Strategyproof and Fair Matching Mechanism for Union of Symmetric M-convex Constraints

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Abstract

In this paper, we identify a new class of distributional constraints defined as a union of symmetric M-convex sets, which can represent a variety of real-life constraints in two-sided matching settings. Since M-convexity is not closed under union, a union of symmetric M-convex sets does not belong to this well-behaved class of constraints in general. Thus, developing a fair and strategyproof mechanism that can handle this class is challenging. We present a novel mechanism called Quota Reduction Deferred Acceptance (QRDA), which repeatedly applies the standard DA mechanism by sequentially reducing artificially introduced maximum quotas. We show that QRDA is fair and strategyproof when handling a union of symmetric M-convex sets. Furthermore, in comparison to a baseline mechanism called Artificial Cap Deferred Acceptance (ACDA), QRDA always obtains a weakly better matching for students and, experimentally, performs better in terms of nonwastefulness.

1 Introduction

The objective of this paper is to identify a new class of distributional constraints in two-sided matching, where a strategyproof and fair mechanism exists. In a two-sided matching problem, two types of agents (e.g., students/schools, hospitals/residents) are matched [Roth and Sotomayor, 1990]. A standard matching market deals with maximum quotas (capacity limits). As the theory has been applied to diverse environments, mechanism designers have faced various forms of distributional constraints, including regional maximum quotas, which restrict the total number of students assigned to a set of schools [Kamada and Kojima, 2015], minimum quotas, which guarantee that a certain number of students are assigned to each school [Biro *et al.*, 2010; Fragiadakis *et al.*, 2016; Goto *et al.*, 2016; Hamada *et al.*, 2017a], and diversity constraints, which guarantee that a school satisfies a balance between different types of students [Hafalir *et al.*, 2013; Ehlers *et al.*, 2014; Kojima, 2012; Kurata *et al.*, 2017]. This topic has been attracting increasing attention from AI researchers [Aziz *et al.*, 2017; Hamada *et al.*, 2017b; Hosseini *et al.*, 2015].

It is well-known that in the presence of distributional constraints, a stable matching may not exist. Matching stability is defined by Gale and Shapley [1962] for two-sided one-to-one matching problems. In the setting of a school-student allocation problem,[1] it is defined as the combination of fairness and nonwastefulness [Balinski and Sönmez, 1999]. Fairness guarantees that when student s is not accepted by school c (which she considers better than her assigned school), then s is ranked lower than any student accepted by c based on c's preference. Nonwastefulness is an efficiency notion that rules out the situations where a student can move unilaterally to her more preferred school without violating the underlying distributional constraints. Given the incompatibility of stability with distributional constraints, mechanism designers encounter a trade-off between fairness and efficiency. Our approach is to investigate whether fairness is achievable under distributional constraints, while achieving as much efficiency as possible. As we describe below, a baseline mechanism called Artificial Cap Deferred Acceptance (ACDA), which uses predetermined artificial maximum quotas, is strategyproof and fair, but it is very wasteful since it sacrifices too much flexibility of the original distributional constraints.

We restrict our attention to strategyproof (SP) mechanisms in which no student has an incentive to misreport her preference over schools. In theory, we can restrict our attention to SP mechanisms without loss of generality due to the well-known revelation principle [Gibbard, 1973], i.e., if a certain property is achieved by a mechanism (more specifically, the property is satisfied in a dominant strategy equilibrium when using the mechanism), it can be achieved by an SP mechanism. An SP mechanism is also useful in practice since a student does not need to speculate about the actions of other students to obtain a good outcome; she only needs to truthfully report her own preference.

Existing works have shown that if constraints belong to a well-behaved class (called M-convex set), then a mechanism called *generalized DA*, which is based on DA, is strategyproof, fair, and retains the flexibility of the original distributional constraints [Kojima *et al.*, 2014; Goto *et al.*, 2017]. Thus, identifying a class beyond an M-convex set, such that

[1]For the sake of presentation, the rest of this paper is described in the context of a school-student allocation problem, but its obtained results are applicable to allocation problems in general.

we can develop a non-trivial strategyproof and fair mechanism, is theoretically interesting/challenging.

In this paper, we introduce a new type of constraints that is represented by a union of symmetric M-convex sets. This class of constraints can represent a variety of real-life constraints that are useful in two-sided matching. Since M-convexity is not closed under union, a union of symmetric M-convex sets is not an M-convex set.

We develop a mechanism called the Quota Reduction Deferred Acceptance mechanism (QRDA), which repeatedly applies the well-known Deferred Acceptance (DA) mechanism [Gale and Shapley, 1962], by sequentially reducing artificially introduced maximum quotas. Fragiadakis and Troyan [2017] use the idea of sequentially reducing maximum quotas for a different goal. In their model, students are partitioned into different types, and the goal is to satisfy type-specific minimum/maximum quotas. Yahiro *et al.* [2018] develop a mechanism (which is also called QRDA) specifically for ratio constraints that specify the acceptable minimum ratio between the least/most popular schools. As described below, the class of distributional constraints that we study is a strict generalization of ratio constraints.

To the best of our knowledge, we are the first to identify a general class of constraints (which can describe many specific types of constraints) beyond an M-convex set, where a non-trivial, strategyproof, and fair mechanism exists. As well as being fair and strategyproof, we show that in terms of student welfare, QRDA outperforms ACDA, both theoretically and experimentally. In terms of nonwastefulness, we experimentally show that QRDA outperforms ACDA.

2 Model

A student-school matching market with distributional constraints is defined by a tuple $(S, C, X, \succ_S, \succ_C, \mathcal{V})$.

- $S = \{s_1, \ldots, s_n\}$ is a finite set of students.
- $C = \{c_1, \ldots, c_m\}$ is a finite set of schools. Let M denote $\{1, 2, \ldots, m\}$.
- $X = S \times C$ is a finite set of all possible contracts. Contract $(s, c) \in X$ means that student s is matched to school c. For $\dot{X} \subseteq X$, $\dot{X}_s$ denotes $\{(s, c) \in \dot{X} \mid c \in C\}$, and $\dot{X}_c$ denotes $\{(s, c) \in \dot{X} \mid s \in S\}$. In other words, $\dot{X}_s$ (resp. $\dot{X}_c$) denotes all contracts in $\dot{X}$ related to s (resp. c).
- $\succ_S = (\succ_{s_1}, \ldots, \succ_{s_n})$ is the profile of the student preferences, where each $\succ_s$ is a strict preference over all contracts that are related to s. For example, if s strictly prefers c over c', it is denoted by $(s, c) \succ_s (s, c')$. We sometimes write $c \succ_s c'$ instead of $(s, c) \succ_s (s, c')$.
- $\succ_C = (\succ_{c_1}, \ldots, \succ_{c_m})$ is the profile of the school preferences, where each $\succ_c$ is a strict preference over all contracts that are related to c. For example, if c strictly prefers s over s', it is denoted by $(s, c) \succ_c (s', c)$. We sometimes write $s \succ_c s'$ instead of $(s, c) \succ_c (s', c)$.
- $\mathcal{V} \subseteq \mathbb{N}_0^m$ is a set of school feasible vectors that reflects distributional constraints. For each $\nu \in \mathcal{V}$, we assume $\sum_{i \in M} \nu_i = n$ holds.

For $\dot{X}$, let $\zeta(\dot{X})$ denote m-element vector $(|\dot{X}_{c_1}|, |\dot{X}_{c_2}|, \ldots, |\dot{X}_{c_m}|)$.

Definition 1 (Feasibility)**.** *For $\dot{X} \subseteq X$, $\dot{X}$ is student-feasible if $|\dot{X}_s| = 1$ for all $s \in S$. We call a student-feasible set of contracts a* matching. *$\dot{X}$ is school-feasible if $\zeta(\dot{X}) \in \mathcal{V}$. $\dot{X}$ is feasible if it is both student-/school-feasible.*

In this market, we assume all schools are acceptable to all students and vice versa.[2]

With a slight abuse of notation, for two matchings, $\dot{X}$ and $\ddot{X}$, we denote $\dot{X}_s \succ_s \ddot{X}_s$ if $\dot{X}_s = \{(s, c')\}$, $\ddot{X}_s = \{(s, c'')\}$, and $c' \succ_s c''$ (i.e., if student s prefers the school she obtained in $\dot{X}$ to the one in $\ddot{X}$). Furthermore, we denote $\dot{X}_s \succeq_s \ddot{X}_s$ if either $\dot{X}_s \succ_s \ddot{X}_s$ or $\dot{X}_s = \ddot{X}_s$.

A mechanism φ is a function that takes a profile of student preferences $\succ_S$ as input[3] and returns a set of contracts. Let $\varphi_s(\succ_S)$ denote $\dot{X}_s$, where $\varphi(\succ_S) = \dot{X}$. Let $\succ_{S \setminus \{s\}}$ denote a profile of the preferences of all students except s, and let $(\succ_s, \succ_{S \setminus \{s\}})$ denote a profile of the preferences of all students, where s's preference is $\succ_s$ and the profile of the preferences of the other students is $\succ_{S \setminus \{s\}}$.

Definition 2 (Strategyproofness)**.** *Mechanism φ is strategyproof if for all s, $\succ_s$, $\succ_{S \setminus \{s\}}$, and $\succ'_s$ (where $\succ'_s$ is an arbitrary preference of student s), $\varphi_s((\succ_s, \succ_{S \setminus \{s\}})) \succeq_s \varphi_s((\succ'_s, \succ_{S \setminus \{s\}}))$ holds.*

Definition 3 (Fairness)**.** *In matching $\dot{X}$, where $(s, c) \in \dot{X}$, student s has justified envy toward another student s' if for some $c' \in C$, $c' \succ_s c$, $(s', c') \in \dot{X}$, and $s \succ_{c'} s'$ hold. Matching $\dot{X}$ is fair if no student has justified envy in $\dot{X}$. A mechanism is fair if it always produces a fair matching.*

In other words, student s has justified envy toward student s' if she is assigned to school c', which is better for s than her current school, even though c' prefers s over s'.

Definition 4 (Nonwastefulness)**.** *In matching $\dot{X}$, where $(s, c) \in \dot{X}$, student s claims an empty seat of c', if $c' \succ_s c$ and $(\dot{X} \setminus \{(s, c)\}) \cup \{(s, c')\}$ is school-feasible. Matching $\dot{X}$ is nonwasteful if no student claims an empty seat in $\dot{X}$. A mechanism is nonwasteful if it always produces a nonwasteful matching.*

In other words, s claims an empty seat of c', which is better than her current school c, if moving her from c to c' does not violate distributional constraints.

In standard matching terminology, fairness and nonwastefulness are combined to form a notion called *stability* [Fragiadakis and Troyan, 2017; Goto *et al.*, 2016; Goto *et al.*, 2017]. However, fairness and nonwastefulness are usually incompatible with distributional constraints [Kojima *et al.*, 2014; Fragiadakis *et al.*, 2016; Goto *et al.*, 2016]. Thus, in

[2]Even though this is a strong assumption, we require it to guarantee the existence of a feasible matching. The same assumption is widely used in existing works [Fragiadakis *et al.*, 2016; Goto *et al.*, 2016; Goto *et al.*, 2017].

[3]We assume that the profile of school preferences is publicly known and focus on strategyproofness for students (the proposing side). Thus, we do not specify it as an input of a mechanism.

this paper, we divide the notion into fairness and nonwastefulness and focus on finding a fair outcome, while reducing wastefulness as much as possible.

If $\mathcal{V}$ is an M-convex set (which is a discrete analogue of maximum elements of a convex set in a continuous domain),[4] then a mechanism called *generalized DA*, based on DA, is strategyproof and fair [Kojima *et al.*, 2014; Goto *et al.*, 2017].

Definition 5 (M-convex set)**.** *Let χ_i denote an m-element unit vector, where its i-th element is 1 and all other elements are 0. A set of m-element vectors $\mathcal{V} \subseteq \mathbb{N}_0^m$ forms an M-convex set, if for all $\nu, \nu' \in \mathcal{V}$, for all i such that $\nu_i > \nu'_i$, there exists $j \in \{k \in M \mid \nu_k < \nu'_k\}$ such that $\nu - \chi_i + \chi_j \in \mathcal{V}$ and $\nu' + \chi_i - \chi_j \in \mathcal{V}$ hold.*

Definition 6 (Symmetry)**.** *$\mathcal{V} \subseteq \mathbb{N}_0^m$ is symmetric if for all $\nu = (\nu_1, \dots, \nu_m) \in \mathcal{V}$, vector ν', which is any permutation of $(\nu_1, \dots, \nu_m)$, also belongs to $\mathcal{V}$.*

In this paper, we concentrate on distributional constraints that are represented as a union of symmetric M-convex sets.

Definition 7 (Union of symmetric M-convex sets)**.** *$\mathcal{V} \subseteq \mathbb{N}_0^m$ is a union of symmetric M-convex sets if it is represented as $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_\ell$, where each $\mathcal{V}_i$ is a symmetric M-convex set.*

3 Example of a Union of Symmetric M-convex Sets

Let us illustrate a variety of real-life distributional constraints that can be represented as a union of symmetric M-convex sets.

First, we identify several distributional constraints that can be represented as a single symmetric M-convex set. If a policymaker thinks that several of these constraints are equally appropriate, she can define the union of these vectors as school-feasible. Then the obtained distributional constraints are represented as a union of symmetric M-convex sets.

Definition 8 (Uniform min-max quotas)**.** *Let q and p respectively be maximum and minimum quotas, i.e., the number of assigned students in each school must be between p and q. Then $\mathcal{V} = \{\nu \in \mathbb{N}_0^m \mid \forall i \in M, p \le \nu_i \le q, \sum_{i \in M} \nu_i = n\}$.*

General min-max quotas can be represented as an M-convex set [Kojima *et al.*, 2014]. Since all schools have identical minimum/maximum quotas, $\mathcal{V}$ must be symmetric.

To introduce another class of constraints that can be represented as an M-convex set, we introduce a notion called the most balanced vectors.

Definition 9 (Most balanced vectors)**.** *$\nu \in \mathcal{V}$ is most balanced if for each $i \in M$, ν_i is either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$. Let $\mathcal{V}^*$ denote the set of all the most balanced vectors.*

Definition 10 (Distance constraints (from the most balanced vectors))**.** *For a given distance threshold d, $\mathcal{V}$ is given as $\{\nu \in \mathbb{N}_0^m \mid \delta(\nu) \le d, \sum_{i \in M} \nu_i = n\}$, where $\delta(\nu)$ is given as $\min_{\nu^* \in \mathcal{V}^*}(\sum_{i \in M} |\nu_i - \nu_i^*|)$.*

Distance constraints from a single most desirable vector can be represented as an M-convex set [Kojima *et al.*, 2014]. Since we consider a set of vectors close enough to the set of all the most balanced vectors (which is symmetric), $\mathcal{V}$ is also symmetric. This class of constraints is appropriate when a policymaker thinks the most balanced vectors are ideal, but can accept any matching that is close enough to the ideal ones.

Next we introduce distributional constraints that are represented as a union of symmetric M-convex sets.

Definition 11 (Ratio constraints)**.** *For a given parameter α (where $0 \le \alpha \le 1$), $\mathcal{V}$ is given as $\{\nu \in \mathbb{N}_0^m \mid r(\nu) \ge \alpha, \sum_{i \in M} \nu_i = n\}$, where $r(\nu)$ is given as $\frac{\min_{i \in M} \nu_i}{\max_{i \in M} \nu_i}$, i.e., $r(\nu)$ is the ratio of student number between the least/most popular schools.*

The ratio constraints can be clearly divided into several uniform min/max quotas. For example, assume $n = 21$, $m = 4$, and $\alpha = 0.5$. Then $\mathcal{V}$ can be represented as a union of $\mathcal{V}_1$ in which the uniform minimum$\sim$maximum quotas are 3$\sim$6, and $\mathcal{V}_2$ in which the uniform minimum$\sim$maximum quotas are 4$\sim$8.

Definition 12 (Difference constraints)**.** *For a given parameter d, $\mathcal{V} = \{\nu \in \mathbb{N}_0^m \mid \gamma(\nu) \le d, \sum_{i \in M} \nu_i = n\}$, where $\gamma(\nu)$ is given as $\max_{i \in M} \nu_i - \min_{i \in M} \nu_i$, i.e., $\gamma(\nu)$ is the difference between the numbers of students allocated to the most/least popular schools.*

It is also easy to see that difference constraints can be divided into several uniform min/max quotas. For example, assume $n = 21$, $m = 4$, and $d = 4$. Then $\mathcal{V}$ can be represented as a union of $\mathcal{V}_1$ in where the uniform minimum$\sim$maximum quotas are 3$\sim$7, and $\mathcal{V}_2$ in which the uniform minimum$\sim$maximum quotas are 4$\sim$8.

Note that even if both ratio and difference constraints can be represented as a union of uniform min/max quotas, they are not interchangeable, i.e., in general, ratio constraints cannot be represented as difference constraints, and conversely. Ratio and difference constraints are alternative ways to specify a well-balanced matching outcome. Which one is more appropriate would be application dependent. If a policymaker cannot decide which one to use, she can declare that both are equally appropriate. Then the obtained school-feasible vectors are also a union of symmetric M-convex sets.

4 Quota Reduction Deferred Acceptance Mechanism (QRDA)

4.1 Mechanism Description

Let us first introduce the standard DA, which is a component of QRDA. A standard market is a tuple $(S, C, X, \succ_S, \succ_C, q_C)$, whose definition resembles a market with distributional constraints. The only difference is that its constraints are given as a profile of maximum quotas: $q_C = (q_c)_{c \in C}$. Matching $\dot{X}$ is school-feasible if for all $c \in C$, $|\dot{X}_c| \le q_c$ holds. The standard DA is defined as follows:

Mechanism 1 (standard DA)**.**

Step 1 *Each student s applies to her most preferred school according to $\succ_s$ from the schools that did not reject her so far.*

[4] To be precise, the condition used in [Kojima *et al.*, 2014; Goto *et al.*, 2017] is an M$^\natural$-convex set, which is a generalization of an M-convex set. When all students must be assigned to schools, it becomes equivalent to an M-convex set. Their results are built upon various earlier works [Fleiner, 2001; Fleiner, 2003; Hatfield and Milgrom, 2005; Fujishige and Tamura, 2006].

Step 2 *Each school c tentatively accepts the top q_c students from the applying students based on $\succ_c$ and rejects the rest of them (no distinction between newly applying students and already tentatively accepted students).*

Step 3 *If no student is rejected, return the current matching. Otherwise, go to **Step 1**.*

Let σ denote the sequence of schools[5] based on the round-robin order $c_1, c_2, \ldots, c_m$. Let $\sigma(k)$ denote the k-th school in σ, i.e., $\sigma(k) = c_j$, where $j = 1 + (k - 1 \mod m)$.

Let ν_{max} be $\max_{\nu \in \mathcal{V}, i \in M} \nu_i$. The Quota Reduction Deferred Acceptance (QRDA) mechanism is defined as follows:

Mechanism 2 (QRDA)**.**

Initialization:

For all $c \in C$, $q_c^1 \leftarrow \nu_{max}$, $k \leftarrow 1$.

Stage k **(**$k \geq 1$**):**

Step 1 *Run the standard DA in market $(S, C, X, \succ_S, \succ_C, q_C^k)$ and obtain matching $\dot{X}^k$.*

Step 2 *If $\dot{X}^k$ is school-feasible, then return $\dot{X}^k$.*

Step 3 *Otherwise, for school $c' = \sigma(k)$, $q_{c'}^{k+1} \leftarrow q_{c'}^k - 1$, and for $c \neq c'$, $q_c^{k+1} \leftarrow q_c^k$. Go to **Stage** $k + 1$.*

4.2 Mechanism Properties

Let us first introduce several useful properties related to the unions of symmetric M-convex sets.

Lemma 1. *If $\nu = (\nu_1, \ldots, \nu_i, \ldots, \nu_j, \ldots, \nu_m) \in \mathcal{V}$ where $\mathcal{V}$ is a union of symmetric M-convex sets and $\nu_i > \nu_j$ holds, then vector $\nu' = (\nu_1, \ldots, \nu_i - 1, \ldots, \nu_j + 1, \ldots, \nu_m)$ and all of its permutations also belong to $\mathcal{V}$.*

Proof. From symmetry, vector ν'', which is obtained by exchanging the i-th and j-th elements of ν, is also feasible. Then by the definition of an M-convex set, since $\nu_i > \nu''_i$ and $\nu_k < \nu''_k$ only holds for $k = j$, based on the condition of an M-convex set, $\nu - \chi_i + \chi_j = \nu'$ must be in $\mathcal{V}$. □

Lemma 2. *If $\mathcal{V}$ is a symmetric M-convex set, it contains all of the most balanced vectors.*

Proof. Let ν be a vector in $\mathcal{V}$. If ν is most balanced, then from symmetry, all of its permutations must belong to $\mathcal{V}$. Thus, $\mathcal{V}$ contains all of the most balanced vectors. If ν is not most balanced, from Lemma 1, we can move one student from a popular school to a less popular school in ν and obtain another vector included in $\mathcal{V}$. By repeatedly applying such modification, we eventually obtain a most balanced vector that belongs to $\mathcal{V}$. □

It is clear that if $\mathcal{V}$ is a union of symmetric M-convex sets, $\mathcal{V}$ includes all the most balanced vectors.

Next we show that QRDA obtains a feasible and fair matching.

[5]For simplicity, we assume σ is based on a fixed round-robin order, but the results in this paper hold for any balanced sequence σ, i.e., for each $\ell \in \mathbb{N}_0$, $\sigma(m\ell + 1), \sigma(m\ell + 2), \ldots, \sigma(m\ell + m)$ is a permutation of $c_1, c_2, \ldots, c_m$. This requirement is necessary to enforce strategyproofness.

Stage	Step	Action
k	1	Student s applies to school c_1.
	2	School c_1 rejects student s_1.
	3	Student s_1 applies to school c_2 (and is accepted).
$k+1$	1	School c_3 rejects student s_2 (due to its quota reduction).
	2	Student s_2 applies to school c_4.
		...

Table 1: Example of a rejection chain

Theorem 1. *QRDA returns a feasible and fair matching when $\mathcal{V}$ is a union of symmetric M-convex sets.*

Proof. If QRDA returns a matching, it is clearly feasible. According to Lemma 2, all the most balanced vectors must be included in $\mathcal{V}$. Eventually, in some stage k', every school's quota q_{c_i} equals $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$, and $\sum_{j=1}^{m} q_{c_j} = n$. In this case, the obtained matching is feasible. Thus, QRDA must terminate in stage $k \leq k'$ and the obtained matching is feasible. The obtained matching is also identical to the matching obtained by the standard DA for the market $(S, C, X, \succ_S, \succ_C, q_C^k)$. Since the standard DA is fair [Gale and Shapley, 1962], QRDA is also fair. □

QRDA's strategyproofness is not trivial. It depends on the fact that quotas are reduced based on the round-robin order. Indeed, a student might have an incentive to secure, at an early stage, a seat from a school that later becomes unavailable. We utilize several lemmas to prove the strategyproofness of QRDA. The first is called *Scenario lemma* and requires two definitions. A *scenario* is a sequence of schools to which a student plans to apply. For example, suppose that student s has scenario $\mathcal{C}_s$, defined as c_1, c_2, c_3. Scenario $\mathcal{C}_s$ means that student s first applies to school c_1, and if she is rejected, then she applies to school c_2, and then to c_3 if she is rejected again. A scenario is not necessarily exhaustive. A *rejection chain* $\mathcal{R}(\mathcal{C}_s)$, based on scenario $\mathcal{C}_s$, is the sequence of all students and schools' actions (applications and rejections) that follow when student s enters the market with scenario $\mathcal{C}_s$, starting when s applies to the first school of this scenario. A rejection chain ends when (i) student s is rejected by the last school in $\mathcal{C}_s$ or (ii) the mechanism terminates. A simple example of a rejection chain is presented in Table 1. For rejection chains, the following property holds:

Lemma 3 (Scenario Lemma)**.** *Consider two scenarios, $\mathcal{C}_s$ and $\mathcal{C}'_s$, of student s starting from the same stage of QRDA. If (1) each school that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$ (the order does not matter), (2) student s applies to all the schools in $\mathcal{C}_s$, and (3) all the actions of $\mathcal{R}(\mathcal{C}'_s)$ happen in the same stage, then all the actions in $\mathcal{R}(\mathcal{C}'_s)$ also happen in $\mathcal{R}(\mathcal{C}_s)$.*

Furthermore, Lemma 4 holds for unions of symmetric M-convex sets.[6] Consider matching $\dot{X}$ obtained in an intermediate stage of QRDA. Assume that, in the following

[6]Due to space limitations, we omit the full proof of Lemmas 3 and 4, and Theorem 2.

stage, school c_r rejects a student (because of quota reduction), which triggers school c_a to accept one more student. We denote by η the *operation* that transforms vector $\zeta(\dot{X})$ into $(\ldots, |\dot{X}_{c_r}| - 1, \ldots, |\dot{X}_{c_a}| + 1, \ldots)$.

Lemma 4. *Consider matching $\dot{X}$ obtained in an intermediate stage of QRDA. Assume that after imposing a sequence of operations $\eta_1, \eta_2, \ldots, \eta_k$ on $\zeta(\dot{X})$, another matching $\dot{X}'$ is obtained, and when imposing the last operation η_k, school c^* accepts one more student. If no student is assigned to c^* in $\eta_1, \ldots, \eta_{k-1}$, and if by imposing only η_k on $\zeta(\dot{X})$ a feasible matching is obtained, then $\dot{X}'$ is also feasible.*

Theorem 2. *QRDA is strategyproof when $\mathcal{V}$ is a union of symmetric M-convex sets.*

Proof sketch. By contradiction, assume student s is assigned to a better school when she misreports. Without loss of generality, we assume that her true preference is $c_1 \succ_s c_2 \succ_s \ldots \succ_s c_m$, and that s is assigned to school c_j in stage k when misreporting, but she is assigned to c_i in stage k' under her true preference, where $j < i$. If $k' \leq k$, s cannot benefit from misreporting, since the standard DA is strategyproof and satisfies resource monotonicity, i.e., DA's outcome is weakly preferred by each student if each school's quota weakly increases [Ehlers and Klaus, 2016]. Thus, $k < k'$ must hold.

Consider an alternative (and equivalent) DA mechanism where, at each stage, we run the standard DA for $S \backslash \{s\}$ under the same quotas, obtain matching $\dot{X}^k$, and then add s to the market. When s is added to the market with her true (resp. misreported) preference at stage k, we assume that s is assigned to school c_o (resp. c_j) and infeasible matching X^k (resp. feasible matching $\ddot{X}^k$) is obtained.

Now we define two scenarios: (a) $\mathcal{C}_s = (c_1, c_2, \ldots, c_{i-1})$, based on the true preference of s, and then the last action in $\mathcal{R}(\mathcal{C}_s)$ must be "school c_{i-1} rejects student s," and (b) $\mathcal{C}'_s$, based on the misreported preferences in which the last school is c_j. For $\mathcal{C}'_s$, the following two cases can occur: (i) c_j is the truly least preferred school for s within $\mathcal{C}'_s$ or (ii) $\mathcal{C}'_s$ contains at least one school that is less preferred by s than c_j.

In case (i), each school c that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$. Thus we can apply Lemma 3, which implies that action "student s applies to school c_j" appears in rejection chain $\mathcal{R}(\mathcal{C}_s)$.

At stage k, when s enters the market (after all other students) with true preferences, there is one school, c_e, such that $|X^k_{c_e}| = |\dot{X}^k_{c_e}| + 1$, and when s misreports, an another school, c_d, satisfies $|\ddot{X}^k_{c_d}| = |\dot{X}_{c_d}| + 1$. When comparing $|X^k_{c_e}|$ and $|X^k_{c_d}|$, three cases can occur : (1) one is strictly greater than $\lceil n/m \rceil$ and the other is strictly lower than $\lfloor n/m \rfloor$, (2) both are greater or equal to $\lfloor n/m \rfloor$, and (3) both are lower or equal to $\lceil n/m \rceil$. Here, we only develop case (1); cases (2) and (3) can be handled in a similar fashion.

Case (1). First we assume that $|X^k_{c_e}| < \lfloor n/m \rfloor$ and $|X^k_{c_d}| > \lceil n/m \rceil$. Since $|\ddot{X}^k_{c_e}| = |X^k_{c_e}| - 1$ and $|\ddot{X}^k_{c_d}| = |X^k_{c_d}| + 1$, by applying Lemma 1 on $\ddot{X}^k$, X^k is also feasible, which is a contradiction. Thus $|X^k_{c_e}| > \lceil n/m \rceil$ and $|X^k_{c_d}| < \lfloor n/m \rfloor$.

Consider that $c_e = c_o$ and $c_d = c_j$; other cases are handled with a similar argument. Now assume that in rejection chain $\mathcal{R}(\mathcal{C}_s)$, student s applies to school c_j in stage k'' ($k \leq k'' \leq k'$), and matching $X^{k''}$ is returned at stage k''. Moreover, assume that student s' is assigned to c_j (strictly) between k and k''. If s' is from c_o, by applying Lemma 4 on $\ddot{X}^k$, a feasible matching is returned when s' is assigned to c_j, which is a contradiction. If s' is from an another school, a similar argument leads to the same contradiction.

Therefore, no student is assigned to c_j between stages k and k''. Thus, student s is accepted in c_j when she applies in stage k'' since there is an available seat in c_j ($q^{k''}_{c_j} \geq \lceil n/m \rceil$ and $|X^{k''}_{c_j}| < \lfloor n/m \rfloor$). However, by Lemma 4, a feasible matching is returned when s is assigned to c_j in stage k'', which is a contradiction.

Finally, case (ii) can be handled in a similar fashion by creating a new scenario $\mathcal{C}''_s$ where we remove all the schools from $\mathcal{C}'_s$ that are truly less preferred than c_j by s. □

4.3 Comparison with Baseline Mechanism

In this subsection, we present a method that sets artificial maximum quotas such that the obtained matching by the standard DA is guaranteed to satisfy distributional constraints: the Artificial Cap Deferred Acceptance mechanism (ACDA). ACDA is used in Japanese medical resident matching programs [Kamada and Kojima, 2015] to handle regional maximum quotas as well as a baseline mechanism in many works related to distributional constraints [Fragiadakis *et al.*, 2016; Goto *et al.*, 2016; Goto *et al.*, 2017].

If we knew beforehand which schools are popular/unpopular, we might be able to set q_C such that distributional constraints are satisfied and students' welfare can be maximized. Otherwise, one simple and reasonable way for finding appropriate q_C is using a most balanced vectors. ACDA based on this idea is defined as follows:

Mechanism 3 (ACDA (based on a most balanced vector))**.**

Initialization:

For each i where $i \leq (n \bmod m)$, $q_{c_i} \leftarrow \lfloor n/m \rfloor$, and for each i where $i > (n \bmod m)$, $q_{c_i} \leftarrow \lceil n/m \rceil$.

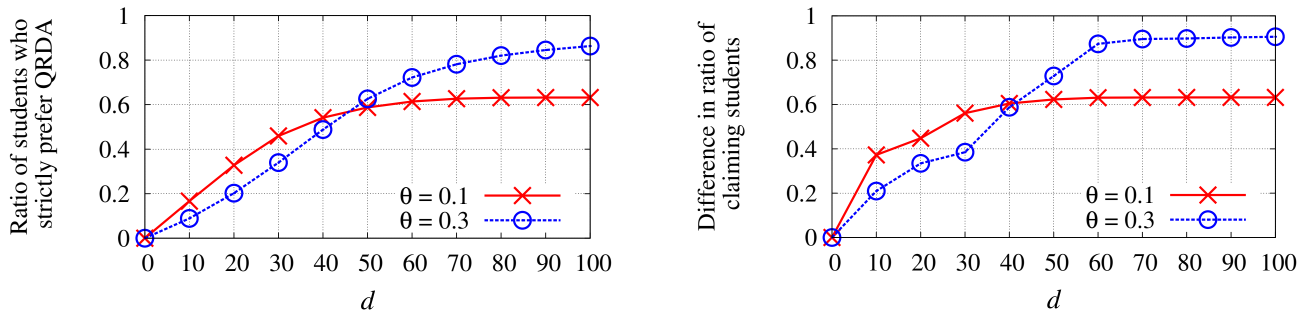
Execution:

Run the standard DA in market $(S, C, X, \succ_S, \succ_C, q_C)$.

Theorem 3. *ACDA (based on a most balanced vector) is strategyproof and returns a feasible and fair matching.*

Proof. Since all the most balanced vectors are included in $\mathcal{V}$, the matching obtained under q_C is clearly feasible. Since the standard DA is fair [Gale and Shapley, 1962], the matching obtained by ACDA is also fair. Furthermore, since q_C is determined independently from $\succ_S$, ACDA is strategyproof since the standard DA is strategyproof. □

Theorem 4. *All students weakly prefer the matching obtained by QRDA over that of ACDA (based on a most balanced vector).*



(a) Ratio of students who strictly prefer QRDA

(b) Difference in ratio of claiming students

Figure 1: Comparison between QRDA and ACDA

Proof. Let k' denote the stage of QRDA where $q_C^{k'}$ becomes identical to the q_C used in ACDA (since the most balanced vector in ACDA is based on the quota reduction sequence σ in QRDA). As shown in the proof of Theorem 1, QRDA terminates at stage k where $k \leq k'$. Since $q_c^k \geq q_c^{k'}$ holds for any $c \in C$ and DA satisfies resource monotonicity (described in the proof of Theorem 2), each student weakly prefers the matching obtained by QRDA over that of ACDA. □

We cannot guarantee that QRDA is less *wasteful* than ACDA. More specifically, Yahiro *et al.* [2018] identify a case where the number of students who claim empty seats in QRDA exceeds that of ACDA under ratio constraints (which is a special case of a union of symmetric M-convex sets).

5 Experimental Evaluation

Under ratio constraints, Yahiro *et al.* [2018] show that in terms of student welfare, QRDA is weakly better than ACDA theoretically and experimentally, and in terms of nonwastefulness, QRDA always experimentally outperforms ACDA. We extend these results by conducting a computer simulation with the *difference constraints* (Def. 12) to quantitatively examine the weak domination of QRDA over ACDA and show that QRDA outperforms ACDA in terms of nonwastefulness.

We considered a market with $n = 800$ students and $m = 20$ schools and generated student preferences with the Mallows model [Tubbs, 1992; Lu and Boutilier, 2014; Drummond and Boutilier, 2013]. We drew strict preference $\succ_s$ of student s whose probability is expressed as follows:

$$\Pr(\succ_s) = \frac{\exp(-\theta \cdot \omega(\succ_s, \succ_{\hat{s}}))}{\sum_{\succ'_s} \exp(-\theta \cdot \omega(\succ'_s, \succ_{\hat{s}}))}.$$

Here $\theta \in \mathbb{R}$ denotes a spread parameter, $\succ_{\hat{s}}$ is a central preference (uniformly randomly chosen from all possible preferences in our experiment), and $\omega(\succ_s, \succ_{\hat{s}})$ represents the Kendall tau distance between $\succ_s$ and $\succ_{\hat{s}}$. The distance is measured by the number of ordered pairs in $\succ_s$ that are inconsistent with those in $\succ_{\hat{s}}$. When $\theta = 0$, it becomes identical to the uniform distribution and converges to $\Pr(\succ_{\hat{s}})$ as θ increases. Similar trends are obtained when conducting experiment in a broad range of θ, thus we only discuss two realistic settings, $\theta = 0.1$ and $\theta = 0.3$. The preference of each school c is drawn uniformly at random. We created 100 problem instances for each parameter setting.

In Fig. 1 (a), we show the proportion of students who strictly prefer QRDA over ACDA depending on the allowed difference d in Def. 12. When $d = 10$ and $\theta = 0.1$, approximately 18% of the students strictly prefer QRDA's outcome; this number increases before plateauing at 60% when $d = 50$. Indeed, as d increases, since the set of feasible sets expands, the potential of improvement with QRDA increases as well. We expect that policymakers will prefer QRDA over ACDA since it outperforms ACDA and a significant amount of students strictly prefer it. When $\theta = 0.3$, the competition among students rises since their preferences tend to be more similar. Compared to $\theta = 0.1$, the improvement by QRDA is smaller when d is small ($d \leq 45$) and is larger when $d \geq 45$.

To compare the nonwastefulness of QRDA and ACDA, we measure the proportion of students who claim an empty seat in both mechanisms and show the difference in Fig. 1 (b), i.e., by plotting $(|S_{\text{ACDA}}| - |S_{\text{QRDA}}|)/n$, where S_{ACDA} (resp. S_{QRDA}) denotes the set of students claiming an empty seat in ACDA (resp. QRDA). This value is always positive for both $\theta = 0.1$ and $\theta = 0.3$, which means that on average more students claim empty seats in ACDA than in QRDA. When $\theta = 0.1$, this number is 40% for $d = 10$ and increases to approximately 60% for $d = 40$ before plateauing. We conclude that experimentally, QRDA is less wasteful than ACDA in the difference constraints setting. Similar to Fig. 1 (a), for both cases of $\theta = 0.1$ and $\theta = 0.3$, QRDA's improvement becomes greater as d increases.

6 Conclusions and Future Works

This paper identified a new class of distributional constraints defined as a union of symmetric M-convex sets. Since M-convexity is not closed under union, a union of symmetric M-convex sets is not an M-convex set. We developed a fair and strategyproof mechanism called QRDA and showed that in terms of student welfare, it theoretically outperforms ACDA. Furthermore, we experimentally showed that QRDA is better than ACDA in terms of student welfare and nonwastefulness.

Future works will clarify whether there exists any class of constraints (broader than a union of symmetric M-convex set) where a non-trivial fair and strategyproof mechanism exists.

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