Altruism in Coalition Formation Games

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Abstract

Nguyen *et al.* [2016] introduced altruistic hedonic games in which agents' utilities depend not only on their own preferences but also on those of their friends in the same coalition. We propose to extend their model to coalition formation games in general, considering also the friends in other coalitions. Comparing the two models, we argue that excluding some friends from the altruistic behavior of an agent is a major disadvantage that comes with the restriction to hedonic games. After introducing our model, we additionally study some common stability notions and provide a computational analysis of the associated verification and existence problems.

1 Introduction

We consider coalition formation games where agents have to form coalitions based on their preferences. For hedonic coalition formation games, Dimitrov *et al.* [2006] proposed the *friends-and-enemies encoding with friend-oriented preferences* which involves a *network of friends*, a (simple) graph whose vertices are the players and where two players are connected by an edge exactly if they are friends of each other. Players not connected by an edge consider each other as enemies. Under friend-oriented preferences, player i prefers a coalition C to a coalition D if C contains more of her friends than D, or C and D contain equally many friends but C contains fewer enemies than D. This is a special case of the *additive encoding* [Aziz *et al.*, 2013]. For more background on these two compact representations, see Section 2 and the book chapter by Aziz and Savani [2016].

Based on friend-oriented preferences, Nguyen *et al.* [2016] introduced *altruistic hedonic games* where agents gain utility not only from their own satisfaction but also from their friends' satisfaction. However, Nguyen *et al.* [2016] specifically considered hedonic games only, which require that an agent's utility only depends on her own coalition. In their interpretation of altruism, the utility of an agent is composed of the agent's own valuation of her coalition and the valuations of all this agent's friends *in this coalition*.

Inspired by the idea of altruism, we extend their model to coalition formation games in general. However, we propose a model where agents behave altruistically to *all their friends*,

Figure 1: A network of friends

not only to the friends in the same coalition. Not restricting to hedonic games, we aim to capture a more natural notion of altruism where none of an agent's friends is excluded from her altruistic behavior. To become acquainted with this idea of altruism, consider the following coalition formation game, represented by the *network of friends* in Figure 1.

Considering coalition structures $\Gamma = \{\{1, 2, 3\}, \{4\}\}$ and $\Delta = \{\{1, 2, 4\}, \{3\}\}$, it becomes clear that player 1 is indifferent between coalitions $\{1, 2, 3\}$ and $\{1, 2, 4\}$ under friend-oriented preferences as both coalitions contain one of her friends and one of her enemies. Under altruistic hedonic preferences [Nguyen *et al.*, 2016], however, player 1 behaves altruistically to her friend 2 (who is friends with 3 but not with 4) and therefore prefers $\{1, 2, 3\}$ to $\{1, 2, 4\}$. Now, consider $\Gamma' = \{\{1\}, \{2, 3\}, \{4\}\}$ and $\Delta' = \{\{1\}, \{2, 4\}, \{3\}\}$. Intuitively, one would still expect 1 to behave altruistically to her friend 2. However, it becomes obvious that under any hedonic preference (which requires that player 1's preference only depends on her own coalition), player 1 would be indifferent between Γ' and Δ'. To model a global level of altruism, we release the restriction to hedonic games and introduce *altruistic coalition formation* where agents behave altruistically to all their friends, independently of their current coalition.

Related Work. Coalition formation games, as considered here, are closely related to the subclass of *hedonic games* which has been broadly studied in the literature, addressing the issue of compactly representing preferences (some representations are listed in Section 2), conducting axiomatic analyses, dealing with different notions of stability, and investigating the computational complexity of the associated problems (see, e.g., the book chapter by Aziz and Savani [2016]). Previously, altruism in games and social contexts of games have been considered mainly for *noncooperative* games. Most prominently, Ashlagi *et al.* [2008] introduced *social context games* where a strategic game is embedded in a social context that is modeled by a graph of neighborhood. Examples include *ranking games* [Brandt *et al.*, 2009] and *coalitional congestion games* [Hayrapetyan *et al.*, 2006; Kuniavsky and Smorodinsky, 2014]. In particular, in the so-

cial context that Ashlagi *et al.* [2008] call "surplus collaboration," players seek to maximize the average payoff of themselves and their friends. Still, their model differs from ours in that they consider *non*cooperative games. Other work studying altruism in noncooperative games is due to Hoefer and Skopalik [2009], Chen *et al.* [2011], Apt and Schäfer [2014], and Rahn and Schäfer [2013]. Based on *altruistic hedonic games* [Nguyen *et al.*, 2016], Schlueter and Goldsmith [2018] defined *super altruistic hedonic games*, related to *social distance games* due to Brânzei and Larson [2011], where friends have a different impact on an agent based on their distances in the underlying network of friends. Monaco *et al.* [2018; 2019] studied hedonic games with social context and modified the fractional hedonic games of Aziz *et al.* [2019] based on *egalitarian* social welfare. While in their models players care also about their friends' happiness, they focus on Nash equilibria, the price of anarchy, and the price of stability.

Our Contribution. Conceptually, we extend the model of altruism proposed by Nguyen *et al.* [2016] from hedonic games to general coalition formation games. We argue how this captures a more global notion of altruism. Technically, we study the common stability concepts in this model and analyze the associated verification and existence problems in terms of their computational complexity.

2 The Model

Coalition Formation Games. Let $N = \{1, \ldots, n\}$ be a set of *agents* (or *players*). Each subset of N is called *coalition*. A *coalition structure* Γ is a partition of N and we denote the set of all possible coalition structures by $\mathcal{C}_N$. For a player $i \in N$ and a coalition structure $\Gamma \in \mathcal{C}_N$, $\Gamma(i)$ denotes the coalition in Γ containing i. The objective of a *coalition formation game* is to form a coalition structure based on the agents' preferences. Hence, a *coalition formation game* is a pair $(N, \succeq)$, where $N = \{1, \ldots, n\}$ is a set of agents, $\succeq = (\succeq_1, \ldots, \succeq_n)$ is a profile of preferences, and every preference $\succeq_i \in \mathcal{C}_N \times \mathcal{C}_N$ is a complete weak order over all possible coalition structures. Note that *hedonic games* are a special case of coalition formation games where an agent's preference relation only depends on the coalitions containing herself. In a hedonic game $(N, \succeq)$, agent $i \in N$ is indifferent between any two coalition structures Γ and Δ as long as her coalition is the same, i.e., $\Gamma(i) = \Delta(i) \implies \Gamma \sim_i \Delta$. Therefore, the preference of agent $i \in N$ is usually represented by a complete weak order over the set of coalitions containing i.

The "Friends and Enemies" Encoding. Since $|\mathcal{C}_N|$, the number of all possible coalition structures, is exponential in the number of agents, it is not reasonable to ask every agent for her complete preference over $\mathcal{C}_N$. Instead, we are looking for a way to compactly represent the agents' preferences. In the literature, many such representations have been proposed for hedonic games such as the *additive encoding* [Sung and Dimitrov, 2010; Aziz *et al.*, 2013; Woeginger, 2013], the *singleton encoding* due to Cechlárová and Romero-Medina [2001] and further studied by Cechlárová and Hajduková [2003], the *"friends and enemies" encoding* due to Dimitrov *et al.* [2006], and FEN-hedonic games due to Lang *et al.* [2015] and also used by Kerkmann *et al.* [2020;

2019] and Rothe *et al.* [2018]. Here, we use the friend-and-enemy encoding due to Dimitrov *et al.* [2006]. We focus on their friend-oriented model and will later adapt it to our altruistic model. In the friend-oriented model, the preferences of the agents in N are given by a network of friends, i.e., a (simple) graph $G = (N, A)$ whose vertices are the players and where two players $i, j \in N$ are connected by an edge $\{i, j\} \in A$ exactly if they are each other's friends. Agents not connected by an edge consider each other as enemies. For an agent $i \in N$, we denote the set of i's friends by $F_i = \{j \in N \mid \{i, j\} \in A\}$ and the set of i's enemies by $E_i = N \setminus (F_i \cup \{i\})$. Under *friend-oriented* preferences as defined by Dimitrov *et al.* [2006], players prefer coalitions with more friends to those with fewer friends and, if there are equally many friends they prefer the coalition with fewer enemies: $C \succeq_i^F D \iff |C \cap F_i| > |D \cap F_i|$ or $(|C \cap F_i| = |D \cap F_i|$ and $|C \cap E_i| \le |D \cap E_i|)$. This can also be represented additively. Assigning a value of n to each friend and a value of -1 to each enemy, agent $i \in N$ values coalition C, containing herself, with $v_i(C) = n|C \cap F_i| - |C \cap E_i|$. Note that $-(n-1) \le v_i(C) \le n(n-1)$, and $v_i(C) > 0$ if and only if there is at least one friend of i's in C. For a given coalition structure $\Gamma \in \mathcal{C}_N$, we also write $v_i(\Gamma)$ for player i's value of $\Gamma(i)$. We, furthermore, denote the sum of the values of i's friends by $\text{sum}_i^F(\Gamma) = \sum_{f \in F_i} v_f(\Gamma)$.

The Altruistic Model. Based on this friend-oriented model, we will define altruistic coalition formation games, while considering three degrees of altruism as defined by Nguyen *et al.* [2016]. However, we adapt them to our model, extending the agents' altruism to all their friends.

- **Selfish First (SF):** Agents rank different coalition structures mainly based on their own valuations. Only in the case of a tie between two coalition structures, their friends' valuations are considered as well.

- **Equal Treatment (EQ):** Agents treat themselves and their friends the same. That means that an agent $i \in N$ and all of i's friends have the same impact on i's utility.

- **Altruistic Treatment (AL):** Agents rank coalition structures based on their friends' valuations. They only consider their own valuations in the case of a tie.

Formally, for an agent $i \in N$ and a coalition structure $\Gamma \in \mathcal{C}_N$, we denote i's utility for Γ under selfish-first preferences by $u_i^{SF}(\Gamma)$, under equal treatment by $u_i^{EQ}(\Gamma)$, and under altruistic treatment by $u_i^{AL}(\Gamma)$. They are defined as

$$u_i^{SF}(\Gamma) = M \cdot v_i(\Gamma) + \text{sum}_i^F(\Gamma) \text{ with } M \ge n^3,$$
$$u_i^{EQ}(\Gamma) = v_i(\Gamma) + \text{sum}_i^F(\Gamma), \text{ and}$$
$$u_i^{AL}(\Gamma) = v_i(\Gamma) + M \cdot \text{sum}_i^F(\Gamma) \text{ with } M \ge n^2.$$

For any coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, agent i's selfish-first preference is then defined by $\Gamma \succeq_i^{SF} \Delta \iff u_i^{SF}(\Gamma) \ge u_i^{SF}(\Delta)$. Her equal- and altruistic-treatment preferences $(\succeq_i^{EQ}; \succeq_i^{AL})$ are defined analogously, using u_i^{EQ} and u_i^{AL}. The factor M, which is used for the selfish-first model and for altruistic treatment, ensures that an agent's utility is first determined by the agent's own valuation in the selfish-first
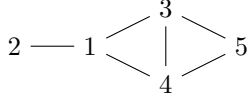
Figure 2: Network of friends for Example 1

$v_i(\Gamma)$	1	2	3	4	5	$\mathrm{sum}_1^F(\Gamma)$
Γ_1	15	3	9	9	0	21
Γ_2	5	5	10	10	10	25

Table 1: Comparison of two coalition structures in Example 1

model and by the friends' valuations in the altruistic model. Similarly as Nguyen *et al.* [2016] prove the corresponding properties in hedonic games, we can show that for $M \geq n^3$, $v_i(\Gamma) > v_i(\Delta)$ implies $\Gamma \succ_i^{SF} \Delta$, and for $M \geq n^2$, $\mathrm{sum}_i^F(\Gamma) > \mathrm{sum}_i^F(\Delta)$ implies $\Gamma \succ_i^{AL} \Delta$. An *altruistic coalition formation game (ACFG)* is a coalition formation game where the agents' preferences were obtained by a network of friends via one of these three cases of altruism. For any ACFG, the players' utilities can obviously be computed in polynomial time.

Comparison to Altruism in Hedonic Games. Nguyen *et al.* [2016] focus on altruism in hedonic games where an agent's utility only depends on her own coalition. As we have already seen from the example in the introduction, there are some aspects of altruistic behavior that cannot be realized by hedonic games. The following example shows that our model crucially differs from the model due to Nguyen *et al.* [2016].

Example 1 *Consider an ACFG with five agents, $N = \{1, 2, 3, 4, 5\}$, and the network of friends in Figure 2. Table 1 shows the values $v_i(\Gamma)$, $1 \leq i \leq 5$, and the sum of the values of player 1's friends for coalition structures $\Gamma_1 = \{\{1, 2, 3, 4\}, \{5\}\}$ and $\Gamma_2 = \{\{1, 2\}, \{3, 4, 5\}\}$.*

Under the selfish-first model, agent 1 prefers Γ_1 to Γ_2 ($\Gamma_1 \succ_1^{SF} \Gamma_2$) because in Γ_1 she is together with all her friends while in Γ_2 she is only together with one friend.

Under altruistic treatment, however, she prefers Γ_2 to Γ_1 ($\Gamma_2 \succ_1^{AL} \Gamma_1$). She would rather form a coalition with only 2 than being with all her friends because 2 doesn't like 3 and 4, and 3 and 4 don't like 2. Since 1 is altruistic to all *her friends, she prefers the coalition structure that is valued higher by her friends. Actually, all of agent 1's friends have a higher valuation for Γ_2. Not paying attention to this fact, the altruistic-treatment model of Nguyen* et al. *[2016] crucially differs. In their model, agents only consider friends in their own coalition. Thus, under their model of altruistic treatment, 1 prefers Γ_1 to Γ_2. She would rather be with 2, 3, and 4 because the average valuation of her friends in her coalition would then be $\frac{21}{3} = 7$ instead of $\frac{5}{1} = 5$ when being alone with 2.*

Considering Example 1, an agent does not really seem to act altruistically if she prefers a coalition structure that makes all her friends worse off. To avoid this kind of behavior, we focus on general coalition formation games. In fact, all preferences that were obtained by one of the three degrees of al-

truism, fulfill friend-oriented unanimity: For coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, we say $\succeq_i$ is *friend-orientedly unanimous* if $v_a(\Gamma) > v_a(\Delta)$ for each $a \in F_i \cup \{i\}$ implies $\Gamma \succ_i \Delta$. Comparing coalition structures by concentrating only on the coalitions that the considered agent is part of, the equal- and altruistic-treatment models by Nguyen *et al.* [2016] are not friend-orientedly unanimous.[1]

3 Stability in ACFGs

The main question in coalition formation games is which coalition structures might form. There are several stability concepts that are well-studied for hedonic games, each indicating whether a given coalition structure would be accepted by the agents or if there are other coalition structures that are more likely to form. Although we consider more general coalition formation games, we can easily adapt these definitions to our framework.

Stability Notions. Let $(N, \succeq)$ be an ACFG with agents $N = \{1, \ldots, n\}$ and preferences $\succeq = (\succeq_1, \ldots, \succeq_n)$ obtained from a network of friends via one of the three degrees of altruism. A coalition structure Γ is said to be

- *Nash stable* if no player prefers moving to another coalition; formally: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C}]$, where $\Gamma_{i \to C}$ denotes the coalition structure that arises from Γ when moving i to C, i.e., $\Gamma_{i \to C} = \Gamma \setminus \{\Gamma(i), C\} \cup \{\Gamma(i) \setminus \{i\}, C \cup \{i\}\}$;

- *individually rational* if no player would prefer being alone: $(\forall i \in N)[\Gamma \succeq_i \Gamma_{i \to \emptyset}]$;

- *individually stable* if no player prefers moving to another coalition and could deviate to it without harming any player in that coalition: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C} \vee (\exists j \in C)[\Gamma \succ_j \Gamma_{i \to C}]]$;

- *contractually individually stable* if no player prefers another coalition and could deviate to it without harming any player in the new or the old coalition: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C} \vee (\exists j \in C)[\Gamma \succ_j \Gamma_{i \to C}] \vee (\exists k \in \Gamma(i))[\Gamma \succ_k \Gamma_{i \to C}]]$;

- *totally individually stable* if no player prefers another coalition and could deviate to it without harming any other player: $(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C} \vee (\exists l \in N \setminus \{i\})[\Gamma \succ_l \Gamma_{i \to C}]]$;

- *core stable* if no nonempty coalition blocks Γ: $(\forall C \subseteq N, C \neq \emptyset)(\exists i \in C)[\Gamma \succeq_i \Gamma_{C \to \emptyset}]$, where $\Gamma_{C \to \emptyset}$ denotes the coalition structure arising from Γ when all players in C leave their coalitions to form the new coalition C, i.e., $\Gamma_{C \to \emptyset} = \Gamma \setminus \{\Gamma(j)|j \in C\} \cup \{\Gamma(j) \setminus C|j \in C\} \cup \{C\}$;

- *strictly core stable* if no coalition weakly blocks Γ: $(\forall C \subseteq N)(\exists i \in C)[\Gamma \succ_i \Gamma_{C \to \emptyset}] \vee (\forall i \in C)[\Gamma \sim_i \Gamma_{C \to \emptyset}]$;

- *popular* if for every other coalition structure Δ, at least as many players prefer Γ to Δ as there are players who

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[1]Note that they define a restricted version of friend-oriented unanimity that only considers the agents' own coalitions.

stability notion	VERIFICATION	EXISTENCE
ind. rationality	in P[1]	trivial: YES[1]
Nash stability	in P[1]	trivial: YES[1]
ind. stability	in P[1]	trivial: YES[1]
core stability	coNP-compl.[2,3]	trivial: YES
str. core stability	in coNP[1]	trivial: YES
popularity	coNP-compl.[2,3]	not trivial[1]
str. popularity	coNP-compl.[2,3]	coNP-hard
perfectness	in P[2,3]	in P[2]

[1] This result also holds for $\succeq^{EQ}$ and $\succeq^{AL}$.
[2] In coNP for $\succeq^{EQ}$.
[3] In coNP for $\succeq^{AL}$.

Table 2: Results for α-VERIFICATION and α-EXISTENCE

prefer Δ to Γ: $(\forall \Delta \in \mathcal{C}_N, \Delta \neq \Gamma)\big[|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| \geq |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|\big]$;

- *strictly popular* if for every other coalition structure Δ, more players prefer Γ to Δ than there are players who prefer Δ to Γ: $(\forall \Delta \in \mathcal{C}_N, \Delta \neq \Gamma)\big[|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| > |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|\big]$; and

- *perfect* if there is no player who prefers any coalition structure to Γ: $(\forall i \in N)(\forall \Delta \in \mathcal{C}_N)[\Gamma \succeq_i \Delta]$.

Note that "totally individual stability" is a new notion which we introduce here. It strengthens the notion of contractually individual stability and makes sense in the context of coalition formation games because players' preferences may also be influenced by coalitions they are not part of.

4 How Hard are Verification and Existence?

We now study the associated *verification* and *existence problems* and conduct a computational analysis for them. Given a stability concept α, these problems are defined as follows.

- α-VERIFICATION: Given an ACFG $(N, \succeq)$ and a coalition structure $\Gamma \in \mathcal{C}_N$, does Γ satisfy α?

- α-EXISTENCE: Given an ACFG $(N, \succeq)$, does there exist a coalition structure $\Gamma \in \mathcal{C}_N$ that satisfies α?

Table 2 summarizes the results for these problems under $\succeq^{SF}$. We will also give results for $\succeq^{EQ}$ and $\succeq^{AL}$ in this section. In Table 2, however, we only mark if the results for $\succeq^{EQ}$ and $\succeq^{AL}$ match those for $\succeq^{SF}$.

Individual Rationality. Verifying individual rationality is easy: We just need to iterate over all agents and compare two coalition structures in each iteration. Since players' utilities can be computed in polynomial time, individual rationality can be verified in time polynomial in the number of agents.

The existence problem is trivial, since $\Gamma = \{\{1\}, \ldots, \{n\}\}$ is always individually rational. Furthermore, we give the following characterization for all three degrees of altruism, omitting the proof due to space constraints.

Theorem 2 *Let $(N, \succeq)$ be an ACFG and let Γ be a coalition structure. Γ is individually rational if and only if it holds for*
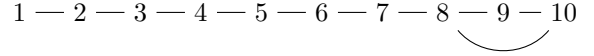
$$1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10$$

Figure 3: Network of friends for Example 4

all players $i \in N$ that $\Gamma(i)$ contains a friend of i's or i is alone, formally: $(\forall i \in N)[\Gamma(i) \cap F_i \neq \emptyset \vee \Gamma(i) = \{i\}]$.

Nash Stability. Since there are at most n coalitions in a coalition structure $\Gamma \in \mathcal{C}_N$, we can verify Nash stability in polynomial time: We just iterate over all agents $i \in N$ ($|N| = n$) and all coalitions $C \in \Gamma \cup \{\emptyset\}$ (at most $n+1$) and check whether $\Gamma \succeq_i \Gamma_{i \to C}$. Since we can check a player's preference over two coalition structures in polynomial time and since we have at most a quadratic number of iterations ($n \cdot (n+1)$), verification is in P for Nash stability.

Existence is trivially in P for Nash stability; indeed, the same example that Nguyen *et al.* [2016] gave for altruistic hedonic games works here as well. Specifically, for $C = \{i \in N \mid F_i = \emptyset\} = \{c_1, \ldots, c_k\}$ the coalition structure $\{\{c_1\}, \ldots, \{c_k\}, N \setminus C\}$ is Nash stable.

Individual Stability. For individual stability, contractually individual stability, and totally individual stability, existence is trivially in P. Nash stability implies all these three concepts, hence, the Nash stable coalition structure from above is also (contractually; totally) individually stable.

Verification is also in P for these concepts. Similarly to Nash stability, we iterate over all players and all coalitions and check the particular conditions in polynomial time.

Core Stability and Strict Core Stability. We now show that the existence problem is trivial for (strict) core stability if the preferences are obtained via the selfish-first model, again omitting the proof due to space constraints.

Theorem 3 *Let $(N, \succeq^{SF})$ be an ACFG where the preferences $\succeq^{SF}$ were obtained from a network of friends G via the selfish-first model. Let further $C_1, \ldots, C_k$ be the vertex sets of the connected components of G. Then $\Gamma = \{C_1, \ldots, C_k\}$ is strictly core stable (and thus core stable).*

However, the coalition structure from Theorem 3 is not always core stable under equal treatment or altruistic treatment.

Example 4 *Let $N = \{1, \ldots, 10\}$ and let the preferences be given by the network of friends G shown in Figure 3.*

Consider the coalition structure consisting of the connected components of G, i.e., $\Gamma = \{\{1, \ldots, 10\}\}$, and the coalition $C = \{8, 9, 10\}$. C blocks Γ under equal and altruistic treatment. To see this, consider how players 7 to 10 value Γ and $\Gamma_{C \to \emptyset}$ and compute the utilities for players 8, 9, and 10. Omitting the details, for all $i \in C = \{8, 9, 10\}$ we have $u_i^{EQ}(\Gamma) < u_i^{EQ}(\Gamma_{C \to \emptyset})$, which implies $\Gamma \prec_i^{EQ} \Gamma_{C \to \emptyset}$, and that $\mathrm{sum}_i^F(\Gamma) < \mathrm{sum}_i^F(\Gamma_{C \to \emptyset})$, which implies $\Gamma \prec_i^{AL} \Gamma_{C \to \emptyset}$. Hence, C blocks Γ for equal and altruistic treatment.

Theorem 5 *(Strict) core stability verification can be done in* coNP *for all three degrees of altruism. For the selfish-first model, core stability verification is even* coNP-*complete.*

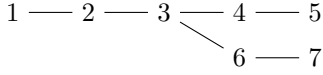The proof of Theorem 5 is omitted due to space constraints.

Figure 4: Network of friends for Example 6

Popularity and Strict Popularity. Under all three degrees of altruism, there does not always exist a (strictly) popular coalition structure.

Example 6 *Let $N = \{1, \ldots, 7\}$ and let the preferences be given by the network of friends shown in Figure 4.*

Then there is no (strictly) popular coalition structure for any of the three degrees of altruism. Since perfectness implies popularity, there is also no perfect coalition structure.

Note that there are 877 possible coalition structures for this game[2] and that we tested this example using brute force.

We now show that under the selfish-first model it is hard to verify if a given coalition structure is (strictly) popular and it is also hard to decide whether there exists a strictly popular coalition structure for a given ACFG.

Theorem 7 *Under the selfish-first model, strict popularity verification is* coNP-*complete, strict popularity existence is* coNP-*hard, and popularity verification is* coNP-*complete.*

Proof. Recall that a coalition structure $\Gamma \in \mathcal{C}_N$ is not strictly popular in an ACFG $(N, \succeq)$ (given by a network of friends G) if and only if there is a coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$, such that $|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| \leq |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}|$. We can nondeterministically guess a coalition structure Δ and verify the above condition in polynomial time, so strict popularity verification is in coNP.

To show coNP-hardness of strict popularity verification, we use a restricted version of EXACT COVER BY 3-SETS (X3C), which we denote by RX3C and which is shown to remain NP-complete by Gonzalez [1985]. We provide a polynomial-time many-one reduction from RX3C to the complement of strict popularity verification. An instance $(B, \mathscr{S})$ of RX3C consists of a set $B = \{1, \ldots, 3k\}$ and a collection $\mathscr{S} = \{S_1, \ldots, S_{3k}\}$ of 3-element subsets of B ($S_i \subseteq B$ and $|S_i| = 3$ for $1 \leq i \leq 3k$). The instance is restricted such that each element of B occurs in exactly three sets in $\mathscr{S}$. Given this instance, the question is whether $\mathscr{S}$ contains an exact cover for B, i.e., a subset $\mathscr{S}' \subseteq \mathscr{S}$ such that every element of B occurs in exactly one set in $\mathscr{S}'$.

Given an instance $(B, \mathscr{S})$ of RX3C, we construct the following ACFG. The set of players is given by $N = \{\alpha_1, \ldots, \alpha_{5k}\} \cup \{\beta_b \mid b \in B\} \cup \{\zeta_S, \eta_S \mid S \in \mathscr{S}\}$.

Let $Alpha = \{\alpha_1, \ldots, \alpha_{5k}\}$, $Beta = \{\beta_b \mid b \in B\}$, and $Q_S = \{\zeta_S, \eta_S\}$ for each $S \in \mathscr{S}$. The network of friends is given in Figure 5, where a dashed circle around a group of players means that all these players are friends of each other:

- All players in $Alpha \cup Beta$ are friends of each other.
- For every $S \in \mathscr{S}$, ζ_S and η_S are friends.
- For every $S \in \mathscr{S}$, ζ_S is friend with every β_b with $b \in S$.
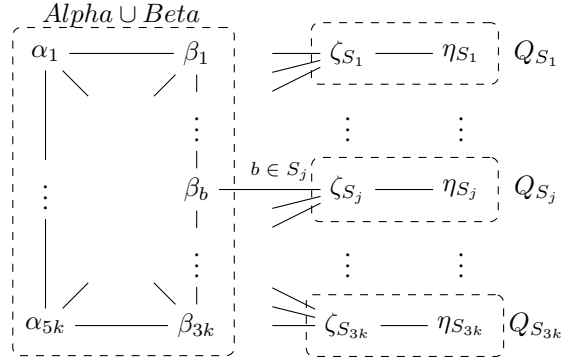
Figure 5: Network of friends in the proof of Theorem 7.

Furthermore, consider the coalition structure $\Gamma = \{Alpha \cup Beta, Q_{S_1}, \ldots, Q_{S_{3k}}\}$. We show that $\mathscr{S}$ contains an exact cover for B if and only if Γ is not strictly popular.

Only if: Assume that there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ for B. Since every set in $\mathscr{S}$ contains three elements of B, we have $|\mathscr{S}'| = k$. Consider the coalition structure $\Delta = \{Alpha \cup Beta \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S | S \in \mathscr{S} \setminus \mathscr{S}'\}$.

All $\beta_b, b \in B$ prefer Δ to Γ since they have $8k - 1$ friends in $\Gamma(\beta_b)$ but $8k$ friends in $\Delta(\beta_b)$.

All $\alpha_l, 1 \leq l \leq 5k$ prefer Γ to Δ because they have the same number of friends in both coalition structures but no enemies in $\Gamma(\alpha_l)$ and $2k$ enemies in $\Delta(\alpha_l)$.

All ζ_S with $S \in \mathscr{S}'$ prefer Δ to Γ because they have one friend in $\Gamma(\zeta_S)$ but four friends in $\Delta(\zeta_S)$. For all ζ_S with $S \in \mathscr{S} \setminus \mathscr{S}'$, it holds that $\Delta(\zeta_S) = \Gamma(\zeta_S)$. Hence, they decide their preferences according to their friends' valuations. They are friends with η_S who values Γ and Δ the same and friends with three $\beta_b, b \in S$, who all value Δ better than Γ. Hence, ζ_S prefers Δ to Γ.

All η_S with $S \in \mathscr{S}'$ prefer Γ to Δ because they have the same number of friends in $\Gamma(\eta_S)$ and $\Delta(\eta_S)$ but less enemies in $\Gamma(\eta_S)$. However, all η_S with $S \in \mathscr{S} \setminus \mathscr{S}'$ are indifferent between Γ and Δ because $\Delta(\eta_S) = \Gamma(\eta_S)$ and their only friend ζ_S values Γ and Δ the same.

We then have $\#_{\Delta \succ \Gamma} = |\{i \in N \mid \Delta(i) \succ_i \Gamma(i)\}| = |\{\beta_1, \ldots, \beta_{3k}, \zeta_{S_1}, \ldots, \zeta_{S_{3k}}\}| = 6k$ and $\#_{\Gamma \succ \Delta} = |\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}| = |\{\alpha_1, \ldots, \alpha_{5k}\} \cup \{\eta_S | S \in \mathscr{S}'\}| = 5k + k = 6k$. Since $\#_{\Delta \succ \Gamma} = \#_{\Gamma \succ \Delta}$, Γ is not strictly popular.

If: Assume that Γ is not strictly popular, i.e., that there is a coalition structure $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$ with $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$.

- For every $\alpha_l, 1 \leq l \leq 5k$, $Alpha \cup Beta$ is her best valued coalition since she is together with all her friends and none of her enemies. Every other coalition is valued worse. Hence, α_l prefers Γ to every coalition structure where she is not in $Alpha \cup Beta$. Furthermore, she is indifferent between Γ and Δ if $\Delta(\alpha_l) = Alpha \cup Beta$.

- If $Alpha \cup Beta$ were a coalition in Δ then some of the players from $Q_{S_1}, \ldots, Q_{S_{3k}}$ would be partitioned in a different way than in Γ. However, this would not cause any player to be happier. There would be at least two players who prefer Γ to Δ but no player who prefers Δ to Γ. This is a contradiction to the assumption.

Hence, $Alpha \cup Beta$ is not a coalition in Δ and all $5k$ α-players prefer Γ to Δ. (But we will show that $Alpha \cup Beta$ is subset of a coalition in Δ.) Furthermore, for every $\eta_S, S \in \mathscr{S}$ it holds that Q_S is her best valued coalition. Again, η_S prefers Γ to Δ if $\Delta(\eta_S) \neq Q_S$ and is indifferent between Γ and Δ if $\Delta(\eta_S) = Q_S$. Let k' be the number Q_S-sets that are not a coalition in Δ, i.e., $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$. Then k' is also the number of η-players who prefer Γ to Δ. The other $3k - k'$ η-players are indifferent between Γ and Δ.

First, observe that $k' \leq k$: For a contradiction, assume that $k' > k$. Then $5k$ α-players and at least $k + 1$ η-players prefer Γ to Δ, so $\#_{\Gamma \succ \Delta} \geq 6k + 1$. However, this is a contradiction to $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$ because there are at most $6k$ players who prefer Δ to Γ, namely $3k$ β-players and $3k$ ζ-players.

Next, observe that $k' \geq k$: For a contradiction, assume that $k' < k$. There are k' ζ_S-players with $\Delta(\zeta_S) \neq Q_S$. Since every ζ_S-player is friends with exactly three β-players, these k' ζ_S-players are friends with at most $3k'$ different β-players. For the other (at least) $3k - 3k'$ β-players it holds that they have none of their ζ-friends in $\Delta(\beta)$. That means, that they have at most $8k - 1$ friends in $\Delta(\beta)$ (all other α- and β-players). And if they have exactly $8k - 1$ friends in $\Delta(\beta)$, then they also have an enemy in $\Delta(\beta)$ because $Alpha \cup Beta$ is not a coalition in Δ. Hence, all these $3k - 3k'$ β-players prefer Γ to Δ. We then have $\#_{\Gamma \succ \Delta} \geq 5k + k' + (3k - 3k') = 8k - 2k' > 8k - 2k = 6k$. This is a contradiction to $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$ and $\#_{\Delta \succ \Gamma} \leq 6k$.

Hence, $k' = k$ and $5k$ α-players plus k η-players prefer Γ to Δ. Then, because of $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$, all β- and ζ-players prefer Δ to Γ. This is only possible if each of the $3k$ β-players has all α- and β-players and a ζ-player as friend in $\Delta(\beta)$. Since there are $k' = k$ ζ_S-players with $\Delta(\zeta_S) \neq Q_S$, each of these ζ_S-players is friends with three different β-players. Hence, $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover for B.

The results for strict popularity existence and popularity verification can be shown by slightly changing the above reduction. The details are omitted here. ❏

Perfectness. Turning now to perfectness, we start with the selfish-first model.

Theorem 8 *Let G be a network of friends on a set of agents N. A coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under the selfish-first model if and only if it consists of the connected components of G and all of them are cliques.*

Proof. From left to right, assume that the coalition structure $\Gamma \in \mathcal{C}_N$ is perfect. It then holds for all agents $i \in N$ and all coalition structures $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$, that i weakly prefers Γ to Δ ($\Gamma \succeq_i^{SF} \Delta$). It follows that $v_i(\Gamma) \geq v_i(\Delta)$ for all $\Delta \in \mathcal{C}_N, \Delta \neq \Gamma$ and $i \in N$. Hence, every agent $i \in N$ has the maximal valuation $v_i(\Gamma) = n \cdot |F_i|$ and is together with all of her friends and none of her enemies. This implies that each coalition in Γ is a connected component and a clique.

The implication from right to left is obvious. ❏

Since it is easy to check this characterization, perfect coalition structures can be verified in polynomial time for the selfish-first model. It follows directly from Theorem 8 that the corresponding existence problem is also in P:
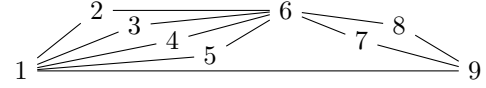

Figure 6: Network of friends for Example 11

Corollary 9 *Let N be a set of agents and G a network of friends on N. There exists a perfect coalition structure under the selfish-first model if and only if all connected components of G are cliques.*

Proposition 10 *Perfectness verification can be done in coNP for all three degrees of altruism. For equal treatment, perfectness existence is in coNP.*

We again omit the proof of Proposition 10. We only mention that membership of perfectness existence in coNP for equal treatment follows from the following observations. If a coalition structure $\Gamma \in \mathcal{C}_N$ is perfect under equal treatment then (a) each agent is together with all her friends (i.e., $(\forall i \in N)[F_i \subset \Gamma(i)]$) and (b) each coalition in Γ has a diameter of at most 2. Combining (a) and (b), Γ consists of the connected components of G and all these components have diameters of at most 2. However, this not an equivalence. The converse does not hold as the following example shows.

Example 11 *Let $N = \{1, \ldots, 9\}$ and let the network of friends G be given by Figure 6.*

$\Gamma = \{\{1, \ldots, 9\}\}$ *consists of the only connected component of G, which has a diameter of 2. However, Γ is not perfect under equal treatment because agent 1 prefers $\Delta = \{\{1, \ldots, 6\}, \{7, 8, 9\}\}$ to Γ: Omitting the details, we can show that $u_1^{EQ}(\Delta) = 113 > 112 = u_1^{EQ}(\Gamma)$.*

5 Conclusions and Open Problems

We have proposed to extend the model of altruistic hedonic games due to Nguyen *et al.* [2016] to coalition formation games in general. We have compared this more general model to altruism in hedonic games and have motivated our work by solving some crucial disadvantages that come with the restriction to hedonic games. We have studied the common stability notions and have initiated a computational analysis of the associated verification and existence problems (see Table 2 for an overview of our results). We also gave characterizations for some of the stability notions, using graph-theoretical properties of the underlying network of friends. For future work, we propose to complete this analysis and get a full characterization for all stability notions. Furthermore, it would be interesting to see if those problems for which we could only show coNP upper bounds are also coNP-complete.

Another interesting research topic could be the consideration of altruism for other representations of the players' preferences such as the friends-and-enemies encoding with enemy-oriented preferences [Dimitrov *et al.*, 2006].

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