Tractable Fragments of Datalog with Metric Temporal Operators

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Abstract

We study the data complexity of reasoning for fragments of DatalogMTL—an extension of Datalog with metric temporal operators over the rational numbers. Reasoning in DatalogMTL is PSPACE-complete, which handicaps its application in practice. To achieve tractability we first study the core fragment, which disallows conjunction in rule bodies, and show that reasoning remains PSPACE-hard. Intractability prompts us to also limit the kinds of temporal operators allowed in rules, and we propose a practical core fragment for which reasoning becomes TC^0-complete. Finally, we show that this fragment can be extended by allowing linear conjunctions in rule bodies (with at most one IDB atom), and show that the resulting fragment is NL-complete, thus no harder than plain linear Datalog.

1 Introduction

DatalogMTL [Brandt *et al.*, 2017; Brandt *et al.*, 2018; Wałęga *et al.*, 2019b] is a temporal extension of the fundamental rule language Datalog [Abiteboul *et al.*, 1995] in which atoms in rules may contain metric temporal logic (MTL) operators interpreted over a rational timeline [Ouaknine and Worrell, 2008; Hunter *et al.*, 2013]. For instance, DatalogMTL rule (1) states that cooling measures should be activated on a device x at time t if the device surpasses a fixed temperature threshold *at some point* in the interval $[t-2.5, t]$; in contrast, rule (2) captures a more conservative requirement, where cooling measures are activated if the aforementioned threshold is exceeded *continuously* within the given time interval:

$$\mathsf{Cool}(x) \leftarrow \diamondminus_{[0,2.5]}\mathsf{Temp}(x, high), \tag{1}$$

$$\mathsf{Cool}(x) \leftarrow \boxminus_{[0,2.5]}\mathsf{Temp}(x, high). \tag{2}$$

MTL is equipped with two alternative semantics: pointwise and continuous; in this paper we consider the latter, which is the one typically adopted in works on DatalogMTL.

DatalogMTL is receiving increasing attention as a suitably expressive language for querying temporal data [Brandt *et al.*, 2018; Artale *et al.*, 2017; Ryzhikov *et al.*, 2019] and reasoning over streams [Wałęga *et al.*, 2019a]. The use of DatalogMTL in applications is, however, handicapped by its high data complexity of reasoning, namely PSPACE-complete for consistency checking [Wałęga *et al.*, 2019b]. A known alternative to regain tractability while still allowing for temporal operators in rules is to preclude recursion altogether; Brandt *et al.* (2018) showed that, if timestamps and intervals are represented as dyadic rational numbers (rationals given as an integer fraction where the denominator is a power of 2), reasoning in nonrecursive DatalogMTL is first-order rewritable and hence AC^0 in data complexity.

In this paper, we study the data complexity of *recursive* fragments of DatalogMTL under continuous semantics. Our aim is to identify natural fragments with good computational properties and sufficient expressive power for data-intensive applications, while gaining a deeper understanding of interactions between constructs leading to intractability.

We first study the core fragment $\mathrm{DatalogMTL}_{\mathrm{core}}$, where each rule with the head different from $\bot$ has a singleton body; clearly, rules (1) and (2) satisfy this condition. Plain core Datalog and its extension with existential quantification in the rule heads are prominent KR languages, which can capture standard KR languages for ontology-based data access (OBDA) [Xiao *et al.*, 2018; Motik *et al.*, 2012]; reasoning in such languages is first-order rewritable, thus ensuring AC^0 data complexity [Calì *et al.*, 2012]. In Section 3 we show that this does not apply to our setting, as $\mathrm{DatalogMTL}_{\mathrm{core}}$ remains PSPACE-hard in data complexity, even if the *since* operator $\mathcal{S}$ is the only temporal operator allowed in rules. Operator $\mathcal{S}$ fully expresses the past diamond operator $\diamondminus$ (but not vice-versa), and hence the rule (1) can be written using only $\mathcal{S}$ as a temporal operator; in contrast, $\mathcal{S}$ does allow us to express $\boxminus$. This prompts us to investigate both $\mathrm{DatalogMTL}^{\diamondminus}_{\mathrm{core}}$—the core fragment where $\diamondminus$ is the only temporal operator allowed—and the fragment $\mathrm{DatalogMTL}^{\boxminus}_{\mathrm{core}}$. We show that reasoning in $\mathrm{DatalogMTL}^{\diamondminus}_{\mathrm{core}}$ is TC^0-complete, and hence both tractable and parallelisable; however, these favourable computational properties do not transfer to $\mathrm{DatalogMTL}^{\boxminus}_{\mathrm{core}}$, for which reasoning is P-hard.

In Section 4 we consider $\mathrm{DatalogMTL}_{\mathrm{lin}}$—the *linear* fragment, where rule bodies must contain at most one IDB conjunct (i.e., a conjunct whose predicate occurs in the head of some rule of the program). For instance, the program consist-

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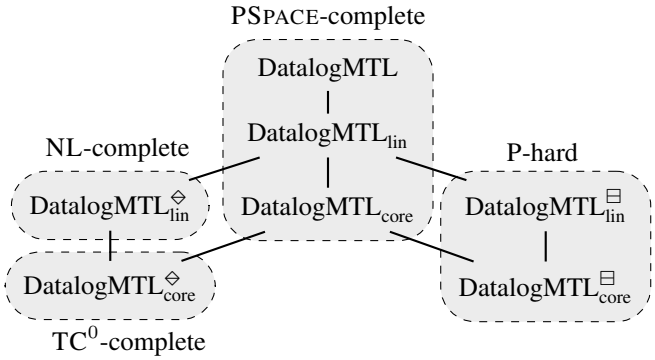


Figure 1: Hasse diagram of DatalogMTL-fragments (line segments link languages and their syntactical fragments).

ing of rule (3), which requires cooling measures to continue being applied if a moderate temperature is detected after cooling measures were already in force, is linear as the conjunction in (3) involves a single conjunct over IDB predicate Cool:

$$\mathsf{Cool}(x) \leftarrow \Diamond_{[0,2]}\mathsf{Cool}(x) \wedge \mathsf{Temp}(x, \mathit{moderate}). \qquad (3)$$

$\text{DatalogMTL}_{\text{lin}}$ extends $\text{DatalogMTL}_{\text{core}}$, and hence all lower bounds proved in Section 4 transfer. To regain favourable computational properties, we focus on $\text{DatalogMTL}^{\Diamond}_{\text{lin}}$: the linear fragment where $\Diamond$ is the only temporal operator allowed. We then show that reasoning in this fragment is NL-complete, and hence no harder than in plain linear Datalog [Dantsin *et al.*, 2001]. Linear Datalog provides the foundation for recursion in SQL, and hence our results open the door to efficient implementations of temporal recursive queries.

Our results are summarised in Figure 1 for reference (recall that $\text{TC}^0 \subseteq \text{NL} \subseteq \text{P} \subseteq \text{PSPACE}$ and $\text{NL} \neq \text{PSPACE}$).

2 Preliminaries

In this section, we define DatalogMTL and its associated reasoning problems [Brandt *et al.*, 2018]. We also recapitulate some important properties of DatalogMTL proven by Wałęga *et al.* (2019b), which we exploit in our technical results.

Intervals. We consider intervals over rationals $\mathbb{Q}$. An interval is denoted by $\langle x, y\rangle$, where the left bracket $\langle$ is [or (, the right bracket $\rangle$ is] or), and $x, y \in \mathbb{Q} \cup \{-\infty, +\infty\}$. Hence,

$$\langle x, y\rangle = \{\, t \in \mathbb{Q} \mid x \leq t \leq y, \\ t \neq x \text{ if } \langle \text{ is } (,\ \text{and } t \neq y \text{ if } \rangle \text{ is }) \,\}.$$

An interval $\langle x, y\rangle$ is *positive* if $x \geq 0$; it is *bounded* if $x, y \in \mathbb{Q}$; and it is *punctual* if it is of the form $[t, t]$, in which case we write $\{t\}$ instead of $[t, t]$. Each number in $\mathbb{Q}$ is represented by an integer numerator and a positive integer denominator, both encoded in binary (we do not require rationals to be dyadic).

Syntax. Assume a function-free first-order vocabulary with constants and predicates equipped with a non-negative arity. An *atom* is of the form $P(\boldsymbol{\tau})$, with P a predicate and $\boldsymbol{\tau}$ a tuple of constants and variables of matching arity. Each predicate in the vocabulary is either *extensional* (EDB) or *intensional* (IDB). A *literal* A is an expression given by the following grammar, with $P(\boldsymbol{\tau})$ an atom and ϱ a positive interval:

$$A ::= P(\boldsymbol{\tau}) \mid \top \mid \bot \mid \Diamond_\varrho A \mid \oplus_\varrho A \mid \boxminus_\varrho A \mid \boxplus_\varrho A \mid A\mathcal{S}_\varrho A \mid A\mathcal{U}_\varrho A.$$

A literal is ***EDB*** if it mentions only EDB predicates and ***IDB*** otherwise. A (DatalogMTL) *rule* is an expression of the form

$$B \leftarrow A_1 \wedge \cdots \wedge A_n, \qquad \text{for } n \geq 0, \qquad (4)$$

where each A_i is a literal and B is a literal not mentioning the operators $\Diamond$, $\oplus$, $\mathcal{S}$, and $\mathcal{U}$. The conjunction of the A_i is the rule *body*, whereas the literal B is the rule *head*. A rule is *safe* if each variable in its head occurs also in its body. A DatalogMTL *program* is a finite set of safe rules. The *greatest common divisor* $\mathsf{gcd}(\Pi)$ of a program Π is the largest rational number dividing all rationals which are endpoints of intervals in Π to integer values (if Π has no numbers, we take $\mathsf{gcd}(\Pi) = 1$ for definiteness). A program is *core* if each of its rules is of the form $B \leftarrow A$ or $\bot \leftarrow A_1 \wedge A_2$. A program is *linear* if each of its rules is either of form (4) with at most one A_i being an IDB literal, or of the form $\bot \leftarrow A_1 \wedge A_2$. We denote the core and linear fragments by $\text{DatalogMTL}_{\text{core}}$ and $\text{DatalogMTL}_{\text{lin}}$, respectively. For $X \in \{\Diamond, \boxminus\}$, and $Y \in \{\text{core}, \text{lin}\}$, we denote by DatalogMTL^X_Y the fragment of DatalogMTL_Y where X is the only temporal operator that may occur in literals. A program is *propositional* if all its predicates are nullary (i.e., propositions), and it is *positive* if it does not mention $\bot$. An expression e (e.g., an atom or a literal) is *ground* if it mentions no variables. A ground expression e' is a *grounding* of e if $e' = e\nu$ for an assignment ν of variables to constants. A *fact* is $\alpha@\varrho$, where α is a ground atom and ϱ is a non-empty interval. A *dataset* is a finite set of facts mentioning only EDB predicates.

Semantics. An *interpretation* $\mathfrak{M}$ specifies, for each ground atom α and each time point $t \in \mathbb{Q}$, whether α is true at t, in which case we write $\mathfrak{M}, t \models \alpha$. This notion extends to ground literals with temporal operators as in Table 1, where $r, s, t \in \mathbb{Q}$. An interpretation $\mathfrak{M}$ is a *model* of a rule of form (4) if, for each grounding assignment ν and each $t \in \mathbb{Q}$, we have $\mathfrak{M}, t \models B\nu$ whenever $\mathfrak{M}, t \models A_i\nu$ for each $i \in \{1, \ldots, n\}$. An interpretation $\mathfrak{M}$ is a *model* of program Π, written $\mathfrak{M} \models \Pi$, if $\mathfrak{M}$ is a model of all rules in Π. Interpretation $\mathfrak{M}$ is a *model* of fact $\alpha@\varrho$, written $\mathfrak{M} \models \alpha@\varrho$, if $\mathfrak{M}, t \models \alpha$ for all time points $t \in \varrho$, and it is a *model* of dataset $\mathcal{D}$, written $\mathfrak{M} \models \mathcal{D}$, if it is a model of all facts in $\mathcal{D}$. Program Π and dataset $\mathcal{D}$ are *consistent* if there is a model of both Π and $\mathcal{D}$. Program Π and dataset $\mathcal{D}$ *entail* fact $\alpha@\varrho$, written $(\Pi, \mathcal{D}) \models \alpha@\varrho$, if $\mathfrak{M} \models \alpha@\varrho$ for each model $\mathfrak{M}$ of Π and $\mathcal{D}$. For brevity, we write $\mathcal{D} \models \alpha@\varrho$ instead of $(\emptyset, \mathcal{D}) \models \alpha@\varrho$.

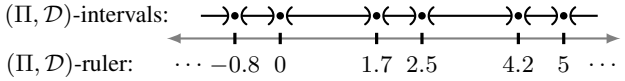
Reasoning Problems. We study the complexity of consistency checking for the linear and core fragments of DatalogMTL. We focus on *data complexity*—the standard measure of complexity in data-intensive applications, where programs are assumed to be fixed. All our complexity results are also applicable to the entailment of punctual facts of the form $\alpha@\{t\}$, because such entailment and inconsistency checking problems are inter-reducible in all the fragments we consider. On the one hand, $(\Pi, \mathcal{D}) \models \alpha@\{t\}$ if and only if $\Pi \cup \{\bot \leftarrow \alpha \wedge P\}$ and $\mathcal{D} \cup \{P@\{t\}\}$ are inconsistent, with

$\mathfrak{M}, t \models \top$	for all $t \in \mathbb{Q}$
$\mathfrak{M}, t \not\models \bot$	for all $t \in \mathbb{Q}$
$\mathfrak{M}, t \models \diamondsuit_\varrho A$	if $\mathfrak{M}, s \models A$ for some s with $t - s \in \varrho$
$\mathfrak{M}, t \models \boxplus_\varrho A$	if $\mathfrak{M}, s \models A$ for some s with $s - t \in \varrho$
$\mathfrak{M}, t \models \boxminus_\varrho A$	if $\mathfrak{M}, s \models A$ for all s with $t - s \in \varrho$
$\mathfrak{M}, t \models \boxdot_\varrho A$	if $\mathfrak{M}, s \models A$ for all s with $s - t \in \varrho$
$\mathfrak{M}, t \models A\mathcal{S}_\varrho A'$	if $\mathfrak{M}, s \models A'$ for some s with $t - s \in \varrho$ and $\mathfrak{M}, r \models A$ for all $r \in (s, t)$
$\mathfrak{M}, t \models A\mathcal{U}_\varrho A'$	if $\mathfrak{M}, s \models A'$ for some s with $s - t \in \varrho$ and $\mathfrak{M}, r \models A$ for all $r \in (t, s)$

Table 1: Semantics of DatalogMTL ground literals

P a fresh proposition; on the other hand, Π and $\mathcal{D}$ are inconsistent if and only if Π and $\mathcal{D}$ entail a fact with a proposition not mentioned in Π or $\mathcal{D}$. In contrast, all known reductions of entailment of facts with arbitrary intervals to inconsistency checking require $\mathcal{S}$ or $\mathcal{U}$, which are not always available in our fragments. We assume without loss of generality that the least number occurring explicitly in each dataset is 0 (for a fixed program, any dataset can be modified to satisfy this assumption while preserving consistency).

Rulers. If a DatalogMTL program Π and a dataset $\mathcal{D}$ are consistent, then they have a unique least model (i.e., the model assigning true as little as possible) [Brandt *et al.*, 2017]. As shown by Wałęga *et al.* (2019b), such a least model can be divided into regularly distributed intervals whose time points satisfy the same ground atoms as follows. The $(\Pi, \mathcal{D})$-*ruler* is the set of time points of the form $x + n \cdot \mathsf{gcd}(\Pi)$, for x a number in $\mathcal{D}$ and $n \in \mathbb{Z}$. A $(\Pi, \mathcal{D})$-*interval* is either a punctual interval over a time point on the ruler, or an interval (t_1, t_2) with t_1 and t_2 consecutive time points on the ruler. For a sequence $\varrho_i, \varrho_{i+1}, \dots, \varrho_j$ of successive $(\Pi, \mathcal{D})$-intervals (i.e., such that $\bigcup_i \varrho_i$ is an interval), with $i \leq j$, we define the (asymmetric) *distances* between ϱ_i and ϱ_j as $\mathsf{dist}(\varrho_i, \varrho_j) = j - i$ and $\mathsf{dist}(\varrho_j, \varrho_i) = -\mathsf{dist}(\varrho_i, \varrho_j)$. As an example, we present below a fragment of the $(\Pi, \mathcal{D})$-ruler for $\Pi = \{\beta \leftarrow \diamondsuit_{[0,2.5)}\alpha, \gamma \leftarrow \diamondsuit_{\{5\}}\beta\}$ and $\mathcal{D} = \{\alpha@(0, 1.7)\}$, where $\mathsf{gcd}(\Pi) = 2.5$ and $\mathsf{dist}(\{0\}, (2.5, 4.2)) = 5$.



The following lemma by Wałęga *et al.* (2019b) shows that points in a given ruler interval satisfy the same ground atoms.

Lemma 1. *For every* DatalogMTL *program* Π*, dataset* $\mathcal{D}$*, atom* α*, and* $t \in \mathbb{Q}$*, if* $(\Pi, \mathcal{D}) \models \alpha@\{t\}$*, then* $(\Pi, \mathcal{D}) \models \alpha@\varrho$*, where* ϱ *is the* $(\Pi, \mathcal{D})$*-interval such that* $t \in \varrho$*.*

The $(\Pi, \mathcal{D})$-ruler can be divided into segments of length $\mathsf{gcd}(\Pi)$ starting from 0, such that each segment contains the same number of $(\Pi, \mathcal{D})$-intervals and the corresponding intervals in different segments have the same lengths. The *density* d of Π and $\mathcal{D}$ is $\mathsf{dist}(\{0\}, \{\mathsf{gcd}(\Pi)\})$; this induces an equivalence relation on $(\Pi, \mathcal{D})$-intervals with d equivalence classes, which we call $(\Pi, \mathcal{D})$-*types* and denote by natural numbers between 0 and $d-1$. The $(\Pi, \mathcal{D})$-type k consists of all $(\Pi, \mathcal{D})$-intervals ϱ such that $\mathsf{dist}(\{0\}, \varrho) - k$ is a multiple of d.

3 Core Fragments

In this section, we study the data complexity of reasoning in the core DatalogMTL fragments. We first show that consistency checking in DatalogMTL$_{\text{core}}$ is PSPACE-hard, which matches the known PSPACE upper bound for the full DatalogMTL language [Wałęga *et al.*, 2019b]. In particular, each problem in PSPACE reduces to checking inconsistency of a dataset and a propositional DatalogMTL$_{\text{core}}$ program where $\mathcal{S}$ is the only temporal operator.

Theorem 2. *Consistency checking in* DatalogMTL$_{\text{core}}$ *is* PSPACE*-hard in data complexity. The result holds already for propositional programs using only* $\mathcal{S}$ *as a temporal operator.*

Proof sketch. Let $\mathcal{P}$ be a problem in PSPACE. Then, there exists a deterministic Turing machine M and a polynomial p such that, for every input word w, M decides $\mathcal{P}$ for w using at most $p(|w|)$ cells. We construct in logarithmic space a dataset $\mathcal{D}_w$ encoding the initial configuration of M on w and a propositional DatalogMTL$_{\text{core}}$ program Π_M simulating the computations of M (on any input word) such that M accepts w if and only if Π_M and $\mathcal{D}_w$ are inconsistent.

Each configuration of M in the run on w is encoded within a unit interval (i.e., of length 1) divided into $p(|w|)$ subintervals, each describing a cell of the tape. Each subinterval is split into two (left-open and right-closed) halves indicated by propositions P and Q, which respectively hold in the first and second half of each subinterval. In addition, we use the following propositions to encode a configuration: P_a—for each alphabet symbol a—which holds in the first half of a subinterval if the corresponding cell contains a; Q_q—for each state q—which holds in the second half of a subinterval if the head is over the corresponding cell and M is in q; and Q_L and Q_R, which hold in the second half of a subinterval if the corresponding cell is, respectively, somewhere to the left and to the right of the head's position.

Program Π_M has two main blocks. The first block consists of the following rules, for each symbol a and state q:

$$P \leftarrow \top\mathcal{S}_{\{1\}}P, \qquad Q \leftarrow \top\mathcal{S}_{\{1\}}Q,$$
$$Q^L_{aq} \leftarrow \top\mathcal{S}_{\{1\}}(Q_q\mathcal{S}_{(0,\infty)}P_a), \qquad P_{aq} \leftarrow P\mathcal{S}_{(0,\infty)}Q^L_{aq},$$
$$Q_{aq} \leftarrow Q\mathcal{S}_{(0,\infty)}P_{aq}, \qquad Q^R_{aq} \leftarrow Q\mathcal{S}_{(0,\infty)}(P\mathcal{S}_{(0,\infty)}Q_{aq}).$$

The first two rules propagate P and Q. The remaining rules copy the description of a configuration into the next unit interval using auxiliary propositions (note that there is a gap between the unit intervals encoding consecutive configurations). The rules ensure that, if P_a and Q_q hold in the two halves of a subinterval, then P_{aq} and Q_{aq} hold in the two halves of the corresponding subinterval in the next unit interval. Moreover, if P_{aq} and Q_{aq} hold in the same subinterval, then Q^L_{aq} and Q^R_{aq} hold in the second halves of the subintervals that are adjacent, respectively, to the left and to the right.

The second block of Π_M encodes the transition function of M (and does not use temporal operators). Each left-moving

transition $(q, a) \mapsto (q', a', L)$ is encoded by the rules

$$P_{a'} \leftarrow P_{aq}, \quad Q_{q'} \leftarrow Q^L_{aq}, \quad Q_R \leftarrow Q_{aq}, \quad Q_R \leftarrow Q^R_{aq}.$$

Right-moving transitions are encoded analogously; similar rules are also needed to propagate Q_L, Q_R, and P_a, for cells without the head. Finally, Π_M contains the rule $\bot \leftarrow Q_q$ for each accepting state q, which ensures inconsistency of Π_M and $\mathcal{D}_w$ when M accepts w. □

The since operator $\mathcal{S}$ is strictly more expressive than the past diamond operator $\Diamond\!\!\!-$. Hence, to regain tractability, we consider the DatalogMTL$^{\Diamond}_{\text{core}}$ fragment and show that, in this case, consistency checking is TC^0-complete. This suggests suitability of DatalogMTL$^{\Diamond}_{\text{core}}$ for data-intensive applications.

We start by defining a normal form. A DatalogMTL$^{\Diamond}_{\text{core}}$ program is *normal* if each of its rules has one of the following forms, where α, α_1, α_2, and β are atoms, and ϱ is a non-empty bounded positive interval:

$$\beta \leftarrow \Diamond\!\!\!-_{\varrho}\alpha, \qquad \beta \leftarrow \top, \qquad \bot \leftarrow \alpha_1 \wedge \alpha_2.$$

Each DatalogMTL$^{\Diamond}_{\text{core}}$ program can be normalised so that the resulting program is a conservative extension of the original one: rules with nested operators are flattened by introducing fresh predicates, rules $\alpha \leftarrow \alpha'$ without metric operators are rewritten as $\alpha \leftarrow \Diamond\!\!\!-_{\{0\}}\alpha'$, and rules $P(\boldsymbol{\tau}) \leftarrow \Diamond\!\!\!-_{[t,\infty)}P'(\boldsymbol{\tau}')$ with unbounded intervals are replaced with rules $Q(\boldsymbol{\tau}) \leftarrow \Diamond\!\!\!-_{\{t\}}P'(\boldsymbol{\tau}')$, $Q(\boldsymbol{\tau}) \leftarrow \Diamond\!\!\!-_{[0,1]}Q(\boldsymbol{\tau})$, and $P(\boldsymbol{\tau}) \leftarrow \Diamond\!\!\!-_{\{0\}}Q(\boldsymbol{\tau})$, for Q a fresh predicate of the same arity as P. Hence, in this section, we concentrate on normal DatalogMTL$^{\Diamond}_{\text{core}}$ programs.

Next, we provide a characterisation of fact entailment for ground normal DatalogMTL$^{\Diamond}_{\text{core}}$ programs in terms of existence of a specific length path in a graph defined below.

Definition 3. Let Π be a ground normal DatalogMTL$^{\Diamond}_{\text{core}}$ program and let $d \in \mathbb{N}$. Then, G^d_Π is the directed edge-weighted multigraph with a vertex v^α_k for each (ground) atom α in Π and each $k \in \{0, \dots, d-1\}$; and an edge $(v^\alpha_k, v^\beta_\ell)$ of weight n whenever $\ell = (k+n) \bmod d$ and Π has a rule $\beta \leftarrow \Diamond\!\!\!-_{\varrho}\alpha$ such that $n \cdot (\mathsf{gcd}(\Pi)/d) \in \varrho$.

The graph G^d_Π is defined so that, if d is the density of Π and a dataset $\mathcal{D}$, then every derivation from Π and $\mathcal{D}$ (i.e., a sequence of punctual facts) corresponds to a path in G^d_Π such that a 1-step derivation of $\beta@\{t_2\}$ from $\alpha@\{t_1\}$ corresponds to an edge $(v^\alpha_k, v^\beta_\ell)$ of weight $\mathsf{dist}(\varrho_1, \varrho_2)$, where $\varrho_1 \ni t_1$ and $\varrho_2 \ni t_2$ are $(\Pi, \mathcal{D})$-intervals of $(\Pi, \mathcal{D})$-types k and ℓ, respectively. Indeed, the following result holds.

Lemma 4. *Let Π be a ground and positive normal* DatalogMTL$^{\Diamond}_{\text{core}}$ *program, $\mathcal{D}$ a dataset, and d the density of Π and $\mathcal{D}$. Then, for each fact $\beta@\varrho$ with a $(\Pi, \mathcal{D})$-interval ϱ, the following are equivalent:*

1. *$(\Pi, \mathcal{D}) \models \beta@\varrho$;*
2. *there exist a fact $\alpha@\varrho'$ with ϱ' a $(\Pi, \mathcal{D})$-interval such that $\mathcal{D} \models \alpha@\varrho'$ (or Π contains a rule $\alpha \leftarrow \top$) and a path in G^d_Π of weight $\mathsf{dist}(\varrho', \varrho)$ from v^α_k to v^β_ℓ, where k and ℓ are the $(\Pi, \mathcal{D})$-types of ϱ' and ϱ, respectively.*

We next show that, given two vertices v and v' in a directed multigraph with weights of edges being non-negative integers (e.g., G^d_Π), we can finitely represent the weights of all paths from v to v'. This lemma relies on the normal form for unary non-deterministic finite automata (NFAs) by Chrobak (1986) and will be essential to prove the TC^0 upper bound.

Lemma 5. *Let G be a directed edge-weighted multigraph with non-negative integer weights and let v, v' be vertices in G. Then, it is possible to construct a finite set S of pairs of non-negative integers such that there is a path in G of weight n from v to v' if and only if $n = a + i \cdot b$ for a pair $(a, b) \in S$ and a non-negative integer i.*

Proof sketch. First, we replace each edge in G of weight $n > 1$ with a chain of n edges of weight 1 by introducing fresh nodes. Then, we construct an NFA $\mathcal{A}$ over a unary alphabet $\{\sigma\}$, where states correspond to vertices: the initial state corresponds to v and the final state to v', and we have a σ-transition for each edge of weight 1 and an ϵ-transition for each edge of weight 0. Then, the weights of paths in G from v to v' are exactly the lengths of words accepted by $\mathcal{A}$. After eliminating ϵ-transitions, we obtain a unary NFA, which we then transform to Chrobak normal form [Chrobak, 1986, Lemma 4.3], where the transitions form a path from the initial state followed by a single nondeterministic choice between several (disjoint) cycles. For every accepting state q in the resulting automaton $\mathcal{A}'$, there exist $a, b \in \mathbb{N}$ such that there is an accepting run of $\mathcal{A}'$ on σ^n ending at q if and only if there is a number i with $n = a + i \cdot b$ (where a and b are the lengths of the simple paths from the initial state to q and of the cycle through q, respectively). □

Lemmas 4 and 5 suggest a TC^0 algorithm for checking whether a fixed ground positive normal DatalogMTL$^{\Diamond}_{\text{core}}$ program Π and a dataset $\mathcal{D}$ entail $\beta@\{t\}$. For this, it suffices to check if there is $\alpha@\varrho$ in $\mathcal{D}$ such that $(\Pi, \{\alpha@\varrho\}) \models \beta@\{t\}$. These checks can be done in parallel for each fact $\alpha@\varrho$ in $\mathcal{D}$ and hence we prove that each of them is feasible in TC^0. Assume that $\mathcal{D}' = \{\alpha@\varrho\}$; we want to check whether $(\Pi, \mathcal{D}') \models \beta@\{t\}$. First, we compute the density d of Π and $\mathcal{D}'$, which is at most 4 since $\mathcal{D}'$ contains only one interval. Then, we compute the $(\Pi, \mathcal{D}')$-interval ϱ' containing t, and the $(\Pi, \mathcal{D}')$-type ℓ of ϱ'. We also compute the left-most, ϱ^L, and the right-most, ϱ^R, $(\Pi, \mathcal{D}')$-intervals contained in ϱ, as well as the set K of $(\Pi, \mathcal{D}')$-types of $(\Pi, \mathcal{D}')$-intervals contained in ϱ. All these computations are feasible in TC^0. Now, by Lemma 4, it suffices to check whether there is a path of weight n from v^α_k to v^β_ℓ in G^d_Π such that $\mathsf{dist}(\varrho^R, \varrho') \le n \le \mathsf{dist}(\varrho^L, \varrho')$ and $k \in K$ (if Π contains rules of the form $\alpha' \leftarrow \top$, some additional but easy to handle checks are required), which by Lemma 5 can also be checked in TC^0.

To check consistency of a ground normal DatalogMTL$^{\Diamond}_{\text{core}}$ program Π and a dataset $\mathcal{D}$, we need to consider rules of the from $\bot \leftarrow \alpha_1 \wedge \alpha_2$, where α_1 and α_2 are ground atoms. In particular, inconsistency arises whenever there is a time point where both α_1 and α_2 hold together in the least model of $\mathcal{D}$ and the positive subset of Π. This leads to technical difficulties since there is an unbounded number of candidate time points. The following theorem shows that these difficulties

can be overcome, and consistency checking is TC^0-complete, where the upper bound holds also for non-ground programs.

Theorem 6. *Consistency checking for* $\mathrm{DatalogMTL}^{\diamondsuit}_{\mathrm{core}}$ *is* TC^0*-complete in data complexity. The lower bound holds for propositional* $\mathrm{DatalogMTL}_{\mathrm{core}}$ *without metric operators.*

Proof sketch. For the lower bound, there is a simple AC^0 reduction of the integer multiplication problem [Hesse, 2001]: $a \cdot b = c$ for positive integers a, b, and c (encoded in binary) if and only if the program $\{\bot \leftarrow A \wedge B\}$, for propositions A and B, and the dataset $\{A@\{a\}, B@\{\frac{c}{b}\}\}$ are inconsistent.

For the upper bound, we first consider a ground normal $\mathrm{DatalogMTL}^{\diamondsuit}_{\mathrm{core}}$ program Π, with Π^+ its positive subset. Since rules in Π with $\bot$ do not contain temporal operators, the $(\Pi, \mathcal{D})$- and $(\Pi^+, \mathcal{D})$-rulers and the corresponding densities coincide, as well as $G^d_{\Pi} = G^d_{\Pi^+}$, for each $d \in \mathbb{N}$.

By linearity of derivations and by Lemma 1, Π and $\mathcal{D}$ are inconsistent if and only if there is a rule $\bot \leftarrow \beta_1 \wedge \beta_2$ in Π and a subset $\mathcal{D}'$ of $\mathcal{D}$ with at most two facts such that $(\Pi^+, \mathcal{D}') \models \beta_i@\varrho$ for both i and some $(\Pi^+, \mathcal{D}')$-interval ϱ. Consider such $\mathcal{D}' = \{\alpha_1@\varrho_1, \alpha_2@\varrho_2\}$. Moreover, for both i, let ϱ_i^L and ϱ_i^R be the left- and the right-most $(\Pi^+, \mathcal{D}')$-intervals in ϱ_i, respectively, let K_i be the sets of the $(\Pi^+, \mathcal{D}')$-types of all $(\Pi^+, \mathcal{D}')$-intervals in ϱ_i, and let d be the density of Π^+ and $\mathcal{D}'$. Then, assuming for simplicity that Π does not have rules $\alpha \leftarrow \top$, we use Lemma 4 to reduce existence of ϱ as above to existence of two paths in $G^d_{\Pi^+}$: one from $v^{\alpha_1}_{k_1}$ to $v^{\beta_1}_{\ell}$ with weight n_1 and another from $v^{\alpha_2}_{k_2}$ to $v^{\beta_2}_{\ell}$ with weight n_2, where $k_i \in K_i$ for both i, ℓ is a $(\Pi^+, \mathcal{D}')$-type, and

$$\mathsf{dist}(\varrho_2^R, \varrho_1^L) \leq n_1 - n_2 \leq \mathsf{dist}(\varrho_2^L, \varrho_1^R). \tag{5}$$

We can check existence of such pairs of paths in parallel for each combination of k_1, k_2, and ℓ. By Lemma 5, we can represent the weights of all paths from $v^{\alpha_i}_{k_i}$ to $v^{\beta_i}_{\ell}$ by a finite set S_i of integer pairs. So to check (5) it suffices to verify if for some $(a_1, b_1) \in S_1$ and $(a_2, b_2) \in S_2$ there exist non-negative integers i_1, i_2 with

$$\mathsf{dist}(\varrho_2^R, \varrho_1^L) \leq (a_1 + i_1 \cdot b_1) - (a_2 + i_2 \cdot b_2) \leq \mathsf{dist}(\varrho_2^L, \varrho_1^R),$$

which, by Bézout's identity, exist if and only if the following interval contains an integer:

$$\left[\frac{\mathsf{dist}(\varrho_2^R, \varrho_1^L) - a_1 + a_2}{\mathsf{gcd}(b_1, b_2)}, \frac{\mathsf{dist}(\varrho_2^L, \varrho_1^R) - a_1 + a_2}{\mathsf{gcd}(b_1, b_2)}\right].$$

The existence of such an integer can be verified in TC^0 (for all $(a_1, b_1) \in S_1$ and $(a_2, b_2) \in S_2$ in parallel); hence, we can check the consistency in TC^0 in the size of $\mathcal{D}$.

Finally, let Π be a non-ground normal $\mathrm{DatalogMTL}^{\diamondsuit}_{\mathrm{core}}$ program. Note that although the grounding $\Pi_{\mathcal{D}}$ of Π with constants from $\mathcal{D}$ can be computed in AC^0, the graph $G^d_{\Pi_{\mathcal{D}}}$ depends not only on Π and d but also on $\mathcal{D}$; this would prevent us from exploiting the TC^0 procedure for ground programs. However, if $\mathcal{D}'$ and $\mathcal{D}''$ are the same modulo renaming of constants not occurring in Π, then $G^k_{\Pi_{\mathcal{D}'}}$ and $G^k_{\Pi_{\mathcal{D}''}}$ are isomorphic. Since we are interested in datasets with at most two facts and with arity of predicates fixed by Π, the number of non-isomorphic graphs for such datasets can be bounded without knowing $\mathcal{D}$. Thus, we can compute the representatives of such non-isomorphic graphs in a data-independent way and reuse our TC^0 procedure for ground programs. □

Now, we show P-hardness for consistency checking in $\mathrm{DatalogMTL}^{\boxminus}_{\mathrm{core}}$, that holds even for propositional programs.

Theorem 7. *Consistency checking in propositional* $\mathrm{DatalogMTL}^{\boxminus}_{\mathrm{core}}$ *is* P*-hard in data complexity.*

Proof sketch. The proof uses some ideas from the hardness proofs for pointwise semantics by Ryzhikov *et al.* (2019). We reduce the well-known P-complete path system accessibility problem (PSA) to inconsistency. An instance G of PSA is a tuple (V, E, S, v_t), where $V = \{v_1, \ldots, v_n\}$ is a finite set of nodes, E is a ternary relation on V such that $i < j < k$ for each $(v_i, v_j, v_k) \in E$, $S \subseteq V$ is a set of (source) nodes, and $v_t \in V$ is a (target) node. The answer to G is affirmative if v_t is accessible, where a node v_k is *accessible* if $v_k \in S$ or there are accessible nodes v_i, v_j such that $(v_i, v_j, v_k) \in E$.

Let Π be the (fixed) program consisting of the following rules over propositions P, P', P'', and Q:

$$P' \leftarrow \boxminus_{\{2\}} P, \qquad P'' \leftarrow \boxminus_{(0,1)} P', \quad P \leftarrow \boxminus_{\{2\}} P'',$$
$$P \leftarrow \boxminus_{\{4\}} P, \qquad \bot \leftarrow P \wedge Q.$$

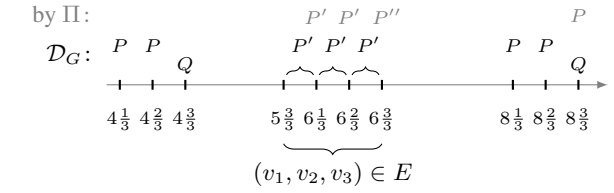
Consider now a PSA instance G as above, and let $e_1, \ldots, e_m$ be an enumeration of E preserving the 'application order'—that is, such that for every edge e_ℓ with node v_k as the third argument, v_k is not the first or second argument of any e_h with $h < \ell$. Let $\mathcal{D}_G$ be a dataset containing the following facts, for every ℓ such that $1 \leq \ell \leq m + 1$:

$$Q@\{4\ell + t/n\} \quad \text{and} \quad P@\{4\ell + s/n\}, \text{ for every } v_s \in S,$$

and the following facts, for every $e_\ell = (v_i, v_j, v_k)$:

$$P'@(4\ell + 1 + k/n, 4\ell + 2 + i/n),$$
$$P'@(4\ell + 2 + i/n, 4\ell + 2 + j/n),$$
$$P'@(4\ell + 2 + j/n, 4\ell + 2 + k/n).$$

For instance, $\mathcal{D}_G$, for G with $V = \{v_1, v_2, v_3\}$, $E = \{(v_1, v_2, v_3)\}$, $S = \{v_1, v_2\}$, and $t = 3$, is as follows.



It can then be easily checked that v_t is accessible if and only if Π and $\mathcal{D}_G$ are inconsistent. □

4 Linear Fragments

Since linear fragments of DatalogMTL are extensions of corresponding core fragments, all lower bounds established in Section 4 for core fragments hold also for linear fragments. To obtain favourable computational properties, we focus on $\mathrm{DatalogMTL}^{\diamondsuit}_{\mathrm{lin}}$ and show NL-completeness of consistency checking in data complexity. The lower bound is inherited

from plain linear Datalog, but we will prove it also for propositional DatalogMTL$^{\diamondsuit}_{\text{lin}}$ programs.

For the upper bound, we will concentrate on programs in a normal form, extending the one for the core case; a DatalogMTL$^{\diamondsuit}_{\text{lin}}$ program is *normal* if it consists only of rules of the following form, where α, α_1, α_2, and β are IDB atoms, ϕ is a possibly empty conjunction of EDB literals, and ϱ is a non-empty bounded positive interval:

$$\beta \leftarrow \diamondsuit_{\varrho}\alpha \wedge \phi, \qquad \beta \leftarrow \top, \qquad \bot \leftarrow \alpha_1 \wedge \alpha_2.$$

Each DatalogMTL$^{\diamondsuit}_{\text{lin}}$ program can be normalised similarly as before—that is, by introducing rules with fresh predicates.

Naively, an inconsistency checking algorithm could guess a derivation of $\bot$ from a normal program and an input dataset. This, however, does not lead to an NL algorithm since the length of a derivation and the representation of its elements (in particular numbers occurring in intervals) are unbounded. We next show how to overcome this problem.

Theorem 8. *Consistency checking for* DatalogMTL$^{\diamondsuit}_{\text{lin}}$ *is* NL*-complete in data complexity. The lower bound holds already for propositional* DatalogMTL$^{\diamondsuit}_{\text{lin}}$.

Proof sketch. The lower bound is by reduction of consistency checking for propositional normal DatalogMTL$^{\diamondsuit}_{\text{core}}$ under *pointwise semantics*, where datasets consist of punctual facts and only the time points explicit in the dataset constitute the temporal universe of interpretations; this problem is NL-complete in data complexity [Ryzhikov *et al.*, 2019]. Given a propositional normal DatalogMTL$^{\diamondsuit}_{\text{core}}$ program Π, we introduce a fresh IDB proposition Q_A for each literal A of the form $\diamondsuit_{\varrho}P$ in Π, and a fresh EDB proposition Q. We then construct a propositional DatalogMTL$^{\diamondsuit}_{\text{lin}}$ program Π' by replacing each literal A as above in Π by Q_A and adding rules $Q_A \leftarrow A \wedge Q$. For each (punctual) dataset $\mathcal{D}$, Π and $\mathcal{D}$ are consistent under pointwise semantics if and only if so are Π' and $\mathcal{D} \cup \{Q@\varrho \mid \varrho \text{ occurs in } \mathcal{D}\}$ under continuous semantics.

For the upper bound, consider a normal DatalogMTL$^{\diamondsuit}_{\text{lin}}$ program Π and a dataset $\mathcal{D}$. By Lemma 1, Π and $\mathcal{D}$ are inconsistent if and only if there are a grounding $\bot \leftarrow \beta_1 \wedge \beta_2$ of a rule in Π and derivations of both facts $\beta_i@\varrho$ for some $(\Pi, \mathcal{D})$-interval ϱ. We can see each derivation as a sequence of facts over ruler intervals with IDB predicates. Since, such a sequence may be of exponential length, to achieve the bound, we will represent such derivations concisely as follows. Let Φ be the set of all groundings of the conjunctions ϕ of EDB literals in rules of Π such that all the atoms in the groundings are in $\mathcal{D}$. For each $\phi \in \Phi$, we compute the maximal intervals in which $\mathcal{D}$ entails ϕ, which is feasible in TC^0. Such intervals allow us to compute in L the partitioning of the whole $\mathbb{Q}$ into *sections*—that is, the maximal intervals such that if $\mathcal{D}$ entails $\phi \in \Phi$ in some time point in a section σ, then $\mathcal{D}$ entails ϕ in the whole σ. Let Φ_σ be the set of all $\phi \in \Phi$ entailed in a section σ by $\mathcal{D}$. We represent each derivation sequence as the subsequence of polynomial length containing only the first and the last facts in the derivation from every section. Each interval ϱ in the subsequence that may have endpoints with exponentially long binary representations is represented concisely as a pair of a pointer to the time point t from $\mathcal{D}$ closest to ϱ and the number $\mathsf{dist}(\{t\}, \varrho)$, which is polynomially representable in the size of $\mathcal{D}$ (special care is required when ϱ is in the first/last section). Hence, such representation of a derivation can be guessed fact by fact in NL.

Next, we show how to verify correctness of a guessed representation. The key step is to check that each fact $\alpha_{i+1}@\varrho_{i+1}$ in the subsequence can be derived from the preceding fact $\alpha_i@\varrho_i$. If ϱ_i and ϱ_{i+1} are in different sections, then we check, in L, whether there is a 1-step derivation of $\alpha_{i+1}@\varrho_{i+1}$ from $\alpha_i@\varrho_i$. Otherwise—that is, when ϱ_i and ϱ_{i+1} are in the same section σ—the derivation of $\alpha_{i+1}@\varrho_{i+1}$ from $\alpha_i@\varrho_i$ can be of exponential length. To deal with this case, we construct in L a program Π_σ by first deleting each rule in the grounding of Π mentioning some EDB conjunction not in Φ_σ, and then deleting all EDB literals from the remaining rules. Then Π_σ is a ground normal DatalogMTL$^{\diamondsuit}_{\text{core}}$ program, so we can show, as in the proof of Lemma 4, that the following are equivalent:

1. there is a derivation using Π and $\mathcal{D}$ that starts with $\alpha_i@\varrho_i$ and ends with $\alpha_{i+1}@\varrho_{i+1}$;
2. there is a path in $G^d_{\Pi_\sigma}$ (for d being the density of Π and $\mathcal{D}$) of weight $\mathsf{dist}(\varrho_i, \varrho_{i+1})$ from $v_\ell^{\alpha_i}$ to $v_\ell^{\alpha_{i+1}}$, where k and ℓ are, respectively, the $(\Pi, \mathcal{D})$-types of ϱ_i and ϱ_{i+1}.

The graph $G^d_{\Pi_\sigma}$ can be constructed in L, while checking existence of an appropriate path is feasible in NL, so existence of a derivation of $\alpha_{i+1}@\varrho_{i+1}$ from $\alpha_i@\varrho_i$ is in NL. □

5 Discussion and Future Work

We have focused on fragments which mention only the past temporal operators $\diamondsuit$ and $\boxminus$, however, analogous results immediately follow for the future temporal operators $\diamondsuit$ and $\boxplus$.

Corollary 9. *Consistency checking is* TC^0*-complete,* P*-hard, and* NL*-complete for* DatalogMTL$^{\diamondsuit}_{\text{core}}$, DatalogMTL$^{\boxplus}_{\text{core}}$, *and* DatalogMTL$^{\diamondsuit}_{\text{lin}}$, *respectively, and the lower bounds hold already for the propositional fragments.*

Low complexity fragments have also been identified for propositional DatalogMTL under the pointwise semantics [Kikot *et al.*, 2018; Ryzhikov *et al.*, 2019]. Results for pointwise and continuous semantics are, however, different: reasoning in DatalogMTL is coNP-complete under pointwise semantics, but PSPACE-complete for continuous semantics.

We see many possibilities for future work. First, it is unclear if the bound in Theorem 7 is tight, as well as whether the upper bound applies to DatalogMTL$^{\boxminus}_{\text{lin}}$. Second, it would be interesting to study also combined complexity. Third, we want to consider DatalogMTL over integer timelines, which may lead to lower complexity and which have been already considered for MTL [Gutiérrez-Basulto *et al.*, 2016]. Finally, our complexity bounds do not directly provide practical reasoning algorithms, which we would like to construct.

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