# On the Complexity of Model Checking for Propositional Default Logics: New Results and Tractable Cases

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## Abstract

We analyse the complexity of standard and weak model checking for propositional default logic; in particular, we solve the open problem of complexity in case of normal default theories and introduce a new ample class of default theories with a tractable model checking problem.

## 1 Introduction and Overview of Results

The complexity of default reasoning is already well understood, however, in search of model-based representations, the complexity of the model checking problem instead of the inference problem needs to be analysed.

As Halpern and Vardi [1991] argue, model checking is a beneficial alternative simplifying reasoning tasks (for instance, in classical propositional logic, model checking can be done using an easy polynomial algorithm, however reasoning is coNP-complete) and allowing for representing the agent's knowledge as a semantic structure instead of a collection of formulae and additionally, this approach introduces a kind of closed-world assumption. Furthermore, the complexity of model checking is closely related to the notion of *representational succinctness* [Gogic *et al.,* 1995] of non-monotonic formalisms.

### 1.1 Complexity of Inference

Gottlob [1992] and Stillman [1992] showed that the complexity of *brave (cautious) reasoning*, i.e. to decide, given a formula $f$ and a default theory $\langle D, W\rangle$, if / is in at least one (all) extension(s) of $\langle D, W\rangle$ is $\Sigma_2^P$-mplete (In-complete), even in case of normal and prerequisite-free default theories, and even if $f$ is a single literal. For related results, see [Papadimitriou and Sideri, 1992].

The complexity decreases one level if *disjunction-free* default theories are considered, i.e. only conjunctions of literals and negated literals are allowed. Kautz and Selman [1991] dealt with the inference problem for such theories: Brave reasoning (for disjunction-free formulae) is NP-complete, even in the case of normal default theories (although finding an extension is polynomial in that case) and even if the formula to be inferred of at least one extension is a single literal. If *W* is a Horn theory, NP-completeness holds even in the case of prerequisite-free normal default theories [Stillman, 1990].

### 1.2 Complexity of Model Checking

An interpretation is a model of a default theory iff it satisfies at least one extension of the theory. Liberatore and Schaerf [1998] show that model checking is $\Sigma_2^P$-complete, even for semi-normal prerequisite-free default theories. In the case of normal default theories, model checking is easier than the corresponding reasoning task - they show that the problem is in $\Delta_2^P$ and $\Delta_2^P[O(\log n)]$-hard, and coNP-complete if defaults are also prerequisite-free.

In general, model checking suffers from two sources of hardness: On the one hand there are $2^{|G|}$ (with $G = \{d \mid \mathcal{M} \models c(d)\}$) possible sets of generating defaults, and the other source of intractability is the hardness of propositional inference. In case of normal default theories, given a particular model, only one subset of *G* needs to be considered and therefore the initial guessing stage is eliminated. In Theorem 4.1 we show with a non-trivial membership proof that this problem is in $\Delta_2^P[O(\log n)]$ and hence due to earlier results $\Delta_2^P[O(\log n)]$-complete. To obtain this theorem we improve techniques of Gottlob [1995] for guessing data-structures.

If the defaults are restricted in such a way that propositional satisfiability and inference are polynomial, the other source of intractability is affected and the problem is due to the necessary guessing of generating defaults NP-complete. If such a default theory is restricted to normal defaults, complexity of model checking is even polynomial. Therefore, in Chapter 5, we introduce the class of *default theories in extended Horn normal form* (abbreviated as "EHNF default theories"), a class containing disjunction-free default theories, for which model checking is still one level easier than for arbitrary default theories. A default theory (D, *W)* is in EHNF iff *W* and all elements of each justification are disjunctions of Horn theories, each prerequisite is a conjunction of dual Horn theories, and each consequent is a Horn theory. This is an ample class of default theories with a tractable model checking problem and hence very useful in practical applications.

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| | General/Semi-N. | Normal |
|---|---|---|
| General | $\Sigma_2^P$-complete* | $\Delta_2^P[O(\log n)]$-cpl. |
| Prerequisite-free | $\Sigma_2^P$-complete* | coNP-complete* |
| EHNF | NP-complete | P-complete |
| EHNF/Prer.-free | NP-complete | P-complete |

**Table 1: Complexity of Model Checking**

| | General/Semi-N. | Normal |
|---|---|---|
| General | $\Sigma_2^P$-complete | $\Sigma_2^P$-complete |
| Prerequisite-free | $\Sigma_2^P$-complete | coNP-complete |
| EHNF | NP-complete | NP-complete |
| EHNF/Prer.-free | NP-complete | P-complete |

Table 2: Complexity of Weak Model Checking

### 1.3 Weak Model Checking and AEL

In Chapter 6 we recall the notion of *weak extensions* and show that *weak model checking,* i.e. deciding if an interpretation satisfies at least one weak extension of a default theory, is, due to the non-constructive nature of the problem, even $\Sigma_2^P$-complete for normal default theories and hence strictly harder than model checking. This issue is also connected to the fact that no modular translation from default logic into autoepistemic logic (AEL) exists [Gottlob, 1995b], since prerequisites are treated in a very different way. The objective parts of stable expansions (N-expansions) of the translated default theory correspond to weak extensions (extensions) of the default theory, therefore we obtain the complexity of model checking for AEL (nonmonotonic logic N) in Chapter 7.

### 1.4 Summary

In Table 1 and Table 2 a summary of our results for model checking and weak model checking with Reiter's default logic is presented (the results already present in [Liberatore and Schaerf, 1998] are marked with *). Additionally, the main contributions of this paper are:

- We solve the open problem of the exact complexity of model checking for normal default theories.
- We introduce a new ample class of default theories with a tractable model checking problem.
- We show that weak model checking is $\Sigma_2^P$-complete, even if restricted to normal default theories.
- We generalize these results to $\Sigma_2^P$-completeness of model checking with AEL and N.
- Finally, in Chapter 8 the complexity results of model checking are used to draw some interesting conclusions in translatability issues.

## 2 Basic Concepts

A prepositional default theory [Reiter, 1980] is a pair *(D, W)* where *W* is a finite set of propositional sentences and *D* is a finite set of defaults. Whenever we use the term *default theory* in the rest of the paper, we mean propositional default theory.

*A default* d is a configuration of the form $\frac{p(d):\beta_1,\ldots,\beta_n}{c(d)}$ where $p(d), \beta_i, c(d)$ are propositional sentences. *p(d)* is called the prerequisite of d, the (non-empty) set $\{\beta_1,\ldots,\beta_n\}$ is referred to as the justification of *d* and denoted by $j(d)$; $c(d)$ is called the consequent of the default *d.* For convenience we define *c(H)* $\{c(d) \mid d \in H\}$ and if $j(d)$ is a singleton we identify it with its only element.

Since Reiter's original definition of extensions [1980] a great number of equivalent characterizations has been introduced. In this paper we normally use a finite quasi-inductive characterization, based on the operator $B^D$ due to Marek and Thuszczyfoki [1993].

We define *cons(A)* as usual as $\{\Phi \mid A \models \Phi\}$. Let *H* be a subset of *D:* $H_0 = \emptyset, H_{k+1} =$

$$\{d \in D \mid W \cup c(H_k) \models p(d), \forall \beta_i \in j(d)\ W \cup c(H) \not\models \neg\beta_i\}.$$

As we limit ourselves to finite default theories, it can easily be seen that at latest after $|D|$ steps a fixed point has been reached. $E = cons(W \wedge c(H_{|D|}))$ is an extension of (D, *W)* iff $H = H_{|D|}$. Every extension is of the form $E = cons(W \cup c(GD(D, E)))$ with *GD* being called the generating defaults of the extension.

In a *normal default theory j(d)* = c(d) for each default. A *semi-normal default theory* is a theory in which each $j$(d) is of the $f(d) \wedge c(d)$ where $f(d)$ are arbitrary propositional formulae.

Definition 2.1 *An interpretation (valuation)* $\mathcal{M}$ *is a model of* $\langle D, W\rangle$ *iff* $\mathcal{M}$ *is model of at least one (consistent) extension of* $\langle D, W\rangle$.

Whenever we use the term *model* we refer to *propositioned Herbrand model* The *Model Checking* problem for default logic is to decide, given an interpretation $\mathcal{M}$ and a default theory $\langle D, W\rangle$, if $\mathcal{M} \models \langle D, W\rangle$.

Short Review of Relevant Complexity Concepts: The notion of completeness we employ is based on many-one polynomial transformability. Recall that the classes $\Delta_k^P, \Sigma_k^P$ and $\Pi_k^P$ of the polynomial time hierarchy (PH)[1] are defined as $\Delta_0^P = \Sigma_0^P = \Pi_0^P = \mathrm{P}$ and for all $k \geq 0$, $\Delta_{k+1}^P = \mathrm{P}^{\Sigma_k^P}, \Sigma_{k+1}^P = \mathrm{NP}^{\Sigma_k^P}, \Pi_{k+1}^P =$ co$-\Sigma_{k+1}^P$. In particular, $\mathrm{NP} = \Sigma_1^P, \mathrm{coNP} = \Pi_1^P$ and $\Delta_2^P = \mathrm{P}^{\mathrm{NP}}$ (see [Papadimitriou, 1993] for details). $\Delta_2^P$ is the class of decision problems that are solvable in polynomial time on a deterministic oracle Turing machine calling an NP-oracle polynomially often.

The classes $\Delta_k^P$ have been refined, depending of how many oracle calls are needed: Of special interest in this paper is the class $\Delta_2^P[O(\log n)]$, also known as $\mathrm{P}^{\mathrm{NP}[O(\log n)]}$ - this is the class of decision problems solvable with a logarithmic number of calls to an NP-oracle.

A survey on already known complexity results for several nonmonotonic logics can be found in [Cadoli and Schaerf, 1992].

[1]We always implicitly assume that $\mathrm{P} \neq \mathrm{NP}$ and that the PH does not collapse.

## 3 Some Useful Tools

Due to lack of space, some proofs in the following chapters are sketched or omitted.

**Lemma 3.1** *Let $\langle D, W\rangle$ be a normal default theory. If $W \wedge c(D)$ is consistent, then there are no mutually incompatible defaults and thus only one generating set.*

**Lemma 3.2** *Let $\langle R, W\rangle$ be a monotonic rule-system (or a normal or justification-free default theory) where W and the consequents $b_i$ of $R = \{\frac{a_1}{b_1}, \dots \frac{a_n}{b_n}\}$ are jointly consistent. Then the statement: "At least $k$ rules do not fire" is equivalent to "It is possible to choose a set $B \subseteq R$ with $|B| = k$ and $k$ interpretations $\mathcal{I}_1, \dots \mathcal{I}_k$ in such a way that $\forall j (1 \leq j \leq k) : \mathcal{I}_j \models W, \forall j\ \forall b \in R \backslash B : \mathcal{I}_j \models b, \forall a \in B\ \exists j : \mathcal{I}_j \not\models a.$"*

The following proposition formalizes a technique (binary search) that is well known in the literature [Wagner, 1990; Papadimitriou, 1993].

Proposition 3.3 *Let $a(I_1, \dots, I_n)$ be a function of $n$ instances of a problem and $a(I_1, \dots, I_n)$ be polynomially bounded in n, i.e. $a(I_1, \dots, I_n) \leq p(n)$. If the problem of deciding $a(I_1, \dots, I_n) \leq r$ (or $a \geq r$) is in* NP, *then the computation of $a(I_1, \dots, I_n)$ is in $\Delta_2^P[O(\log n)]$.*

Proposition 3.4 *M is a model of a default theory {D,W} iff there exists an extension of (D,W) generated by a subset $G_1 \subseteq G$ where $G = \{d \mid \mathcal{M} \models c(d)\}$. In a normal default theory at most one extension E such that $\mathcal{M} \models E$ exists.*

Proposition 3.5 *Let E be an extension of the default theory (G, W) and G $\subseteq$ D. Then, E is an extension of $\langle D, W\rangle$ iff for each default in $D \backslash G$ it holds that $W \wedge c(GD(G, E)) \not\models p(d)$ or $W \wedge c(GD(G, E)) \models \neg\beta_i$ for at least one $i$ with $\beta_i \in j(d)$.*

## 4 Model Checking for Normal Default Theories is $\Delta_2^P[O(\log n)]$-complete

Theorem 4.1 *Let $\mathcal{M}$ be an interpretation and $\langle D, W\rangle$ be a normal default theory. Deciding whether $\mathcal{M} \models \langle D, W\rangle$ is in $\Delta_2^P[O(\log n)]$.*

Proof. We describe a Turing machine *M* which decides this problem in polynomial time using an NP-oracle for only O(logn) times where n is the number of defaults. *M* works in four steps.

<u>*Step 1:*</u> *M* rejects if $\mathcal{M} \not\models$ *W.* Let *G* $\subseteq$ *D* be the set of all defaults *d* with $\mathcal{M} \models$ c(d). This step determines *G* by sorting out all unwanted ("bad") defaults *B = D\G.* None of the defaults of *B* shall fire or $\mathcal{M}$ is not a model of this extension (Proposition 3.4). Checking if $\mathcal{M} \models$ *W* and constructing *G* can be achieved in polynomial time. In the following, let *m* be the cardinality of *G.*

<u>*Step 2:*</u> *M* computes the cardinality of the set *G* of generating defaults of the extension, i.e. those defaults in *G* which are applicable because their prerequisites can be inferred; due to Lemma 3.1, G1 is unique. *M* assumes that *E = cons{W* U *c{G1))* is an extension, and in step 3 and 4 *M* will verify if no "bad" defaults have to be used. To compute the cardinality of *G1*, *M* determines the number $|G_2| = |G \backslash G_1|$ of defaults that do not fire. We can identify $\langle D, W\rangle$ with a monotonic rule-system $\langle R, W\rangle$ in which all consequents are jointly consistent.

**Claim** The problem to decide if at least $r$ rules of a monotonic rule-system (in n rules) do not fire is in NP.

*Proof of Claim:* Machine *M* guesses a data-structure $\langle G_2', \{\mathcal{I}_1, \dots, \mathcal{I}_r\}\rangle$ with $|G_2'| = r$. $G_2'$ is a set of defaults (rules), the $\mathcal{I}_j (1 \leq j \leq r)$ are interpretations, and proves in polynomial time (using the monotonic rule system syntax of Lemma 3.2): $\forall j : \mathcal{I}_j \models W$, $\forall j\ \forall b \in R \backslash G_2' : \mathcal{I}_j \models b, \forall a \in G_2'\ \exists j : \mathcal{I}_j \not\models a$. Due to Lemma 3.2, this is equivalent to the question if at least $r$ rules do not fire, i.e. if $|G_2| \geq r$. ◇

From Proposition 3.3 the number $t$ of defaults which do not fire can be computed in polynomial time using O(logm) calls to an NP-oracle. After concluding this step *M* knows $|G_2| = t$, hence $|G_1| = m - t$.

<u>*Step 3:*</u> If it can be shown that $W \cup c(G_1) \models p(d)$ and $W \wedge c(G_1) \wedge j(d)$ is consistent for at least one *d* є B, then *G1* is not a set of generating defaults and the given interpretation $\mathcal{M}$ is not a model for *(D, W)* (Proposition 3.5). Two types of "bad" defaults need to be distinguished: *B* is consisting of defaults in which the prerequisites are not applicable, formally $B_1 = \{d \in B \mid W \cup c(G_1) \not\models p(d)\}$; and $B_2 = B \backslash B_1$. *M* determines the exact number $u$ of defaults in *B* and in the fourth step will check for each default in *B2,* if the justifications are consistent with $W \wedge c(G_1)$ (after guessing the right *G1* and *B1).*

**Claim** The problem to decide, given $|G_1|$, if the number of defaults in *B1* is at least *s,* is in NP.

*Proof of Claim:* For a given ,s, *M* guesses a data-structure $\langle G_1 \subseteq G, \{\mathcal{N}_1, \dots, \mathcal{N}_t\}, B_1' \subseteq B, \{\mathcal{O}_1, \dots, \mathcal{O}_s\}\rangle$ with $|G_1| = m - t$, $|B_1'| = s$ and $\mathcal{N}_e$ and $\mathcal{O}_y$ are interpretations. Now *M* proves:

- $\forall d \in G_2\ \exists \mathcal{N}_e (1 \leq e \leq t) : \mathcal{N}_e \not\models p(d), \forall e : \mathcal{N}_e \models W \wedge c(G_1)$ (In step 2 *M* was provided with the information of the cardinality of *G1*, therefore *M* now just has to guess a *G* of suitable cardinality and check if it is the right one.).
- $\forall d \in B_1'\ \exists \mathcal{O}_y (1 \leq y \leq s) : \mathcal{O}_y \not\models p(d), \forall y : \mathcal{O}_y \models W \wedge c(G_1)$. If it is possible to choose the $\mathcal{O}_y$ this way, then there are at least *s* not applicable prerequisites of defaults, hence $|B_1| \geq s$ (This statement is equivalent to $\forall d \in B_1' : W \cup c(G_1) \not\models p(d)$.).

Both steps can be concluded in polynomial time and are due to the initial guess in NP. ◇

Hence, $u = |B_1|$ can be computed with $\log(n - m)\rceil$ oracle calls (Proposition 3.3).

<u>*Step 4:*</u> In the previous steps *M* has determined the cardinality of $B_1$ and *G*. Now *M* finally has to check that $W \wedge c(G_1) \wedge j(d)$ is inconsistent for all defaults *d* є *B2* (if *yes,* then *G* is a generating set). To this aim, let us show that the converse problem is in NP and introduce a machine *N* solving it.

**Claim** The problem to decide if $W \wedge c(G) \wedge j(d)$ is consistent for at least one *d* є *B2,* given the cardinality of *G1* and *B1,* is in NP.

*Proof of Claim:* Machine $N$ guesses a data-structure $\langle G_1 \subseteq G, \{\mathcal{N}_1, \ldots, \mathcal{N}_t\}, B_1 \subseteq B, \{\mathcal{P}_1, \ldots, \mathcal{P}_u\}, \mathcal{Q}\rangle$ with $|G_1| = m - t$ and $|B_1| = u$. Now $N$ tests in polynomial

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- $\forall d \in G_2\ \exists \mathcal{N}_e (1 \leq e \leq t) : \mathcal{N}_e \not\models p(d), \forall e : \mathcal{N}_e \models W \wedge c(G_1)$.
- $\forall d \in B_1\ \exists \mathcal{P}_z (1 \leq z \leq u) : \mathcal{P}_z \not\models p(d), \forall z : \mathcal{P}_z \models W \wedge c(G_1)$.
- Now $N$ just needs to prove that $\exists d \in B_2 : \mathcal{Q} \models W \wedge c(G_1) \wedge c(d)$ (with $c(d) = j(d)$). This states the fact that there is a default which is consistent with *cons(W* ∪ c(G1)).

All steps can be concluded in polynomial time and are due to the initial guess in NP. ◇

This means that the output *yes* of machine $N$ is achieved iff there exist defaults of *B2* which fire (with guessing the only suitable *G1* and *B*, and $\mathcal{Q}$). The converse problem therefore needs merely one (negative) oracle call.

Hence, step 1 to 4 are feasible in polynomial time with $\lceil \log(m(n-1)) \rceil + 1 = O(\log n)$ calls to three different oracles in NP. With standard techniques these three oracles can be replaced with a single oracle. □

## 5 EHNF Default Theories

A default theory $\langle D, W\rangle$ is *disjunction-free,* if *W,* all *p(d),* all elements of each $j(d)$, and all consequents *c(d)* are conjunctions of (negated and not negated) literals. A *Horn clause* is a disjunction of literals with at most one positive literal. A *Horn theory* is a conjunction of Horn clauses. A *dual Horn clause* is a conjunction of literals containing at most one negative literal. A *dual Horn theory* is a disjunction of dual Horn clauses.

**Definition 5.1** *A default theory (D,W) is in extended Horn normal form (EHNF) iff W and all elements of each j(d) are disjunctions of Horn theories, each p(d) is a conjunction of dual Horn theories, and each consequent c(d) is a Horn theory. If (D,W) is normal, then, trivially, the only element of each j(d) is only a Horn theory, too.*

This is an ample class of default theories containing many restricted cases with an intractable reasoning problem even for normal default theories. However, model checking turns out to be tractable for normal EHNF default theories.

**Theorem 5.2** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D, W\rangle$ *an EHNF default theory. Deciding whether* $\mathcal{M} \models$ *(D,W) is* in **NP.**

**Proof.** (Sketch) Guess $G \subseteq D$, compute $G_{|D|}$, show that $G = G_{|D|}$ and $\mathcal{M} \models W \wedge c(G)$. All satisfiability checks are of the form *sat*$(W \wedge c(G) \wedge \beta_i)$ and due to the deduction theorem, the inference checks are equivalent to $\models \neg W \vee \neg c(G_k) \vee p(d)$. All these checks are polynomial, because satisfiability of a Horn theory and tautology checking of a dual Horn theory are tractable; furthermore, the question *sat*$(f_1 \vee \ldots \vee f_m)$ is equivalent to *sat*$(f_1)$ or ... or *sat*$(f_m)$ and *taut*$(f_1 \wedge \ldots \wedge f_m)$ is equivalent to *taut*$(f_1)$ and ... and *taut*$(f_m)$; therefore each check can be splitted to a quadratic number of checks in the number of disjunctions of $W$ and $p(d)$ resp. $j(d)$ (However, if we would have allowed the consequents to be *disjunctions of* Horn theories, too, the number of disjunctions would be exponential in the input length, even if only a logarithmic number of defaults would be allowed to have this form.). □

**Theorem 5.3** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D, W\rangle$ *be an EHNF default theory. Deciding whether* $\mathcal{M} \models \langle D, W\rangle$ *is* **NP-Ziard**, *even if* $\langle D, W\rangle$ *is prerequisite-free, semi-normal, disjunction-free and all consequents are single literals.*

**Proof.** (Sketch) Use a reduction of 3SAT. Transform $F(t_1, \ldots, t_m) = F_1 \wedge \ldots \wedge F_i \wedge \ldots \wedge F_n$ with $F_i = x_{i1} \vee x_{i2} \vee x_{i3}$ into $\langle D, W\rangle$ with $W = \emptyset$, $D = \bigcup_k \{\frac{:u_k \wedge \neg t_k}{\neg t_k}, \frac{:t_k \wedge \neg u_k}{\neg u_k}\} \cup \bigcup_i \{\frac{:f(x_{i1}) \wedge \neg y_i}{\neg y_i}, \frac{:f(x_{i2}) \wedge \neg y_i}{\neg y_i}, \frac{:f(x_{i3}) \wedge \neg y_i}{\neg y_i}, \frac{:y_i}{y_i}\}$ with $f(x_{ij}) = \neg t_k \wedge u_k$ if $x_{ij} = \neg t_k$ and $f(x_{ij}) = \neg u_k \wedge t_k$ if $x_{ij} = t_k$. Intuitively, the new atoms $u_k$ are introduced because the empty set shall be a model regardless to which evaluation of $(t_1, \ldots, t_m)$ satisfies *F.* Each extension has to contain, for each $i$, either $\neg y_i$ or $y_i$; iff *F* is satisfiable it can contain all $\neg y_i$, i.e. *F* is satisfiable iff $\emptyset \models \langle D, W\rangle$. □

Instead of using a prerequisite-free default theory, in the above proof an "almost" normal default theory with prerequisites could be used, however the first set of defaults of the above proof cannot be presented as normal defaults unless using weak extensions (Chapter 6) or *disjunctive default logic* of Gelfond *et al* [1991] (with $\forall k: \neg u_k | \neg t_k$).

We show that in the case of normal EHNF default theories, model checking is tractable, and moreover belongs to the hardest problems in P w.r.t. logspace-reductions.

**Theorem 5.4** *Let* $\mathcal{M}$ *be an interpretation and* (D, *W) be an EHNF normal default theory. Deciding whether* $\mathcal{M} \models \langle D, W\rangle$ *is in* P.

**Proof.** We describe the algorithm given in Figure 1: In the *while* loop, each default of the set G is tested and *G* is created. To this aim, the check $\models \neg W \vee \neg c(G_1) \vee p(d)$, where $W = W_1 \vee \ldots \vee W_k \vee \ldots \vee W_{dw}$ (each $W_k$ is a Horn theory, hence the negation a dual Horn theory) and each $p(d) = p_1(d) \wedge \ldots \wedge p_l(d) \wedge \ldots \wedge p_{dp(d)}(d)$, is splitted to $dw \times$ *dp* tautology checks of dual Horn theories, which is polynomial (the boolean functions *taut(dualHorntheory)* and *sat{Horntheory)* are assumed to be given). If $x = dw \times dp$, then each check was successful and the default is added to *G*. After at most $n$ steps, all defaults with applicable prerequisites w.r.t. *G* have been added to the generating set *G* of $\langle G, W\rangle$. To show, that this is a generating set of $\langle D, W\rangle$, check that no default of *D\G* is applicable, i.e. either the prerequisite cannot be inferred or the justification (which is in case of normal EHNF default theories only a singleton and only a Horn

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Input: M, an EHNF normal default theory ⟨D,W⟩.
Result: "Accept" if M ⊨ ⟨D,W⟩; "Reject" otherwise.

begin
G := {d | M ⊨ c(d)}; G1 := ∅; n := |G|; B := D\G;
dw := number of disj. of Horn theories in W; i := 0;
while i ≤ n do
  i := i + 1;
  for each d ∈ G do
    dp := no. of cnj. of dual Horn th. in p(d); x := 0;
    for k := 1 to dw
    for l := 1 to dp
      if taut(¬Wk ∨ ¬c(G1) ∨ pl(d)) then x := x + 1;
    rof; rof;
  if x = dw × dp then G1 := G1 ∪ {d} and G := G\{d};
  od;
od;
for each d ∈ B do
  dp := no. of cnj. of dual Horn th. in p(d); x := 0; y := 0;
  for k := 1 to dw
    for l := 1 to dp
      if taut(¬Wk ∨ ¬c(G1) ∨ pl(d)) then x := x + 1;
    rof;
    if sat(Wk ∧ c(G1) ∧ j(d)) then y := 1;
  rof;
  if x = dw × dp and y = 1 then reject and halt;
od;
accept;
end.
```

Figure 1: Deterministic Algorithm of Theorem 5.4

theory) is not consistent, which is implemented in the *for each* $d \in B$ loop. If $x = dw \times dp$ (expressing that the prerequisite is inferred) and $y = 1$ (expressing that $sat(W \wedge c(G_1) \wedge j(d))$, because one disjunction is satisfiable, i.e. the justification is consistent with *G\),* then at least one of the "bad" defaults has to be used and $\mathcal{M}$ is not a model. □

**Theorem 5.5** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D,W\rangle$ *be an EHNF normal default theory. Deciding whether* $\mathcal{M} \models \langle D,W\rangle$ *is P-hard, even if* $\langle D,W\rangle$ *is either disjunction-free or prerequisite-free.*

Proof. In the case of disjunction-free default theories use a reduction of the P-complete problem of reasoning in propositional not-free logic programming [Dantsin *et al*., 1997]. Let $\langle D,W\rangle$ be the default theory where each clause $b \leftarrow a_1 \wedge \ldots \wedge a_n$ of *P* is translated into a normal default $\frac{a_1 \wedge \ldots \wedge a_n : b}{b}$ and the default $\frac{:\neg g}{\neg g}$ is added which is feasible in logarithmic space since only one new default is added. Then $\mathcal{M}_{full} \models \langle D,W\rangle$ iff $g$ is in the stable model of *P.* In the case of prerequisite-free EHNF default theories use a reduction of the (essentially the same) P-complete problem if a Horn theory is satisfiable and use the default theory $\langle \emptyset, W\rangle$ with *W* being a Horn theory. □

If a normal default-theory is disjunction-free *and* prerequisite-free the problem no longer remains P-hard and can be shown to be in non-deterministic logspace.

## 6 Complexity of Weak Model Checking

Let $\langle D,W\rangle$ be a default theory and $H, J \subseteq D$. $J = \{d \in D \mid W \cup c(H) \models p(d), \forall \beta_i \in j(d)\ W \cup c(H) \not\models \neg\beta_i\}$. $E =$ *cons*$(W \wedge c(H))$ is called a *weak extension* of $\langle D,W\rangle$ iff $H = J$ (see, e.g. [Marek and Truszczynski, 1993]) and $H$ is the weak generating set of this weak extension.

In prerequisite-free default theories extensions and weak extensions coincide. $\mathcal{M}$ is a *weak model* of $\langle D,W\rangle$ iff $\mathcal{M}$ is a model of at least one (consistent) weak extension. Every extension is a weak extension, therefore every model is a weak model. However, the converse does not hold, since not every extension is a subset of a weak extension.

**Theorem 6.1** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D,W\rangle$ *be a default theory. Deciding if* $\mathcal{M}$ *is a weak model of* $\langle D,W\rangle$ *is* $\Sigma_2^P$*-complete, even if* $\langle D,W\rangle$ *is semi-normal and prerequisite-free.*

Lemma 3.1 does not hold for weak generating sets, and for normal default theories it is neither sufficient to look at the largest weak generating set nor at the smallest w.r.t. G, hence an exponential number of subsets needs to be considered:

**Theorem 6.2** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D,W\rangle$ *be a normal default theory. Then the problem to decide if* $\mathcal{M}$ *is a weak model of* $\langle D,W\rangle$ *is* $\Sigma_2^P$*- complete.*

Proof. (Sketch) We use a transformation from the $\Sigma_2^P$-complete problem $\mathrm{QBF}_{2,\exists}$, i.e. the problem of checking whether there exists an evaluation of the $p_i \in$ P, such that for each evaluation of the $q_j \in Q$ *F(P,Q)* is true, into $\langle D,W\rangle$ with $W = \emptyset$, $D =$

$$\bigcup_i \left\{ \frac{w \Rightarrow p_i : w \Rightarrow p_i}{w \Rightarrow p_i}, \frac{w \Rightarrow \neg p_i : w \Rightarrow \neg p_i}{w \Rightarrow \neg p_i} \right\} \cup \left\{ \frac{\neg w : v}{v}, \frac{: w \wedge \neg F}{w \wedge \neg F} \right\}$$

with *w* and $v$ being new atoms. In the full paper we show that $\emptyset \models \langle D,W\rangle$ iff $F \in \mathrm{QBF}_{2,\exists}$. □

**Theorem 6.3** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D,W\rangle$ *be a normal prerequisite-free default theory. Then the problem to decide if* $\mathcal{M}$ *is a weak model of* $\langle D,W\rangle$ *is* **coNP** *-complete.*

**Theorem 6.4** *Let* $\mathcal{M}$ *be an interpretation and* $\langle D,W\rangle$ *be an EHNF default theory. Then the problem to decide if* $\mathcal{M}$ *is a weak model of* $\langle D,W\rangle$ *is* NP*-complete even if the theory is disjunction-free and either semi-normal and prerequisite-free or normal, and P-complete if the theory is prerequisite-free and normal.*

## 7 Complexity of Model Checking with AEL and N

An interpretation $\mathcal{M}$ in the language $\mathcal{L}$ is a propositional AEL (N)-model of a set of premises in $\mathcal{L}_m$ ($\mathcal{L}$ extended with a modal operator) iff $\mathcal{M}$ satisfies an objective part of at least one stable expansion (N-expansion).

**Theorem 7.1** *Let* $\Sigma$ *be a set of premises in* $\mathcal{L}_m$ *and* $\mathcal{M}$ *be an interpretation in* $\mathcal{L}$. *Deciding if* $\mathcal{M}$ *is a propositional AEL (N)-model is* $\Sigma_2^P$*-complete.*

**Proof.** Hardness: Weak extensions (extensions) of a default theory correspond to objective parts of stable expansions (N-expansions) of the translated theory [Marek and TVuszczyriski, 1989; 1993].

Membership: In [Gottlob, 1992] it is shown that deciding if a formula is not occuring in all extensions is in $\Sigma_2^P$ (using the finitary characterization of Niemela [1991] with modal subformulae). Consider the propositional language $\{a_1, \ldots, a_n\}$ and any mode$\mathcal{M}$ontaining some of these atoms. Then $\mathcal{M} \models E$ iff $E \not\models l_1 \vee \ldots \vee l_n$ with $l_i = \neg a_i$ if $a_i \in \mathcal{M}$ and $l_i = a_i$ if $a_i \notin \mathcal{M}$. □

## 8 Translatability Issues

A pfm-function / : $A \mapsto B$ is a function embedding formalism *A* into formalism *B* fulfilling the additional criteria of polynomiality, faithfulness (extensions/expansions coincide in some way) and modularity (a propositional subtheory can independently be translated) [Janhunen, 1998; Gottlob, 1995b].

We refer to [Eiter and Gottlob, 1995] for details of disjunctive logic programming (DLP) (reasoning is as hard as in default logic) and to [Inoue and Sakama, 1993] for results that DLP can be pfm-embedded into default logic.

Theorem 8.1 *Unless the PH collapses, there exists no pfm-function embedding default logic, normal default logic or disjunction-free default logic into DLP.*

Proof. Assume the existence of a pfm-function embedding default logic into disjunctive logic programming; then a given default theory would admit the same models as a pfm-translation into a corresponding disjunctive program. However, as model checking with disjunctive logic programming is only coNP-complete, this implies that $\Sigma_2^P$ = coNP. Contradiction. □

## 9 Future Research

We plan to investigate the complexity of model checking for further interesting variants of default logic and other nonmonotonic logics. We hope that this will allow us to clarify a number of translatability issues, akin the result of Chapter 8.

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