (GI for short) which can be looked as the inference rule of the logic Bystem while the material implication can be looked as a logical connective. Their relevant logic has neither WF nor SF, but its semantic model is not clear and the definition of GI in it needs several axioms.

Let $\vdash_L$ represent the GI in logic L and W stand for the set of all well formed formulas, the belief set of a formula set S is defined by $Th_L(S) = \{G \mid (G\in W)\wedge(S\vdash_L G)\}$. Call a formula set S meaningless under $\vdash_L$ iff $Th_L(S)=W$. Call a formula set S contradictory under $\vdash_L$ iff there exists a formula A such that $S\vdash_L A$ and $S\vdash_L\sim A$. Call logic L or the GI $\vdash_L$ paraconsistent iff there exists a formula set, which is contradictory, but not meaningless under $\vdash_L$. For example, the paraconsistent logic LP proposed by Priest [1979] has no SF, thus the paradoxical implication $\sim A\wedge A\vdash_{LP}B$ does not hold true for arbitrary B in it. The paraconsistent logic can reason rationally when the premise of a theory is contradictory.

Logic L or the GI $\vdash_L$ is monotonic iff $\forall A,B,C\in W$, if $A\vdash_L C$, then $A\wedge B\vdash_L C$. $\vdash_L$ is reflexive iff $\forall A\in W, A\vdash_L A$. $\vdash_L$ is transitive iff $\forall A,B,C\in W$, if $A\vdash_L B$ and $B\vdash_L C$, then $A\vdash_L C$. We say that $\vdash_L$ has

WF (or SF) iff L has WF (or SF). $\vdash_L$ is GIPF iff L is GIPF.

The extensions of classical logic into nonclassical logics can be classified into the following two types: one extends the truth value set or the logical connectives; the other modifies the GI. The former improves the expressive power of the logic, the latter modifies the reasoning ability of the logic. For example, fuzzy logic [Lee, 1972] extends classical logic's truth value set {0, 1} to [0, 1] so as to reason with uncertain knowledge. The entailments of preferential logics [Shoham, 1987; Jiang, 1990; Benferhat, 1993] are preferential implications, which are nonmonotonic.

In section 2 of this paper, we will analyse the rationality of some existing fuzzy implications. Section 3 presents two new fuzzy preferential implications and section 4 discusses the decision procedures for these implications.

For convenience, the formulas discussed here will be restricted within the propositional version of Lee's system [1972]. Let $T_I(G)$ represent the truth value of a formula G under an interpretation I, then $\forall G, H \in W$, $T_I(G) \in [0, 1]$, $T_I(\sim G) = 1 - T_I(G)$, $T_I(G \wedge H) = \min\{T_I(G), T_I(H)\}$, $T_I(G \vee H) = \max\{T_I(G), T_I(H)\}$, $T_I(G \rightarrow H) = T_I(\sim G \vee H)$.

2 Fuzzy Generic Implications

Mukaidono [1982] has argued that: "In researching fuzzy reasoning, the following two pionts should be made clear: (1) What form is adopted as a inference rule for deriving a logical consequence from the premise and how the fuzzy implication is defined? (2) What significance is postulated to fuzzy inference?".

There have been only a few attempts [Liu and Xiao, 1985; Liu, 1990; Liu etal, 1991; Lee, 1972; Mukaidono, 1982; Yager, 1985] to extend fuzzy logic to automated reasoning. The GIs of these systems are of the following four types:

① $G \models_f H$ iff $\forall I, T_I(G) < T_I(H)$

② $G \Rightarrow H$ iff $\forall I, T_I(\sim G \vee H) > 0.5$

③ $G \Rrightarrow H$ iff $\forall I$, if $T_I(G) > 0.5$, then $T_I(H) > 0.5$

④ $G \Rrightarrow' H$ iff $\forall I$, if $T_I(G) > 0.5$, then $T_I(H) > 0.5$

Where G and H are formulas. The antecedents of above implications can be formula sets also. A formula set S is regarded as a conjunction of all formulas in S. The truth value of a formula set S is defined by the smallest one of the truth values of formulas in S.

The significance standard of fuzzy implication should be changed when we need the fuzzy inference engine capable of dealing with the incomplete or inconsistent information just like human being. We think that the GI with fewer paradoxes is more significant. We will discuss the properties of the above GIs in this section.

Let I be an interpretation of a formula (set) G, we say that I satisfies G or I is a model of G iff $T_I(G) > 0.5$; if $T_I(G) < 0.5$, then I is said to falsify G. G is said to be valid iff any interpretation I satisfies G. G is unsatisfiable iff any interpretation I falsifies G. Obviously, I satisfies a formula set iff I satisfies every formula in it.

From above definitions, we can prove the following properties:

Property 1 Suppose the formulas G and H share no atom, if $G \models_f H$ then G is unsatisfiable, H is valid; if $G \Rrightarrow H$ then H is valid; if $G \Rrightarrow' H$ then G is unsatisfiable; if $G \Rightarrow H$ then G is unsatisfiable or H is valid.

If H is valid i.e. $\forall I, T_I(H) > 0.5$ then $\forall G \in W, G \Rrightarrow H$ holds true, i.e. $\Rrightarrow$ has WF; if G is unsatisfiable i.e. $\forall I, T_I(G) \ngtr 0.5$ then $\forall H \in W, G \Rrightarrow' H$ holds true, i.e. $\Rrightarrow'$ has SF; but the logic system adopting $\models_f$ has neither WF nor SF, i.e. $\models_f$ is GIPF.

Property 2. $\Rrightarrow$ and $\Rrightarrow'$ are not paraconsistent; $\models_f$ and $\Rightarrow$ are paraconsistent.

Property 3. $\forall G, H \in W$, $G \Rightarrow H$ holds true in Lee's fuzzy logic iff $G \rightarrow H$ is valid in classical logic.

So $\Rightarrow$ has the implicational paradoxes just like the material implication in classical logic and the resolution system in Lee's fuzzy logic [1972] has no formal difference from that of classical logic. The difference between fuzzy proposition and classical proposition exists only in the truth values of atoms.

Property 4. $\Rrightarrow$, $\Rrightarrow'$, $\models_f$ and $\Rightarrow$ are all reflexive, transitive and monotonic.

As a result of the monotonic property, even if we have obtained new evidence, we cannot change the conclusions that were derived from the initial belief set, so the logics lack the cognitive or adaptive capabilities. The preferential entailment [Jiang, 1990; Shoham, 1987] is important in the research of nonmonotonic reasoning. We will present two nonmonotonic, paraconsistent fuzzy implications in the following section.

3 Fuzzy Preferential Implications

Definition 1. $\forall v_1, v_2 \in [0, 1]$, if $|v_1 - 0.5| > |v_2 - 0.5|$, then we say that v_2 is not more exact than v_1, denoted by $v_1 \boxed{\geq} v_2$; if $|v_1 - 0.5| > |v_2 - 0.5|$, v_1 is said to be more exact than v_2, denoted by $v_1 \boxed{>} v_2$.

Definition 2. Let M_1 and M_2 be two models of the formula (set) S, $M_1 \boxed{\geq} M_2$ iff for any atom P occurring in S, $T_{M_1}(P) \boxed{\geq} T_{M_2}(P)$; $M_1 \boxed{>} M_2$ iff for any atom P occurring in S, $T_{M_1}(P) \boxed{\geq} T_{M_2}(P)$, and there exists at least one atom Q occurring in S such that $T_{M_1}(Q) \boxed{>} T_{M_2}(Q)$.

Definition 3. A model M_1 of S is called a preferential model iff there exists no model M_2 of S satisfying $M_2 \boxed{>} M_1$.

It merits attention that there are no constraints on the atoms that do not occur in S for the preferential models of S.

Definition 4. If every preferential model of S is a model of G, we say that the formula (set) S preferentially entails the formula G, denoted by $S \Rrightarrow_P G$; otherwise, we say that $S \not\Rrightarrow_P G$.

For example: ① $\{P, P \rightarrow Q\} \Rrightarrow_P Q$, i.e. modus ponens holds true in the inference system using $\Rrightarrow_P$.

② $\{P, P \rightarrow Q, \sim R \wedge R\} \Rrightarrow_P Q$, i.e. $\Rrightarrow_P$ can work reasonably well while the inference is not affected by the contradictions of the premise.

③ {P, P→Q, ~P} $\not\Rightarrow_P$ Q, i.e. if a new evidence ~P contradicts the initial supposition {P, P→Q}, under $\Rightarrow_P$, we can withdraw the conclusion Q that is derived from the outdated or incomplete premise {P, P→Q}. This example illustrates the cognitive process.

④ {P, P→Q, ~Q} $\not\Rightarrow_P$ Q, i.e. under $\Rightarrow_P$, we can withdraw the conclusion Q that contradicts a new evidence ~Q. So, $\Rightarrow_P$ can limit the propagation of contradictions.

Property 5. $\Rightarrow_P$ is reflexive and paraconsistent.

Proof: For any formula G, the preferential model of G is obviously a model of G, so G $\Rightarrow_P$ G, i.e. $\Rightarrow_P$ is reflexive.

Since {P, P→Q, ~P} $\Rightarrow_P$ P, {P, P→Q, ~P} $\Rightarrow_P$ ~P; {P, P→Q, ~P} is contradictory under $\Rightarrow_P$. {P, P→Q, ~P} $\not\Rightarrow_P$ R referring to the following preferential model of S: I={P=0.5, Q=1, R=0}, so {P, P→Q, ~P} is not meaningless under $\Rightarrow_P$. Thus, $\Rightarrow_P$ is paraconsistent. (Q.E.D.)

For $\Rightarrow_P$ is reflexive, P∧(P→Q)∧(~P) $\Rightarrow_P$ P∧(P→Q); from example ①, P∧(P→Q) $\Rightarrow_P$ Q; but from example ③, P∧(P→Q)∧(~P) $\not\Rightarrow_P$ Q, so $\Rightarrow_P$ is neither transitive nor monotonic.

Let **Atomset(S)** denotes the set of atoms occurring in a formula set S; **Lit(S)**={P, ~P | P∈Atomset(S)} represent the literal set concerning S.

Definition 5. Let M be a model of formula (set) S, the inconsistent set of M is defined by M!:

M! = {P | P∈Atomset(S) and T_M(~P∧P)=0.5}.

Property 6. A formula set S is contradictory under $\Rightarrow_P$ iff the inconsistent set of any model of S is not empty.

Property 7. Let M_1 and M_2 be two models of formula set S, if $M_1 \trianglerighteq M_2$, then $M_1! \subseteq M_2!$.

Because any interpretation is a model of a valid formula, S $\Rightarrow_P$ ~Q∨Q for any formula set S, i.e. $\Rightarrow_P$ has WF, so $\Rightarrow_P$ is not GIPF.

$\models_r$ is GIPF, but it lacks some reasonable composition. For example, modus ponens does not hold true under $\models_r$, i.e. {P, P→Q} $\not\models_r$ Q because under I={P=0.5, Q=0}, T_I({P, P→Q})=min{T_I(P), T_I(P→Q)}= 0.5 $\not\leq$ 0=T_I(Q).

Definition 6. Let S be a formula set and H be a formula. If T_I(H) > T_I(S) under any preferential model I of S, we say that S $\models_{rP}$ H; otherwise, we say that S $\not\models_{rP}$ H.

For example: ① {P, P→Q} $\models_{rP}$ Q, because under the unique preferential model I={P=1, Q=1} of {P, P→Q}, T_I({P, P→Q}) =1 < T_I(Q)=1, i.e. $\models_{rP}$ can accomplish the rational inference that can not be accomplished by $\models_r$.

② {P, P→Q} $\not\models_{rP}$ ~R∨R, because under the preferential model I={P=1, Q=1, R=0.5} of {P, P→Q}, T_I(~R∨R)=0.5 $\not\geq$ 1=min{T_I(P), T_I(P→Q)}. i.e. $\models_{rP}$ can avoid the paradox that cannot be avoided by $\Rightarrow_P$.

Similarly, ③ {P, P→Q, ~R∧R} $\models_{rP}$ Q;

④ {P, P→Q, ~P} $\not\models_{rP}$ Q; {P, P→Q, ~Q} $\not\models_{rP}$ Q.

It can be proved that for any formulas G and H, if G |=fp H holds true and G and H have no sharing atom, then G is unsatisfiable and H is valid, has neither WF nor SF, 1. e. |=rp is GIPF.

Property 8. The preferential implication |=cp is reflexive, paraconsistent and GIPF.

Similar to =>P, l=rp is neither transitive nor monotonic.

The comparision among the above fuzzy Implications is listed in Table 1:

Properties of fuzzy implications	Having WF or not?	Having SF or not?	Monotonic?	Paraconsistent?
=>	y	y	y	n
=>'	n	y	y	n
≡>	y	n	y	y
$\models_r$	n	n	y	y
$\Rightarrow_P$	y	n	n	y
$\models_{rP}$	n	n	n	y

Table 1.

4 Decision Tableaux

The resolution procedure can be used as a decision procedure for => in Lee's system[1972]: suppose S is a Bet of clauses, H is a clause, S=>H iff there is a resolution deduction of the empty clause from SU{~~H},

Both Mukaidono[1982] and Yager [1985] think that $|=_f$ is meaningful, Lee[1972] has proved that, if both the truth values of the parent clauses C_1 and C_a are greater than 0.5 under an interpretation I, a resolvent clause $R(C_{1(}C_3)$ derived by the resolution principle is significant, i.e. Ti (C.ACXTi GKCi.CJ). Mukaidono showed an interpretation that, even if the truth value of one parent clause is not greater than 0.5, a resolvent clause is meaningful in the sense of reducing ambiguity, i.e. Ti (CACARiC, C_a))[>Jr, (dAC_a) under any interpretation I. Yager also found that in fuzzy logic the resolution law doeB not hold true in its usual form and presented some modified inference laws, but didn't prove the completeness of these laws. Liu and Xiao [1986J proposed Operator Fuzzy Logic (OFD which can represent vague knowledge by fuzzy operators explicitly and without resorting to the use of intermediate truth values at the semantic level. In order to prove that >--resolution is a sound decision procedure for inconsistent clause set in OFL, Liu[1990] introduced the GI that is ^>' when the threshold is 0.5. [Liu etal, 1991] chose the inference rule that is => when the threshold is 0.5, but in order to keep the significance of λ-resolution in OFL, they need the λ-pseudoreduction that cannot apply to Lee's fuzzy logic which has no fuzzy operator. Till now, nobody has discussed the proof theory of $\Longrightarrow$ or $\Longrightarrow'$. In brief, the resolution procedure may not be suitable for deciding whether $\models_r$, $\Longrightarrow$ or $\Longrightarrow'$ holds true, and the decision problems of $\models_r$, $\Longrightarrow$ and $\Longrightarrow'$ are still open.

Semantic tableaux [Smullyan, 1968], which are dual forms of Gentzen's sequent calculi, are widely used in the field of automatic theorem proving. Using the unified notation, we can classify the non-literal propositional formulas of Lee's system[1972] into the following two types:

Type α: $A \wedge B$, $\sim(A \vee B)$, $\sim(A \rightarrow B)$, $\sim\sim A$

Type β: $\sim(A \wedge B)$, $A \vee B$, $A \rightarrow B$

We define the direct descendents of them in Table 2 and Table 3 respectively:

α	α_1	α_2
$A \wedge B$	A	B
$\sim(A \vee B)$	$\sim A$	$\sim B$
$\sim(A \rightarrow B)$	A	$\sim B$
$\sim\sim A$	A	A

Table 2.

β	β_1	β_2
$\sim(A \wedge B)$	$\sim A$	$\sim B$
$A \vee B$	A	B
$A \rightarrow B$	$\sim A$	B

Table 3.

Definition 7. Let S be a set of formulas, the tableau of S, which is denoted by Tbl(S), is a binary tree with formula sets as nodes constructed by the following rules: 1) The root of Tbl(S) is S.

2) For any node S_i of Tbl(S), we generate its direct descendent node(s) by the following rules:

① α **rule:** If $S_i = \{\alpha\} \cup S_j$, then S_i has one direct descendent node: $S_i' = \{\alpha_1, \alpha_2\} \cup S_j$. In this case, S_i is called a type α node.

② β **rule:** If $S_i = \{\beta\} \cup S_j$, then S_i has two direct descendent nodes: $S_i' = \{\beta_1\} \cup S_j$ and $S_i'' = \{\beta_2\} \cup S_j$. In this case, S_i is called a type β node.

Obviously, any terminal node of Tbl(S) contains literals only.

Lemma 1. Suppose all of the terminal nodes of Tbl({A}) are $\mathbf{a}_1, \cdots, \mathbf{a}_n$, the literals in $\mathbf{a}_i$ are $a_{i\,1}, \cdots, a_{i\,mi}$ for $i=1, \cdots, n$, then $A = (a_{1\,1} \wedge \cdots \wedge a_{1\,m1}) \vee \cdots \vee (a_{n\,1} \wedge \cdots \wedge a_{n\,mn})$.

Definition 8. A terminal node is called closed if it contains some complementary literal pair. A type α node is called closed if its direct descendent node is closed. A type β node is called closed if both its direct descendent nodes are closed. A tableau is called closed iff its root is closed.

By lemma 1, we can derive the following conclusion:

Proposition 1. Let A be a formula, A is unsatisfiable iff Tbl({A}) is closed.

Theorem 1. $\forall A, B \in W$, $A \Longrightarrow B$ iff Tbl({A, ~B}) is closed.

In order to analyse the causal link between the premise and the conclusion, we'll construct tableaux for the premise and the negation of the conclusion respectively to decide whether a implication holds true.

Definition 9. Suppose S is a set of formulas, A is a formula, the terminal nodes of Tbl(S) are $S_1, \cdots, S_n$, the terminal nodes of Tbl({~A}) are $\mathbf{a}_1, \cdots, \mathbf{a}_m$. If for any $i \in \{1, \cdots n\}$, $j \in \{1, \cdots m\}$, there exists some complementary literal pair in $S_i \cup \mathbf{a}_j$ and the complementary literals belong to S_i and $\mathbf{a}_j$ respectively or both belong to $\mathbf{a}_j$, then we say that the dual tableaux of S and A are closed for $\Longrightarrow$.

Theorem 2. Let S be a formula set, A be a formula, $S \Longrightarrow A$ iff the dual tableaux of S and A are closed for $\Longrightarrow$.

Proof: Let the terminal nodes of Tbl(S) be $S_1, \cdots, S_n$; the terminal nodes of Tbl({~A}) be $\mathbf{a}_1, \cdots, \mathbf{a}_m$ and the literals in $\mathbf{a}_j$ be $a_{j\,1}, \cdots, a_{j\,kj}$ for $j=1, \cdots, m$.

Suppose the dual tableaux of S and A are closed for $\Longrightarrow$. By lemma 1, any model I of S satisfies at least one terminal node S_{i_0} of Tbl(S) where $i_0 \in \{1, \cdots n\}$. Since the dual tableaux of S and A are closed for $\Longrightarrow$, for any $j \in \{1, \cdots m\}$ there exists some complementary literal pair in $S_{i_0} \cup \mathbf{a}_j$ and the complementary literals belong to S_{i_0} and $\mathbf{a}_j$ respectively or both belong to $\mathbf{a}_j$.

If the complementary literals belong to S_{i_0} and $\mathbf{a}_j$ respectively, without losing generality, we can suppose $L_B \in S_{i_0}$, $L_A \in \mathbf{a}_j$, $L_B = \sim L_A$. Because I satisfies S_{i_0}, $T_I(L_B) > 0.5$, i.e. $T_I(\sim L_A) > 0.5$.

If both the complementary literals belong to $\mathbf{a}_j$, we can suppose $L_A \in \mathbf{a}_j$ and $\sim L_A \in \mathbf{a}_j$. Because any interpretation satisfies at least one of L_A and $\sim L_A$, without losing generality, we suppose $T_I(\sim L_A) > 0.5$.

By the arbitrariness of $\mathbf{a}_j$, for every terminal node $\mathbf{a}_j$ of Tbl({~A}), I satisfies the conjugate of some literal $a_{j\,lj}$ in $\mathbf{a}_j$ where $l_j \in \{1, \cdots, k_j\}$, $j \in \{1, \cdots, m\}$.

Thus I satisfies $\sim a_{1\,l1} \wedge \cdots \wedge \sim a_{j\,lj} \wedge \cdots \wedge \sim a_{m\,lm}$ where $l_1 \in \{1, \cdots, k_1\}, \cdots, l_j \in \{1, \cdots, k_j\}, \cdots, l_m \in \{1, \cdots, k_m\}$.

By lemma 1, $\sim A = (a_{1\,1} \wedge \cdots \wedge a_{1\,k1}) \vee \cdots \vee (a_{m\,1} \wedge \cdots \wedge a_{m\,km})$

$A = \sim(\sim A) = \sim((a_{1\,1} \wedge \cdots \wedge a_{1\,k1}) \vee \cdots \vee (a_{m\,1} \wedge \cdots \wedge a_{m\,km}))$

(by the De. Morgan's Law in fuzzy logic)

$= (\sim a_{1\,1} \vee \cdots \vee \sim a_{1\,k1}) \wedge \cdots \wedge (\sim a_{m\,1} \vee \cdots \vee \sim a_{m\,km})$

(by the Complete Distributive Law in fuzzy logic)

$= (\sim a_{1\,1} \wedge \cdots \wedge \sim a_{m\,1}) \vee \cdots \vee (\sim a_{1\,l1} \wedge \cdots \wedge \sim a_{m\,lm}) \vee \cdots \vee (\sim a_{1\,k1} \wedge \cdots \wedge \sim a_{m\,km})$. So, I satisfies A. By the arbitrariness of I, $S \Longrightarrow A$ holds true.

Suppose $S \Longrightarrow A$ holds true, from the above analysis, we can see that any model I of S satisfies at least one conjunction $(\sim a_{1\,l1} \wedge \cdots \wedge \sim a_{j_0\,lj_0} \wedge \cdots \wedge \sim a_{m\,lm})$ where $l_1 \in \{1, \cdots, k_1\}, \cdots, l_{j_0} \in \{1, \cdots k_{j_0}\}, \cdots, l_m \in \{1, \cdots, k_m\}$. If the dual tableaux of S and A are not closed for $\Longrightarrow$, there is at least one pair of nodes S_{i_0} and $\mathbf{a}_{j_0}$ ($i_0 \in \{1, \cdots, n\}$, $j_0 \in \{1, \cdots, m\}$) that includes no complementary literal pair satisfying the condition that at least one of the complementary literals belongs to a_{jo}. Suppose L_A is a literal of a_{jo}, L_A' is the conjugate of L_A, thus either $L_A \in S_{io}$ and $L_A' \notin S_{io}$ or $L_A \notin S_{io}$ and $L_A' \notin S_{io}$, i.e. the conjugate of any literal of a_{jo} doesn't belong to S_{io}.

We can construct the following interpretation M of S. For any $P \in$ Atomset(S):

if $P \in S_{io}$, $\sim P \in S_{io}$, then let $T_M(P) = 0.5$;
if $P \in S_{io}$, $\sim P \notin S_{io}$, then let $T_M(P) = 1$;
if $P \notin S_{io}$, $\sim P \in S_{io}$, then let $T_M(P) = 0$;
if $P \notin S_{io}$, $\sim P \notin S_{io}$, then let $T_M(P) = 0$.

Obviously, M satisfies S_{io}, by lemma 1, M satisfies S. Since the conjugate of any literal of a_{jo} doesn't belong to S_{io}, the conjugate of any literal of a_{jo} is assigned the truth value 0 under M. Specially, $T_M(\sim a_{jo\ ljo}) = 0$, this contradicts the assumption that M satisfies $(\sim a_{1\ l1} \wedge \cdots \wedge \sim a_{jo\ ljo} \wedge \cdots \wedge \sim a_{m\ lm})$. Thus we can see, if $S \Longrightarrow A$, then the dual tableaux of S and A are closed for $\Longrightarrow$. (Q.E.D.)

If every terminal node of Tbl({~A}) is closed, the dual tableaux of S and A are closed for $\Longrightarrow$ for any formula set S. By proposition 1, any valid formula A is a WF of $\Longrightarrow$, so $\Longrightarrow$ is not GIPF.

Corresponding to the preferential implications, we must distinguish the nodes that determine the preferential models.

Let $BA(S) = \{P \mid \sim P, P \in S\}$ for any literal set S.

Suppose $PS = \{S_1, \cdots, S_n\}$, S_i is a literal set for any $i \in \{1, \cdots n\}$, S_i is called minimum contradictory in PS if there is no S_j ($j \in \{1, \cdots, n\}$) satisfying $BA(S_j) \subset BA(S_i)$.

Suppose S is a set of formulas, PS is the set of the terminal nodes of Tbl(S), the minimum contradictory element in PS are called the minimum contradictory terminal nodes of Tbl(S).

Definition 10. Suppose S is a set of formulas, A is a formula; the minimum contradictory terminal nodes of Tbl(S) are $S_1, \cdots, S_n$; the terminal nodes of Tbl({~A}) are $a_1, \cdots, a_m$. If for any $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, m\}$, there exists some complementary literal pair in $S_i \cup a_j$ and the complementary literals belong to S_i and a_j respectively or both belong to a_j, then we say that the dual tableaux of S and A are closed for $\Longrightarrow_P$.

Lemma 2. Suppose S is a formula set, ① any preferential model M of S satisfies at least one of the minimum contradictory terminal nodes of Tbl(S); ②any minimum contradictory terminal node of Tbl(S) is satisfied by some preferential model of S.

Proof: Let the terminal nodes of Tbl(S) be $S_1, \cdots, S_n$; $S_Q, S_{1Q}, \cdots, S_{nQ}$ be the conjunctions of the formulas in S, $S_1, \cdots, S_n$ respectively, by lemma 1, $S_Q = S_{1Q} \vee \cdots \vee S_{nQ}$.

①Suppose M_1 is a preferential model of S, by lemma 1, M_1 satisfies a terminal node S_i ($i \in \{1, \cdots, n\}$) of Tbl(S). If M_1 satisfies no minimum contradictory terminal node of Tbl(S), there must exist S_j ($j \in \{1, \cdots, n\}$) satisfying $BA(S_j) \subset BA(S_i)$, thus there exists $Q \in$ Atomset(S) satisfiying $Q \in BA(S_i)$ and $Q \notin BA(S_j)$.

We can construct the following interpretation M_2 of S. For any $P \in$ Atomset(S):

if $P \in S_j$, $\sim P \in S_j$, then let $T_{M2}(P) = 0.5$;
if $P \in S_j$, $\sim P \notin S_j$, then let $T_{M2}(P) = 1$;
if $P \notin S_j$, $\sim P \in S_j$, then let $T_{M2}(P) = 0$;
if $P \notin S_j$, $\sim P \notin S_j$, then let $T_{M2}(P) = 0$.

Obviously, M_2 satisfies S_j, by lemma 1, M_2 satisfies S. For any $P \in$ Atomset(S), $T_{M2}(P) = 0.5$ means $P \in BA(S_j)$; because $BA(S_j) \subset BA(S_i)$, this means $P \in BA(S_i)$; since M_1 is a model of S_i, $T_{M1}(P) = 0.5$. $Q \notin BA(S_j)$ means $T_{M2}(Q) \neq 0.5$; since M_1 satisfies S_i, $Q \in BA(S_i)$ means $T_{M1}(Q) = 0.5$; thus $M_2 \boxed{>} M_1$. This result contradicts the assumption that M_1 is a preferential model of S.

Thus we have shown, any preferential model M of S satisfies at least one of the minimum contradictory terminal nodes of Tbl(S).

②Suppose S_j ($j \in \{1, \cdots, n\}$) is a minimum contradictory terminal node of Tbl(S), we can construct the following interpretation M_1 of S. For any $P \in$ Atomset(S):

if $P \in S_j$, $\sim P \in S_j$, then let $T_{M1}(P) = 0.5$;
if $P \in S_j$, $\sim P \notin S_j$, then let $T_{M1}(P) = 1$;
if $P \notin S_j$, $\sim P \in S_j$, then let $T_{M1}(P) = 0$;
if $P \notin S_j$, $\sim P \notin S_j$, then let $T_{M1}(P) = 0$.

Obviously, M_1 satisfies S_j, by lemma 1, M_1 satisfies S. If M_1 is not a preferential model of S, there must exist a preferential model M_2 of S satisfying $M_2 \boxed{>} M_1$, by property 7 of section 3, $M_2! \subseteq M_1!$. The truth values of the atoms under M_1 can only be 0, 0.5, 1 and no truth values can be more exact than 0 or 1, so $M_2! \subset M_1!$. By lemma 1, M_2 must satisfies some terminal node S_i ($i \in \{1, \cdots, n\}$) of Tbl(S). $BA(S_i) \subseteq M_2! \subset M_1! = BA(S_j)$. This contradicts the assumption that S_j is a minimum contradictory terminal node of Tbl(S). Thus we can see, any minimum contradictory terminal node of Tbl(S) is satisfied by some preferential model of S. (Q.E.D.)

By lemma 2, we can prove the following theorem:

Theorem 3. Let S be a formula set, A be a formula, $S \Longrightarrow_P A$ iff the dual tableaux of S and A are closed for $\Longrightarrow_P$.

Similarly, we can solve the decision problem of $\models_{FP}$.

Definition 11. Suppose S is a formula set, A is a formula; the minimum contradictory terminal nodes of Tbl(S) are $S_1, \cdots, S_n$; the terminal nodes of Tbl({~A}) are $a_1, \cdots, a_m$. If for any $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, m\}$, there exists some complementary literal pair in $S_i \cup a_j$ and the complementary literals belong to S_i and a_j respectively or both belong to a_j and the complementary literals belong to Lit(S) or S_i is closed and a_j is closed, then we say that the dual tableaux of S and A are closed for $\models_{FP}$.

Lemma 3. Suppose S is a set of formulas; the minimum contradictory terminal nodes of Tbl(S) are $S_1, \cdots, S_n$; $S_Q, S_{1Q}, \cdots, S_{nQ}$ are the conjunctions of the formulas in S, $S_1, \cdots, S_n$ respectively; then under any preferential model I of S, $T_I(S_Q) = T_I(S_{1Q} \vee \cdots \vee S_{nQ})$.

Proof: Suppose PS is the terminal node set of Tbl(S). If $PS = \{S_1, \cdots, S_n\}$, by lemma 1, $T_I(S_Q) = T_I(S_{1Q} \vee \cdots \vee S_{nQ})$. Otherwise, we can derive the conclusion as follows:

Firstly, the truth values of the atoms occurring in S under a preferential model I of S can only be 0, 0.5, 1.

Secondly, by lemma 2, any preferential model I of S satisfies at least one minimum contradictory terminal node S_i of Tbl(S) $(i \in \{1, \cdots, n\})$, i.e. $T_I(S_{i\Omega}) > 0.5$; and for any $S_L \in PS - \{S_1, \cdots, S_n\}$, S_L contains some complementary literal pairs, so $T_I(S_{L\Omega}) < 0.5$. By lemma 1, $T_I(S_\Omega) = \max\{T_I(S_{L\Omega}) \mid S_L \in PS\} = \max\{T_I(S_{L\Omega}) \mid L \in \{1, \cdots, n\}\} = T_I(S_{1\Omega} \vee \cdots \vee S_{n\Omega})$. (Q.E.D.)

By lemma 3, we can prove the following thoerem:

Theorem 4. Let S be a formula set, A be a formula, $S \models_{rP} A$ iff the dual tableaux of S and A are closed for $\models_{rP}$.

Similar to the decision procedures given by theorem 2, 3 or 4, we have the following decision methods for $\models_r$ and $=>'$ respectively:

$S \models_r A$ iff for any terminal node S_i of Tbl(S) and any terminal node a_j of Tbl({~A}), there exists some complementary literal pair in $S_i \cup a_j$ and the complementary literals belong to S_i and a_j respectively or S_i is closed and a_j is closed.

$S =>' A$ iff for any terminal node S_i of Tbl(S) and any terminal node a_j of Tbl({~A}), there exists some complementary literal pair in $S_i \cup a_j$ and the complementary literals belong to S_i and a_j respectively or both belong to S_i.

As space is limited, the proofs will be omitted here.

By the different conditions required in theorem 2, 3 or 4, we can see that the above fuzzy implications have different paradoxes.

5 Applications and Conclusions

The dual tableaux method presented in this paper is important in the research of the model semantics of the generic implications. It is easy to adapt it for other logic systems, such as [Liu, 1990; Shoham, 1987; Priest, 1979].

Our treatment of inconsistency gives a robust semantics for logic programs and the decision tableaux for $\models_{rP}$ can be used as a framework for the nonmonotonic, paraconsistent fuzzy Prolog [ISHIZUKA and KANAI, 1985]. The dual tableaux method, as compared with resolution deduction, has more potential parallelism to exploit. While querying a given large database S many times, the minimum contradictory terminal nodes of Tbl(S) can be stored beforehand; for any simple query A, the decision problem whether $S \models_{rP} A$ holds true can be solved quickly.

$\models_{rP}$ is nonmonotonic, paraconsistent and GIPF, so is more reasonable. It is capable of reasoning by the inconsistent, incomplete or imprecise knowledge without taking away any piece of knowledge [Benferhat etal, 1993]. Thus it applies to the design of adaptive expert systems or very large knowledge bases where inconsistent information is often present.

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