Preduction: A Common Form of Induction and Analogy

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Abstract

Deduction, induction, and analogy pervade all our thinking. In contrast with deduction, understanding logical aspects of induction and analogy is still an important and challenging issue of *artificial intelligence.* This paper describes a logical formalization, called *production,* of common conjectural reasoning of both induction and analogy. By introduction of preduction, analogical reasoning is refined into "preduction + deduction" and (empirical) inductive reasoning is refined into "preduction + mathematical induction". We examine generality of preduction through applications to various examples on induction and analogy.

1 Introduction

Deduction, induction, and analogy are most common patterns of our thinking. While *deduction* infers a property about a specific individual from a general property which every individual satisfies, *inductive reasoning* infers an unknown property which every individual will satisfy commonly from specific properties about individuals. *Analogical reasoning* infers an unknown property about an individual from known properties about similar others. Because of their generality and importance in our intelligent activities, understanding their reasoning processes is indispensable for embodying *artificial intelligence.*

More formally, inferences by deduction, induction, and analogy are typically represented as Table I[1]. Deduction(D.I) expresses "(D.I.I) *a* is *P.* (D.1.2) all *P*-things are *Q.* Therefore (D.1.3) *a* is Q." Induction(I.I) which we call *mathematical* expresses the usual axiom schema of induction in the arithmetic axioms. "(1.1.1) the case of 0 satisfies *P.* (1.1.2) if the case of *x* satisfies *P,* the succeeding case of *x* also satisfies *P.* Consequently, (1.1.3) any case will satisfy *P."* Induction (1.2) expresses more *empirical* reasoning than (1.1); the same consequence is inferred not from a general assertion like (1.1.2) but from an observation that (1.2.2) every case of 0 to *n* satisfies *P.* Analogy (A.I) and (A.2) express "(A.*.I) the *base case b* satisfies *P.* (A.*.2) the *target case t* is similar to *b.* Thus, (A.*.3) *t* also satisfies *P."* They are different in that the *similarity* between a target and a base is explicit as a property *S* in (A.2).

Deduction(D.1)	
(D.1.1)	$P(a)$
(D.1.2)	$\forall x(P(x) \supset Q(x))$
(D.1.3)	$Q(a)$

Induction(I.1)		**Analogy(A.1)**	
(I.1.1)	$P(0)$	(A.1.1)	$P(b)$
(I.1.2)	$\forall x(P(x) \supset P(s(x)))$	(A.1.2)	$t \sim b$
(I.1.3)	$\forall x.P(x)$	(A.1.3)	$P(t)$

Induction(I.2)		**Analogy(A.2)**	
(I.2.1)	$P(0)$	(A.2.1)	$P(b)$
(I.2.2)	$P(1) \wedge \cdots \wedge P(n)$	(A.2.2)	$S(b) \wedge S(t)$
(I.2.3)	$\forall x.P(x)$	(A.2.3)	$P(t)$

Table 1: Deduction, induction, and analogy

In this paper, we especially focus on two logical aspects of induction and analogy; their consistency and their relationship on inference. In contrast with (D.I) and (1.1), each inference rule of (1.2), (A.I), and (A.2) has at least one critical logical defect. The former, each of (D.I) and (1.1), *preserves consistency* (i.e., only consistent theorems are inferred from consistent axioms), while the latter, each of (1.2), (A.I), and (A.2) does not. In spite of this fact, the latter inference rules seem to be natural to our common sense. For example, when we infer a general rule from individual observations, where knowledge as (1.1.2) comes from? Can we directly recognize the knowledge from observations? Although we may recognize knowledge such as (1.2.2) from our environments, we do not recognize such an (1.1.2) at least from our daily life. From our view, it is the heart of our empirical induction to infer (1.1.2) from (1.2.1) and (1.2.2), and it is the process which we should investigate and formalize.

Conjectural reasoning which always brings a consistent conclusion at present but possibly inconsistent in

[1]Deduction (D.I) is a derivative rule from V-elimination and modus ponens.

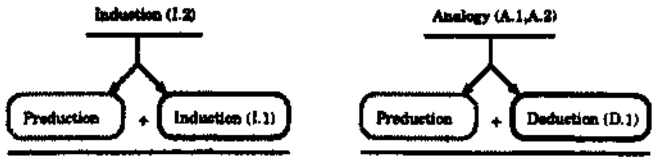

Figure 1: Preduction + (Deduction/Induction)

the future immediately implies that it is *non-monotonic.* Analogy and empirical induction often bring us conclusions turned to be wrong after we know more. In such cases, their conclusions are invalidated in our belief. This non-monotonicity is quite common to our reasoning. McCarthy introduced *ctrcumscription [McCarthy,* 1980] for the purpose of formalizing non-monotonicity in common-sense reasoning. Circumscription of a predicate *P* makes its extension minimized; anything is not *P* unless it is stated *P* by given axioms. Helft[Helft, 1988] used minimization of all predicates to formalize inductive reasoning. By means of minimizing all predicates, although preferably inferred when $P(a) \wedge S(a)$, a generalization $\forall x(S(x) \supset P(x))$ is no more inferred when $P(a) \wedge S(a) \wedge S(b)$ that normally happens whenever we want to deduce a useful conclusion from the generalization. Inductive reasoning (and analogical reasoning) should be naturally interpreted as a particular expansion of the extension of a predicate rather than minimization. Our approach follows circumscription just for formalization of non-monotonicity, but does not inherit the idea of minimality.

Relation between induction and analogy is another point to be investigated. Analogy (A.2) can be viewed as a two-step argument[Davies and Russell, 1987]; $\forall x(S(x) \supset P(x))$ by a *single-instance* induction from $S(b)$ in (A.2.2) and $P(b)$ in (A.2.1), and then by deduction with $S(t)$ in (A.2.2), we obtain (A.2.3). A similar idea is suggested in [Peirce, 1932; Mostow, 1983]. Unlike to their views, we consider there is a common inferential structure behind induction (1.2), analogy (A.I), and (A.2). Analogy includes a projection of information from a similar known object to an unknown object. Induction similarly includes a projection of information from known previous cases. We discuss more about this common denominator in the next section. We formalize this type of projection based on a transitive relation between objects. We call the formalization *preduction* because the introduction of preduction allows empirical induction and analogy to be broken into a stable part, (1.1) and (D.I), and the *precedtng* more conjectural part corresponding to the preduction (Figure 1).

Formalizing consistent *preduction* has at least two significant points. One is (of course) to allow us to use empirical induction and analogy free from inconsistency. If consistency of preduction guaranteed, because of consistency of deduction and mathematical induction, the whole processes of empirical induction and analogy would become consistent.

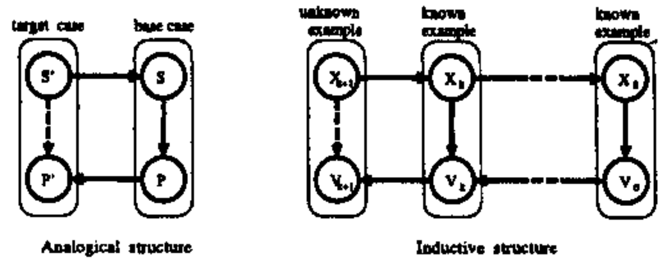

Figure 2: Common structure to analogy and induction

The other is to bring us a better understanding of logical aspects of induction and analogy by stepping into their insides. Their inferences are reduced into two clear pieces; a common denominator, preduction, and each of their residues, mathematical induction and deduction that are well investigated. It allows us to focus on an unexplored central part of thinking related to induction and analogy by removing differential and well-explored parts from them.

This paper is organized as follows. Section 2 gives a formal view about the common denominator between induction and analogy. Section 3 proposes a form of preduction. Section 4 shows its generality by applying the form to various examples over induction and analogy. Section 5 proposes its model theory and shows that the form preserves consistency. Section 6 concludes this.

2 Common Denominator

Figure 2 illustrates two typical processes by analogy and by induction from another point of view than their logical aspects. In analogical reasoning, an unknown property P' about a *target* case is inferred by finding, based on a property S' of the target case, a *base* case which satisfies a corresponding property S to S', and by projecting a relevant base case property P to the target case. In inductive reasoning, we see a similar inference which is made possible by a result of induction rather than induction itself. By a result of induction, an unknown attribute value v_{k+1} of an example indexed by x_{k+1} becomes possible to be inferred based on the known value v_k of its *preceding* example indexed by x_k. (e.g., the value, v_{k+1}, of factorial of $(k+1)$, x_{k+1}, is computed from the preceding case, the value of factorial of k.) To get the unknown value v_{k+1}, we may have to trace back (recursively) to some example indexed by x_0 whose value is known (e.g., the value of factorial of 0 is 1). Let us identify this known example in induction with a *base* case in analogy, and the unknown example indexed by x_{k+1} with a *target* case. Then, although the number of repetition of inference is different, we can identify these structures where information is mapped from a base case into a target case.

Now, in order to formalize a more general common structure related to both analogy and induction, we abstract these in two ways: number of repetition and parameters of their cases/examples. Let us represent *inner-relation* among the n-parameters inside a case/example by an n-ary predicate P ($n \geq 1$), and their parameters by a tuple of n-arguments of P. In this representation, we do not distinguish a parameter for an index from a parameter for a value. This abstraction is needed to capture an inference about a concept which is naturally represented by a predicate without parameter for value (e.g., a predicate, "Is a number"). Let us denote *outer-relation* between the parameters in a case/example and the parameters in a preceding case/example by a 2 • n-ary predicate $.R.$. Then, letting x and y be n-tuples of variables, the following sentence absorbs the number of repetition,

$$\forall x, y(P(x) \land xRy \supset P(y)),$$

which we call the *R-expansite sentence of P.* This implies that, if there is a P-thing, any entities which relate to it by R are also P-things. That is, by this sentence the property P will be (recursively) projected from a case /example known to be P to the unknown successive case/example.

Returning to the schemas in Table 1, let us see how this sentence relates with them. If we substitute $y = s(x)$ for xRy, we get

$$\forall x, y(P(x) \land y = s(x) \supset P(y)) \quad \cdots \text{(i)}$$

which is equivalent to (I.1.2) by the nature of $=$, and if $x \sim y$ and $S(x) \land S(y)$ for xRy, we get

$$\forall x, y(P(x) \land x \sim y \supset P(y)), \quad \cdots \text{(ii)}$$

$$\forall x, y(P(x) \land S(x) \land S(y) \supset P(y)) \quad \cdots \text{(iii)}$$

with which (A.1.3) and (A.2.3) can be deduced from their premises, respectively. Now let us assume an inferential schema, *preduction*, which can conclude each R-expansile sentence of (i), (ii), and (iii) from the premises of (I.2), (A.1), and (A.2), respectively. Then by this new schema, Induction (I.2) can be broken down into two steps; *1)* from the premises (I.2.1) and (I.2.2), conclude the expansile sentence (i) by preduction, and *2)* from (I.2.1) and (i), conclude (I.2.3) by Induction (I.1). Each of Analogy (A.1) and (A.2), on the other hand, becomes a derivative rule from preduction and deduction; *1)* from the premises, conclude each R-expansile sentence (ii)/(iii) by preduction, and *2)* from (ii)/(iii) together with their premises, conclude (A.1.3)/(A.2.3) by deduction, respectively. Thus, preduction is a common inference of (I.2), (A.1), and (A.2) (Figure 1).

3 Preduction

Preduction is a formal representation of the following concept: *"If every entity known to be P can be traced back to some roots of P along a relation R, then the unknown descendants of their roots will satisfy P similarly. "* Here, a set of *roots* corresponds to a set which includes a base case/an example X_0 in the previous section.

Before describing a form of the concept, we introduce some terminology. For a transitive relation $\prec$ and a pair of objects e and e', when $e \prec e'$, we say e is an *ancestor* of e' and e' is a *descendant* of e with respect to $\prec$. An n-ary *predicate* U is generally expressed by $\lambda x Q$, where x is a tuple of n object variables, Q is a sentence in which no object variables except variables in x occur free.

For a $2 \cdot n$-ary predicate $.R.$ and $. \prec_R .$, let $Tr(R; \prec_R)$ express that $\prec_R$ is the transitive closure of R, that is, $\prec_R$ is the minimal predicate which satisfies

$$\forall x, y(xRy \supset x \prec_R y)$$
$$\land \forall x, y, z(x \prec_R z \land z \prec_R y \supset x \prec_R y).$$

Let P be an n-ary predicate symbol and A a first order sentence. Let $. \prec .$ be a $2 \cdot n$-ary transitive predicate in which P does not occur. The *preductive sentence* of P on $\prec$ in A, written $Pd(A; P; \prec)$, is

$$\exists \Phi(\forall x(\Phi(x) \supset P(x)) \land \exists x, y(\Phi(x) \land x \prec y \land P(y))$$
$$\land A[\lambda x(\Phi(x) \lor \exists z(\Phi(z) \land z \prec x))])$$
$$\supset \forall x, y(P(x) \land x \prec y \supset P(y)),$$

where $A[\lambda x(\Phi(x) \lor \exists z(\Phi(z) \land z \prec x))]$ expresses the result of substituting the predicate $\lambda x(\Phi(x) \lor \exists z(\Phi(z) \land z \prec x))$ for all occurrences of P in A.

In the left side of the implication of the preductive sentence, a predicate variable Φ represents a concept of *root* of P-things with respect to a relation $\prec$. The first conjunct expresses that Φ is a sub-class of P. The second conjunct confirms that there is an entity of Φ that is an ancestor (with respect to $\prec$) of an entity of P. The third conjunct expresses that P can be interpreted as a set of entities who have their roots in Φ. If a predicate Φ satisfies all these three conditions, $Pd(A; P; \prec)$ tells us every $\prec$-descendants of an entity satisfying P satisfies P similarly.

Let P do not occur in R. Then, the *preduction of P on R from A* is the sentence,

$$A \land Tr(R; \prec_R) \land Pd(A; P; \prec_R),$$

denoted by $Preduct(A; P; R)$, where $\prec_R$ is a new predicate symbol which does not occur in A.

Preduction is guaranteed to maintain consistency, which will be proved in Section 5. The following properties are straightforward and used in the next section. The conclusion of a preductive sentence, $\forall x, y(P(x) \land x \prec_R y \supset P(y))$, implies the R-expansile sentence of P, $\forall x, y(P(x) \land xRy \supset P(y))$, because of $Tr(R; \prec_R)$. $Tr(R; \prec_R)$ can be expressed formally by a second order sentence which *circumscribes* $\prec_R$ alone in the above two sentences of $Tr(R; \prec_R)$ [McCarthy, 1980]. If R is transitive, the transitive closure of R is R itself. Thus, in this case $Preduct(A; P; R)$ is simplified to $A \land \forall x, y(xRy \equiv x \prec_R y) \land Pd(A; P; R)$.

4 Examples: Induction and Analogy

In the following examples, we implicitly use a classical logic with equality axioms and with the *unique name axioms* [Clark, 1978] which tells us that different ground terms denote different objects (or you may always add these axioms to A_i in these examples).

4.1 Induction

Example 1. Let A_1 and S be

$$A_1 \equiv N(0) \wedge N(s(s(0)))$$

and

$$xSy \equiv y = s(x).$$

Let $\prec_S$ denote the transitive closure of S by $Tr(S;\prec_S)$. Then, $Pd(A_1;N;\prec_S)$ is

$$\begin{aligned}&\exists\Phi(\ \forall x(\Phi(x) \supset N(x))\\ &\wedge\exists x,y(\Phi(x) \wedge x \prec_S y \wedge N(y))\\ &\wedge(\Phi(0) \vee \exists z(\Phi(z) \wedge z \prec_S 0))\\ &\wedge(\Phi(s(s(0))) \vee \exists z(\Phi(z) \wedge z \prec_S s(s(0)))))\\ &\supset \forall x,y(N(x) \wedge x \prec_S y \supset N(y)).\end{aligned}$$

The sentence scoped by $\exists\Phi$, if substituted $\Phi(x) \equiv x = 0$ ("x is a root iff x is 0"), becomes

$$N(0) \wedge \exists y(0 \prec_S y \wedge N(y)) \wedge 0 \prec_S s(s(0))$$

and follows from $A_1 \wedge Tr(S;\prec_S)$. Thus, as the left side of the implication of $Pd(A_1;N;\prec_S)$ hold, the preduction of P on S from A_1 concludes

$$\forall x,y(N(x) \wedge x \prec_S y \supset N(y)),$$

and, by $\forall x,y(y = s(x) \supset x \prec_S y)$ (in $Tr(S;\prec_S)$) and by the equality axioms,

$$\forall x(N(x) \supset N(s(x))).$$

As this example shows, the preduction from "0 and 2 are natural numbers" gives "the successor of each natural number is a natural number" and thus, $N(1)$, $N(3)$, $\cdots$ are also theorems of the production. It shows the production expands the extension of N.

The production is non-monotonic with respect to A, that is, the theorems of some production from A does not always include the theorems of the production from a sub-set theory of A. The following shows an example where a theorem of a production from a sub-set A_1 of A_2 is not the theorem from A_2.

Example 2. Let $A_2 \equiv A_1 \wedge \neg N(s(0))$ and S is the same as in Example 1. $Pd(A_2;N;\prec_S)$ is the result obtained just by attaching the following sentence to the left side of the implication of $Pd(A_1;N;\prec_S)$ in the scope of $\exists\Phi$;

$$\neg(\Phi(s(0)) \vee \exists z(\Phi(z) \wedge z \prec_S s(0))),$$

which, by distribution of the negation, becomes

$$\neg\Phi(s(0)) \wedge \neg\exists z(\Phi(z) \wedge z \prec_S s(0))).$$

This second conjunct telling us "any less than 1 does not satisfy Φ". This contradicts $(\Phi(0) \vee \exists z(\Phi(z) \wedge z \prec_S 0))$ in the left side of the implication of $Pd(A_1;N;S)$ by the transitivity of $\prec_S$. Thus, for Φ, no predicate satisfies the left side of the implication of $Pd(A_2;N;S)$, and $Pd(A_2;N;S)$ is seen to be tautology.

Example 3. In Inductive Logic Programming (ILP), the inference of the following form is most commonly* used in generalizing a clause (*absorption* [Muggleton, 1991], *saturation* [Rouveirol and Puget, 1990J, *folding* [Lu and Arima, 1996], and etc.):

$$\begin{array}{ll}\text{Induction(I.3)} & \\ \text{(I.3.1)} & \forall x(\alpha(x) \supset P(x))\\ \text{(I.3.2)} & \forall x(\alpha(x) \supset \beta(x))\\ \hline \text{(I.3.3)} & \forall x(\beta(x) \supset P(x)),\end{array}$$

where P is a predicate symbol and α, β are conjunctions of literals. Unfortunately, this rule also does not maintain consistency. Thus, to check consistency is always necessary to discard an inconsistent conclusion produced by this rule. (It corresponds to *over-generalization* in ILP where a *negative example* e of P is covered by the newly obtained generalized clause H, that is, $A \vdash \neg P(e)$ but $A, H \vdash P(e)$.) The preduction can work as a consistent version of this rule.

Proposition 1 *Let P do not appear in β. Let xBy be $\beta(x) \wedge \beta(y)$. Then the following is a theorem of $Preduct(A;P;B)$:*

$$\begin{aligned}&\forall x(\alpha(x) \supset P(x)) \wedge \forall x(\alpha(x) \supset \beta(x))\\ &\wedge\exists x\alpha(x) \wedge\ A[\beta]\\ &\supset \forall x(\beta(x) \supset P(x)),\end{aligned}$$

Proof. B is transitive. Thus, $Pd(A;P;B)$ is a theorem of $Preduct(A;P;B)$. Assuming $\forall x(\alpha(x) \supset P(x))$, $\forall x(\alpha(x) \supset \beta(x))$, and $\exists x\alpha(x)$, it is sufficient to prove that $A[\beta] \supset \forall x(\beta(x) \supset P(x))$ is a theorem of $Pd(A;P;B)$.

Consider a preductive sentence $Pd(A;P;B)$ where α substituted for Φ. The first condition of the sentence is just the same as the first assumption $\forall x(\alpha(x) \supset P(x))$. The second condition,

$$\exists x,y(\alpha(x) \wedge \beta(x) \wedge \beta(y) \wedge \alpha(y)),$$

is equivalent to $\exists x(\alpha(x) \wedge \beta(x))$, which follows from the second and the third assumptions. The third condition of the preductive sentence becomes simply $A[\beta]$ under the second and the third assumptions, because

$$(\alpha(x) \vee \exists z(\alpha(z) \wedge \beta(z) \wedge \beta(x))) \equiv \beta(x).$$

Consequently, the conditions of the preductive sentence is equivalent to $A[\beta]$. Also, these assumptions give

$$\exists x(P(x) \wedge \beta(x)),$$

which simplifies the conclusion of the preductive sentence $\forall x,y(P(x) \wedge \beta(x) \wedge \beta(y) \supset P(y))$ to $\forall x(\beta(x) \supset P(x))$.

As Proposition 1 shows, the preduction proposes two more conditions than the two premises of Induction(I.3). The former condition $\exists x\alpha(x)$ relates to the *justification* of induction (I.3). If $\exists x\alpha(x)$ is not required like Induction(I.3), it allows the case $A \vdash \neg\exists x\alpha(x)$, where (I.3.1) and (I.3.2) hold always, because these preconditions are always false. Thus Induction(I.3) yields, no matter what β is, $\forall x(\beta(x) \supset P(x))$. There would be no reason to justify such an inference. The latter condition keeps consistency. This will be shown in Theorem 2.

4.2 Analogy

A consistent version of Analogy (A.2) is also deduced from a preduction. In the next proposition, as in (A.2), *S* corresponds to similarity and *P* a projected property by the similarity.

Proposition 2 *Let P do not appear in S. Let xCy be* $S(x) \wedge S(y)$. *Then the following is a theorem of* $Preduct(A;P;C)$:

$$\exists z(P(z) \wedge S(z)) \wedge A[S] \supset \forall x(S(x) \supset P(x)),$$

Proof. C is transitive. Thus, $Pd(A;P;C)$ is a theorem. Assuming $P(v) \wedge S(v)$, consider the sentence obtained by substituting $\lambda x(x = v)$ for Φ in $Pd(A;P;C)$. The first condition $\forall x(x = v \supset P(x))$ of the sentence is satisfied because $P(v)$, and the second condition $\exists y(S(v) \wedge S(y) \wedge P(y))$ is satisfied because of v as y. The third condition becomes simply $A[S]$, because $(x = v \vee vCx) \equiv S(x)$ from $S(v)$. The conclusion of the preductive sentence is simplified to $\forall x(S(x) \supset P(x))$ because $\exists x(P(x) \wedge S(x))$ from the assumption. Consequently, $Preduct(A;P;C) \wedge P(v) \wedge S(v)$ gives

$$A[S] \supset \forall x(S(x) \supset P(x)),$$

which proves the proposition.

Example 4. Let A_3 be

$$human(t) \wedge mortal(b) \wedge human(b).$$

We define *C* by their similarity, *human*(= *S*), that is,

$$xCy \equiv human(x) \wedge human(y).$$

Then, as Proposition 2 shows, $Preduct(A_3; mortal; C)$ yields

$$\exists z(mortal(z) \wedge human(z)) \wedge A_3[human]$$
$$\supset \forall x(human(x) \supset mortal(x)).$$

Substituted the target case b for z, the left side of the implication is

$$mortal(b) \wedge human(b)$$
$$\wedge human(t) \wedge human(b) \wedge human(b)$$

which follows from A_3. Thus, $Preduct(A_3; mortal; C)$ yields

$$\forall x(human(x) \supset mortal(x)),$$

which tells us that every human is mortal. Therefore, the preduction of *mortal* on *C* can derive an analogical conclusion *mortal(t)* by the fact *human(t)*.

Attempts to understand analogical reasoning are rigorously continued in Philosophy, Cognitive Science, and Artificial Intelligence[Helman, 1988]. Theschemas (A.1) and (A.2) are too simple description of the process of analogy. Davies and Russell clearly argue that there should be more premises in their schemas by posing the following example: we will not infer that one *(t)* of two cars of the same model (5) is painted red *(P)* just because the other (b) is painted red, although we may guess a price (P') of the one just because the other same model car is valued at the price (P')[Davies and Russell, 1987]. That is, we prefer the latter inference, ($P'(t)$ just for $P'(b) \wedge S(b) \wedge S(t)$) than the former ($P(t)$ just for $P(b) \wedge S(b) \wedge S(t)$) by the difference of the properties (P and P') although there is no difference in applying the schemas to the cases. The missing premise should be weaker than $\forall x(S(x) \supset P(x))$, because, if otherwise, $S(t)$ is enough for the conclusion $P(t)$ and thus information about the base case (b) becomes unnecessary. Davies and Russell proposed the following premise:

$$\forall p, s(\exists x(\Sigma(x,s) \wedge \Pi(x,p)) \supset \forall y(\Sigma(y,s) \supset \Pi(y,p))),$$

which allows a particular pair of s (as a similarity) and p (as a projected property) to work in analogical reasoning based on a base case (x). (e.g., $\Sigma = Model$ and $\Pi = Price$). They call this sentence *determination rule*.

Although weaker than $\forall x(S(x) \supset P(x))$, this preduction is not an answer for the query of the missing premise in analogical schema. Instead, preduction provides a consistent way to infer determination itself. Given a certain triplet, a sentence *A*, a predicate *P* and a relation *R*, the preduction can yield the determination rule as a theorem. This would be enough by showing the fact that the determination rule is an expansile sentence.

Example 5. In the R-expansile sentence, substituting $\lambda x, p\Pi(x,p)$ for P, $\lambda x, p, y, p' \exists s(\Sigma(x,s) \wedge \Sigma(y,s) \wedge p = p')$, for R, $\{x,p\}$ for x and $\{y,p'\}$ for y results in

$$\forall x, y, p(\Pi(x,p) \wedge \exists s(\Sigma(x,s) \wedge \Sigma(y,s)) \supset \Pi(y,p))$$

by the nature of =, which is arranged to the determination rule.

5 Model Theory

Let U and φ be sets such that $\varphi \subseteq U$. Let $\prec$ be a transitive relation on U. We say a set ψ is *the expansile set of* φ *with respect to* $\prec$ if ψ is the union of φ with the (ascending) segment by φ with respect to $\prec$, that is,

$$\psi = \varphi \cup \{e \mid \text{ there exists } z \in \varphi, \ z \prec e\}.$$

For a predicate P and a structure $\mathcal{M}$, we write $P^{\mathcal{M}}$ for the extension of P in $\mathcal{M}$. Let $\mathcal{M}$ and $\mathcal{N}$ be structures. For a predicate P and a predicate R in which P does not occur, we say that $\mathcal{M}$ is an *R-expansile structure of* $\mathcal{N}$ *in* P, if i) $\mathcal{M}$ and $\mathcal{N}$ have the same universe, ii) all other predicate symbols and function symbols besides P have the same extensions in $\mathcal{M}$ and $\mathcal{N}$ (thus, $R^{\mathcal{M}} = R^{\mathcal{N}}$), iii) but, for some subset ϕ of $P^{\mathcal{N}}$ such that an entity $e \in \phi$ and an entity $e' \in P^{\mathcal{N}}$ satisfy $e \prec_R e'$, $P^{\mathcal{M}}$ is the expansile set of ϕ with respect to $\prec_R$, where $\prec_R$ is the transitive closure of $R^{\mathcal{N}}$.

Let $\mathcal{M}$ be a model of A. We say that $\mathcal{M}$ is a *preductive model* of A in P on R, if i) any R-expansile structure of $\mathcal{M}$ is not a model of A, or otherwise ii) $\mathcal{M}$ is an R-expansile structure of a model of A.

Theorem 1 We write $A \models_R^P f$ if a sentence f is true in all preductive models of A in P on R. Then,

$$A \models_R^P f \text{ if } Preduct(A; P; R) \vdash f$$

Proof. Let $\mathcal{M}$ be a preductive model of A in P on R. Let φ be a predicate which, for the transitive closure $\prec_R$ of R, satisfies the left side of the implication of $Pd(A; P; \prec_R)$ when substituted for Φ. Then the extension of φ is a subset of the extension of P and includes an entity which is an ancestor of an entity of P with respect to $\prec_R$, and its expansile set with respect to $\prec_R$ is an alternative extension of P. Thus, there is a model of A that is an R-expansile structure of $\mathcal{M}$. It implies that $\mathcal{M}$ should also be an R-expansile structure of a model of A by the definition of the preductive model. If the right side of $Pd(A; P; R)$ were not true on $\mathcal{M}$, there would exist a pair of entities, (e_1, e_2), such that they satisfies R and that e_1 is an entity of P but e_2 is not of P. This would contradict that $\mathcal{M}$ is an R-expansile structure.

Theorem 2 *(consistency): A predicate P does not occur in a predicate R. If A is consistent, the preductive sentence $Preduct(A; P; R)$ is consistent.*

Proof. Assume that A is consistent but that $Preduct(A; P; R)$ is inconsistent. $A \wedge Tr(R; \prec_R)$ is consistent because $Tr(R; \prec_R)$ just defines a new predicate $\prec_R$ to be the transitive closure of R. By the assumption, for any model $\mathcal{N}$ of $A \wedge Tr(R; \prec_R)$, $\mathcal{N} \models \neg Pd(A; P; \prec_R)$. Thus, because the left side of $Pd(A; P; \prec_R)$ is true on $\mathcal{N}$, there is a predictive model of $A \wedge Tr(R; \prec_R)$ that is an R-expansile structure of $\mathcal{N}$. Let this be $\mathcal{M}$. Now, by the assumption, the negation of the right side of $Pd(A; P; \prec_R)$ is also true on $\mathcal{M}$, that is,

$$\mathcal{M} \models \exists x, y(P(x) \wedge x \prec_R y \wedge \neg P(y)).$$

$\mathcal{M}$ agrees with $\mathcal{N}$ on all symbols (including R and $\prec_R$) except P and, on P, agrees with the expansile set of the extension of Φ on $\mathcal{N}$ with respect to $\prec_R$. Therefore, letting $\Phi^{\mathcal{N}} = \phi^{\mathcal{M}}$,

$$\mathcal{M} \models \exists x, y((\phi(x) \vee \exists z(\phi(z) \wedge z \prec_R x)) \wedge x \prec_R y \wedge (\neg\phi(y) \wedge \neg\exists z(\phi(z) \wedge z \prec_R y))).$$

$$\mathcal{M} \models \exists x, y((\phi(x) \wedge x \prec_R y \vee \exists z(\phi(z) \wedge z \prec_R x \wedge x \prec_R y)) \wedge \neg\exists z(\phi(z) \wedge z \prec_R y)).$$

The both disjuncts in the first conjunct contradict the second conjunct by the transitivity of $\prec_R$. Thus, there is no such a model $\mathcal{M}$. This contradicts the assumption.

6 Conclusion

This paper proposes a common form of analogy and (empirical) induction. The form, called *preduction,* preserves consistency and brings a view that analogy and induction come from the same type of inference and diverge depending on types of sequent inference, deduction and mathematical induction, respectively. The generality of this form is verified by its broad application ranging over analogy and induction in the logical approach of artificial intelligence. Although the form will not contribute directly to a design of a general inference machine, we hope that this promotes us to devise it by better understanding logical aspects of analogy and induction.

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