Inconsistency Measurement for Paraconsistent Inference

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Abstract
One of the main aims of the methods developed for reasoning under inconsistency, in particular paraconsistent inference, is to derive informative conclusions from inconsistent bases. In this paper, we introduce an approach based on inconsistency measurement for defining non-monotonic paraconsistent consequence relations. The main idea consists in adapting properties of classical reasoning under consistency to inconsistent propositional bases by involving inconsistency measures (IM). We first exhibit interesting properties of our consequence relations. We then study situations where they bring about consequences that are always jointly consistent. In particular, we introduce a property of inconsistency measures that guarantees the consistency of the set of all entailed formulas. We also show that this property leads to several interesting properties of our IM-based consequence relations. Finally, we discuss relationships between our framework and well-known consequence relations that are based on maximal consistent subsets. In this setting, we establish direct connections between the latter and properties of inconsistency measures.

1 Introduction
The main problem that we face in inconsistency handling is the principle of explosion. It is a law stating that any formula can be proven from a contradiction. This principle shows that classical inference deals with inconsistent bases as non-informative. A logical consequence relation is said to be paraconsistent if it does not validate the principle of explosion (e.g. see [Middelburg, 2011; Arruda, 1980]). Inasmuch as it avoids to treat inconsistency as triviality, paraconsistency is appropriate for contradiction-tolerant reasoning and can be used for deriving useful information from inconsistent bases.

Non-monotonic inference deals with the problem of deriving plausible conclusions with the possibility to retract them in conformity with new information (e.g. see [Kraus et al., 1990]). Many problems in artificial intelligence, such as argumentation, belief revision and planning, involve this type of reasoning. Although paraconsistent consequence relations are usually monotonic, the definition of non-monotonic paraconsistent relations has received some attention in the literature according to different approaches. We can first mention the relation proposed in [Rescher and Manor, 1970], which is based on the use of maximal consistent subsets to derive plausible conclusions. Other variants of this relation that are both non-monotonic and non-explosive have also been introduced in the literature (e.g. [Benferhat et al., 1993; Benferhat et al., 1997; Konieczny et al., 2019]). Moreover, using a three-valued semantics, the consequence relation of Priest’s paraconsistent logic, called minimally inconsistent logic of paradox, is non-monotonic [Priest, 1991]. Inspired by Shoham’s idea [Shoham, 1987; Shoham, 1988] of preferential models for making inferences, [Arieli and Avron, 2000] presented an approach that is also based on a multi-valued semantics for defining non-monotonic paraconsistent relations. It is also interesting to mention the approach of adaptive logics that uses dynamic proof theories [Batens, 2013]. While all of these approaches have interesting properties, none of them can be considered as appropriate for reasoning about inconsistency in all situations. This explains the interest in continuing to investigate new approaches to pave the way for more applications of non-monotonic paraconsistent relations.

In this work, we use the notion of inconsistency measure (e.g. see [Hunter and Konieczny, 2010]) as a stepping stone for defining non-monotonic paraconsistent consequence relations. We show that our new approach can be seen as a simple and natural way to adapt properties of classical reasoning under consistency to inconsistent propositional bases. These properties concern the consistency of the consequence and the inconsistency of its negation in combination with the considered base. Additionally, our approach can greatly benefit from the variety of inconsistency measures proposed in the literature. Indeed, plenty of proposals for measures and systems to define them have been made by considering different conflict forms (e.g. see [Grant and Hunter, 2013; Besnard, 2014; Thimm, 2016; Bona et al., 2018]).

Our contribution is manifold. First, we introduce an IM-based framework for defining three types of consequence relations. Second, because our consequence relations do not always bring about jointly consistent consequences, we propose a property, called DISJUNCT MINIMALITY, that, if satisfied by an inconsistency measure, guarantees that any consequence relation based on this measure always generates
consistent sets of formulas. Furthermore, we show that this property allows characterizing IM-based consequence relations closed under conjunction introduction, which means that this property is necessary in specific cases to avoid conflicts between the entailed formulas and provide a reasoning framework similar to classical logic. We also show that the measures that satisfies DISJUNCT MINIMALITY produce consequence relations that have desired properties of non-monotonic relations mentioned in [Gabbay, 1984; Kraus et al., 1990]. Finally, we discuss relationships between our consequence relations and some well-known relations that are based on the use of maximal consistent subsets (MCS) [Rescher and Manor, 1970; Benferhat et al., 1993]. In this context, we establish direct connections between properties of inconsistency measures and the considered MCS-based relations. In particular, we introduce a new property of inconsistency measures that allows characterizing skeptical reasoning.

2 Preliminary Definitions and Notations

Given a countable set of propositional variables \( \text{Prop} \), a propositional formula \( \phi \) has the form: \( \phi ::= \bot | p | \neg \phi | \phi \land \phi \), where \( \bot \) denotes false, \( p \) belongs to \( \text{Prop} \), \( \neg \) is the negation connective and \( \land \) the conjunction connective. The constant \( \top \) (true) and the connectives \( \lor, \rightarrow \) and \( \leftrightarrow \) are defined as usual in classical propositional logic. For the set of propositional formulas we write \( \text{Form} \). Notationally, we use, possibly with subscripts and/or superscripts, the letters \( p, q \) and \( r \) to denote the propositional variables and the greek letters \( \phi, \psi \) and \( \chi \) to denote the propositional formulas. Moreover, given a formula \( \phi \) (resp. a set of formulas \( S \)), we use \( \forall \phi \) (resp. \( \forall S \)) to denote the set of propositional variables occurring in \( \phi \) (resp. \( S \)). For the cardinality of a set \( S \) we write \( |S| \).

An interpretation \( w \) is a function from \( \text{Prop} \) to \( \{0, 1\} \). It is inductively extended to the propositional formulas as follows: \( w(\bot) = 0, w(\neg \phi) = 1 - w(\phi) \) and \( w(\phi \land \psi) = \min(w(\phi), w(\psi)) \). For the set of interpretations defined over \( \text{Prop} \) we write \( \text{W}_{\text{Prop}} \). A model of a formula \( \phi \) is an interpretation \( w \) that makes \( \phi \) true, that is, \( w(\phi) = 1 \). The set of models of \( \phi \) is denoted \( \text{Mod}(\phi) \), that is, \( \text{Mod}(\phi) = \{ w \in \text{W}_{\text{Prop}} : w(\phi) = 1 \} \). A formula is said to be consistent if it admits a model. Further, we say that a finite set of formulas \( S \) is consistent if its corresponding formula \( \wedge_{\phi \in S} \phi \), written \( \wedge S \), is consistent; otherwise it is inconsistent. We use \( \text{In}(S) \) to refer to the set of inconsistent formulas that belong to \( S \).

A formula \( \phi \) is said to be a logical consequence of a finite set of formulas \( S \), written \( S \models \phi \), if \( \text{Mod}(\wedge S) \subseteq \text{Mod}(\phi) \). For convenience, if \( S \) contains only one formula \( \psi \), we write \( \models \psi \) instead of \( \{\psi\} \models \phi \). We use \( \equiv \) if we have both \( \models \phi \land \psi \) and \( \models \phi \land \psi \). Moreover, we write \( \text{Eq}(S, \phi) \) for the set of formulas in \( S \) equivalent to \( \phi \), that is, \( \{ \psi \in S : \psi \equiv \phi \} \).

Two finite sets of formulas \( S \) and \( S' \) are said to be equivalence equivalent, written \( S \equiv e S' \), if there is a bijection \( f : S \rightarrow S' \) such that \( \phi \equiv f(\phi) \) holds for each \( \phi \in S \). Clearly, \( \equiv e \) is an equivalence relation. In addition, given a set \( X \) of sets of formulas (resp. of sets of formulas), we use \( X/\equiv (\text{resp. } X/\equiv e) \) to denote the quotient of \( X \) by \( \equiv (\text{resp. } \equiv e) \), that is, the set of all equivalence classes of \( X \) w.r.t. \( \equiv (\text{resp. } \equiv e) \).

A beliefbase is defined as a finite set of formulas. For the set of beliefbases we write \( K_{\text{Form}} \).

**Definition 1** (MIS). A minimal inconsistent subset (MIS) of a beliefbase \( K \) is a subset \( M \) of \( K \) such that \( M \subseteq K, M \models \bot \), and for each \( \phi \in M, M \setminus \{ \phi \} \not\models \bot \).

**Definition 2** (MCS). A maximal consistent subset (MCS) of a beliefbase \( K \) is a subset \( M \) of \( K \) such that \( M \subseteq K, M \not\models \bot \), and for each \( \phi \in K \setminus M, M \cup \{ \phi \} \not\models \bot \).

We use \( \text{MCS}(K) \) and \( \text{MIS}(K) \) to denote the set of all maximal consistent subsets and that of all minimal inconsistent subsets of \( K \), respectively.

**Definition 3** (Free Formula). A formula \( \phi \) in a beliefbase \( K \) is said to be free in \( K \) if \( \phi \notin M \) for every \( M \in \text{MIS}(K) \).

For the set of free formulas in \( K \) we write \( \text{Free}(K) \).

A formula \( \phi \) is said to be problematic in a beliefbase \( K \) if \( \phi \) does not belong to \( \text{Free}(K) \).

Among the inconsistency measures considered in this work, the measure \( I_C \), introduced in [Konieczny et al., 2003], is based on Priest's three-valued logic [Priest, 1991]. This logic has a semantics with an additional truth-value \( \bot \) representing intuitively both true and false. The three truth-values are ranked as follows: \( 0 < B < 1 \). A three-valued interpretation \( w \) of a formula \( \phi \) is a function from \( \text{Prop} \) to \( \{0, B, 1\} \). It is extended to formulas as follows: \( w(\bot) = 0 \), \( w(\neg \phi) = 1 - w(\phi) \) if \( w(\phi) \in \{0, 1\} \), \( w(\neg \phi) = B \) otherwise; and \( w(\phi \land \psi) = \min(w(\phi), w(\psi)) \). We use \( \text{both}(w) \) to denote the set \( \{p \in \text{Prop} : w(p) = B\} \). A three-valued model of a formula \( \phi \) is an interpretation \( w \) such that \( w(\phi) \neq 0 \). For the set of three-valued models of \( \phi \) we write \( 3VM(\phi) \).

3 Inconsistency Measures

In this paper, we focus on the use of inconsistency measures for defining consequence relations. An inconsistency measure is a function that allows quantifying the intensity of contradiction. Here, we consider a formal definition that takes into account specific rationality postulates introduced in the literature [Hunter and Konieczny, 2010; Besnard, 2014].

We write \( \mathbb{R}_+^\infty \) for the set of positive real numbers augmented with a greatest element denoted \( \infty \).

**Definition 4.** An inconsistency measure is a function \( I : K_{\text{Form}} \rightarrow \mathbb{R}_+^\infty \) that satisfies the following properties for every \( K, K' \in K_{\text{Form}} \) and every \( \psi, \psi' \in \text{Form} \):

- CONSISTENCY: \( I(K) = 0 \) iff \( K \not\models \bot \);
- MONOTONICITY: if \( K \subseteq K' \) then \( I(K) \leq I(K') \);
- EQUIVALENCE: if \( \phi \equiv \psi \) then \( I(K \cup \{\phi\}) = I(K \cup \{\psi\}) \); and
- TAUTOLogy: if \( \top \models \phi \) then \( I(K \cup \{\phi\}) = I(K) \).

CONSISTENCY states that an inconsistency measure must enable distinguishing between consistent and inconsistent beliefbases. MONOTONICITY says that the amount of conflicts cannot decrease by adding new formulas. EQUIVALENCE requires that equivalent formulas bring the same intensity of
contradiction. TAUTOLOGY decrees that tautologies do not impact inconsistency values.

Note that there is a broad consensus in the literature on CONSISTENCY and MONOTONICITY, which is not the case of the two other properties. In our work, the benefit of including EQUIVALENCE in the definition of inconsistency measure is to avoid deriving a formula and rejecting one of its equivalent formulas from the same beliefbase. Regarding TAUTOLOGY, it is used to prevent any change in the obtained consequences by adding tautologies.

Let us mention that the literature is rich in more or less disputed postulates (e.g. see [Besnard, 2014; Ammoura et al., 2017; Thimm, 2018]). Among them, we are particularly interested in the following properties:

- DOMINANCE: if $\phi \not\rightarrow \bot$ and $\phi \vdash \psi$ then $I(K \cup \{\phi\}) \geq I(K \cup \{\psi\})$;
- FREE FORMULA: if $\phi \in \text{Free}(K)$ then $I(K \setminus \{\phi\}) = I(K)$;
- ENALTY: if $\phi \in K$ and $\phi \notin \text{Free}(K)$ then $I(K \setminus \text{Eq}(K,\phi)) < I(K)$.

The definition of ENALTY is an adaptation of the property named PENALTY in the literature to take into account the presence of EQUIVALENCE in the definition of inconsistency measure.

The three foregoing properties are particularly considered to point out their impact on the introduced IM-based consequence relations.

Let us now recall specific inconsistency measures from the literature:

- $I_D(K) = 1$ if $K \vdash \bot$, and $I_D(K) = 0$ if $K \not\vdash \bot$ [Hunter and Konieczny, 2008];
- $I_M(K) = |\text{MIS}(K)/\equiv_0|$ [Hunter and Konieczny, 2008];
- $I_A(K) = |\text{MCS}(K)/\equiv_0| + \alpha(K) - 1$, where $\alpha(K) = 1$ if $\text{Inc}(K) \neq \emptyset$, and $\alpha(K) = 0$ if $\text{Inc}(K) = \emptyset$ [Grant and Hunter, 2011];
- $I_P(K) = |(K \setminus \text{Free}(K))/\equiv_1|$ [Grant and Hunter, 2011];
- $I_HS(K) = \min\{|H_1| : H \subseteq \bigcup_{\phi \in K} \text{Mod}(\phi) \land \forall \phi \in K, \exists w \in H, w(\phi) = 1\} - 1$ with $\min\{}$ = $\infty$ [Thimm, 2016];
- $I_C(K) = \min\{\text{both}(w) : w \in \text{3VM}(K)\}$ if $\text{Inc}(K) = \emptyset$, and $I_C(K) = \infty$ if $\text{Inc}(K) \neq \emptyset$ [Konieczny et al., 2003];
- $I_{\text{Dalal}}(K) = \min\{d(\text{Dalal}(\phi), w) : w \in \text{WProp}\}$ [Grant and Hunter, 2017].

where $d$ corresponds to the Dalal distance between interpretations defined as follows: for all $w, w' \in \text{WProp}$, $d(w,w') = |\{p \in \text{Prop} : w(p) \neq w'(p)\}|$. It is extended to sets of interpretations as follows: $d(S,w) := \min\{d(w',w) : w' \in S\}$ if $S \neq \emptyset$, $d(S,w) := \infty$ otherwise.

Note that some of the measures above are adapted to satisfy EQUIVALENCE. For instance, the original versions of $I_M$ and $I_P$ count the number of minimal inconsistent subsets and the number of problematic formulas, respectively.

### 4 IM-based Consequence Relations

In the following, we present three types of consequence relations that involve inconsistency measurement. They are inspired by specific properties satisfied by the classical consequence relation under consistency.

**Definition 5 (IM-based Consequence Relations).** Let $I$ be an inconsistency measure. The consequence relations $\vdash_I^+, \vdash_I^-$ and $\vdash_I^3$ are defined as follows: for each $K \in \mathcal{K}_{\text{Form}}$ and each $\phi \in \text{Form},$

- $K \vdash_I^+ \phi$ iff $I(K \cup \{\neg \phi\}) > I(K)$;
- $K \vdash_I^- \phi$ iff $I(K \cup \{\phi\}) = I(K) \land I(K \cup \{\neg \phi\}) > I(K)$;
- $K \vdash_I^3 \phi$ iff $I(K \cup \{\neg \phi\}) > I(K \cup \{\phi\})$.

In order to motivate our approach, consider a consistent beliefbase $K$. For every formula $\phi$, we have $K \vdash \phi$ if and only if $K \cup \{\neg \phi\} \not\vdash \bot$, which means that the negation of the consequence brings contradiction in a conflict-free beliefbase. Using the notion of inconsistency measure, this is extended to inconsistent beliefbases in a natural way by the relation $\vdash_I^+$, which says that a formula is a consequence if its negation brings new conflicts in the beliefbase. However, note that $\vdash_I^+$ does not take into account the conflicts that may arise from the entailed formula. This explains the proposal of $\vdash_I^-$ and $\vdash_I^3$. The consequence relation $\vdash_I^-$ considers the additional property that the entailed formula must not bring any new conflict in the beliefbase. To some extent, it is similar to the fact that $K \cup \{\phi\} \not\vdash \bot$ results from $K \vdash \phi$ whenever $K$ is consistent (the consequence keeps the beliefbase conflict-free). As regards the consequence relation $\vdash_I^3$, it is defined by requiring that each entailed formula must bring less conflicts than its negation. Clearly, it is weaker than the relation $\vdash_I^-$. That is, for every beliefbase $K$ and every formula $\phi$, if $K \vdash_I^2 \phi$ then $K \vdash_I^3 \phi$; the converse is not necessarily true. Moreover, the relation $\vdash_I^+$ is weaker than both $\vdash_I^-$ and $\vdash_I^3$.

Notationally, for the set $\{\phi \in \text{Form} : K \vdash_I^+ \phi\}$ we write $\text{On}(K,I,i)$.

**Example 1.** Consider the beliefbase $K = \{p \land q, \neg p \land q, r \land \neg q, \neg r \land r, r \land q\}$ and the inconsistency measure $I_M$. We have $I_M(K) = 4$. The formula $q$ is free in $K \cup \{q\}$, and it follows $I_M(K) = I_M(K \cup \{q\}) = 4$. However, the MISes of $K \cup \{\neg q\}$ are $\{p \land q, \neg p \land q\}, \{g \land \neg q\}, \{p \land q, \neg q\}, \{r \land q, \neg q\}, \{\neg p \land q, \neg q\}, \{\neg r, r \land q\}$ and $\{\neg r, r \land q\}$; hence, $I_M(K \cup \{\neg q\}) = 7$ holds. Therefore, we obtain $K \vdash_I^+ q$, $K \vdash_I^- q$ and $K \vdash_I^3 q$. Additionally, $I(K \cup \{p\}) = 5$ and $I(K \cup \{\neg p\}) = 5$ yield $K \vdash_I^3 p, K \vdash_I^- p$ and $K \vdash_I^3 p$. Moreover, $K \vdash_I^+ r, K \vdash_I^- r$ and $K \vdash_I^3 r$ ensue from $I(K \cup \{r\}) = 5$ and $I(K \cup \{\neg r\}) = 6$.

Note that the three consequence relations coincide in the case of the inconsistency measure $I_D$. In fact, a formula $\phi$ is a logical consequence of a beliefbase $K$ w.r.t. any consequence relation based on $I_D$ if and only if $K \not\vdash \bot$ and $K \vdash \phi$.

In other words, the consequence relation in this case is the most restrictive paraconsistent relation that preserve classical reasoning under consistency. (The relation that decrees that nothing is entailed from inconsistent beliefbases.)
We show below that our IM-based consequence relations preserve classical reasoning under consistency. This explains why our approach can be seen as an extension of the properties related to entailment under consistency mentioned previously. Note that we only need the postulate CONSISTENCY to show the following proposition.

**Proposition 1.** Let I be an inconsistency measure. Then, for every \( K \in K_{\text{Form}} \) such that \( K \not
\perp \perp \) and every \( \phi \in \text{Form} \), we have \( K \vdash_1 \phi \iff K \vdash \phi \) for each \( i \in \{1, 2, 3\} \).

**Proof.** We only consider the case of \( \vdash_1 \), the others being similar. Using CONSISTENCY and \( K \not
\perp \perp \), \( I(K) = 0 \) holds. Then, \( K \vdash_1 \phi \iff I(K \cup \{\neg \phi\}) > 0 \). Hence, in view of, again, CONSISTENCY, we obtain \( K \vdash \phi \iff K \cup \{\neg \phi\} \not
\perp \perp \). We finally have \( K \cup \{\neg \phi\} \not
\perp \perp \iff K \vdash \phi \). \( \square \)

The following proposition provides some common properties of the IM-based consequence relations.

**Proposition 2.** The following properties are satisfied for every inconsistency measure I, every \( i \in \{1, 2, 3\} \) and every beliefbase \( K \in K_{\text{Form}} \):

1. there exists a formula \( \phi \) s.t. \( K \not
\vdash_1 \phi \);
2. \( \vdash_1 \) is non-monotonic;
3. for every \( \phi \in K \), \( K \not
\vdash_1 \neg \phi \);
4. for every \( \phi, \psi \in \text{Form} \) with \( \phi \equiv \psi \), \( K \vdash_1 \phi \iff K \vdash_1 \psi \);
5. for every \( \phi, \psi, \chi \in \text{Form} \) with \( \phi \equiv \psi \), \( K \cup \{\phi\} \vdash_1 \chi \iff K \cup \{\psi\} \vdash_1 \chi \).

Property 1 says that there is no beliefbase where the IM-based consequence relations explodes into triviality; it ensures that these relations are paraconsistent for any inconsistency measure. Property 2 shows that all IM-based relations are non-monotonic (a consequence relation \( \vdash_1 \) is monotonic if \( K \vdash_1 \psi \) implies \( K \cup \{\phi\} \vdash_1 \psi \), for any formula \( \phi \)). Property 3 states that it is not possible to entail the negation of a formula that belongs to the beliefbase. Properties 4 and 5 say that no difference is made between equivalent formulas.

Note that non-monotonicity can be easily seen from the fact that we have \( \{p\} \vdash_1 p \) (see Proposition 1) without having \( \{\neg p\} \vdash_1 p \) (Property 3).

Now, we show that \( \vdash_1 \) and \( \vdash_2 \) prevent contradiction between consequences.

**Proposition 3.** Let I be an inconsistency measure. Then, for every \( K \in K_{\text{Form}} \) and every \( \phi \in \text{Form} \), \( K \not
\vdash_1 \phi \) or \( K \not
\vdash_1 \neg \phi \) for \( i \in \{1, 2, 3\} \).

The foregoing proposition is not satisfied by \( \vdash_1 \). Consider, for instance, the beliefbase \( K = \{p \land q, \neg p \land q\} \). Then, we obtain \( I_M(K) = 1 \), \( I_M(K \cup \{p\}) = 2 \) and \( I_M(K \cup \{\neg p\}) = 2 \). Consequently, \( I_M(K \cup \{p\}) > I_M(K) \) and \( I_M(K \cup \{\neg p\}) > I_M(K) \) yield \( K \vdash_1 p \) and \( K \vdash_1 \neg p \).

To illustrate the presence of relationships between IM properties and IM-based consequence relations, we point out in the proposition below the impact of Dominance.

**Proposition 4.** Let I be an inconsistency measure that satisfies Dominance. Then, the following properties are satisfied for every \( i \in \{1, 2, 3\} \), every \( K \in K_{\text{Form}} \), and every \( \phi, \psi \in \text{Form} \) s.t. \( \phi \not
\perp \perp \) and \( \phi \vdash \psi \):

1. if \( K \vdash \phi \) then \( K \vdash_1 \psi \), and
2. \( K \cup \{\phi\} \not
\vdash_1 \neg \psi \).

**5 Consistent Consequence Relation**

In the general case, our consequence relations do not bring about jointly consistent formulas. We discuss in this section an IM property, called DISJUNCT MINIMALITY, that guarantees the consistency of the associated consequence relations. It deals with disjunction and is satisfied by several inconsistency measures. We also show that this property allows characterizing IM-based consequence relations closed under conjunction introduction. This shows that DISJUNCT MINIMALITY is necessary in specific cases to avoid conflicts between entailed formulas and provide a reasoning framework similar to classical logic.

Let us recall the compactness theorem that states that a set of formulas is satisfiable if and only if all of its finite subsets are satisfiable. This mainly allows us to deal with the satisfiability of a set that corresponds to all consequences of a beliefbase.

We define the property **DISJUNCT MINIMALITY** as follows: for every \( K \in K_{\text{Form}} \) and every \( \phi, \psi \in \text{Form} \),

\[
I(K \cup \{\phi \lor \psi\}) \geq \max(I(K \cup \{\phi\}), I(K \cup \{\psi\}))
\]

It can be seen as the complement of the following property introduced in [Besnard, 2014]: \( I(K \cup \{\phi \lor \psi\}) \leq \max(I(K \cup \{\phi\}), I(K \cup \{\psi\})) \).

The proposition below is a consequence of the fact that each (three-valued) interpretation satisfies \( \phi \lor \psi \) if and only if it satisfies \( \phi \) or \( \psi \).

**Proposition 5.** The inconsistency measures \( I_C \), \( I_{HS} \) and \( I_{\text{Dalal}} \) satisfy DISJUNCT MINIMALITY.

For the sake of illustration, let us show that \( I_M \) does not satisfy DISJUNCT MINIMALITY. Consider the belief base \( K = \{p \lor \neg p, p \land q, \neg p \land q\} \). Then, \( I_M(K) = 1 \) holds. However, we have \( I(\{p, p \land q, \neg p \land q\}) = 2 \) and \( I(\{\neg p, p \land q, \neg p \land q\}) = 2 \).

**Theorem 1.** Let I be an inconsistency measure that satisfies DISJUNCT MINIMALITY. Then, the consequence relations \( \vdash_1 \), \( \vdash_2 \) and \( \vdash_3 \) coincide.

**Proof.** We only need to show that \( \vdash_1 \subseteq \vdash_2 \subseteq \vdash_3 \), since we already have \( \vdash_2 \subseteq \vdash_3 \subseteq \vdash_1 \). Let K be a beliefbase and \( \phi \) a formula s.t. \( K \vdash_1 \phi \). Thus, \( I(K) < I(K \cup \{\phi\}) \). Using Tautology, we have \( I(K) = I(K \cup \{\phi \lor \neg \phi\}) \). Then, using DISJUNCT MINIMALITY, we obtain \( I(K \cup \{\phi \lor \neg \phi\}) \geq \min(I(K \cup \{\phi\}), I(K \cup \{\neg \phi\})) \). Consequently, \( I(K) \geq I(K \cup \{\phi\}) \) holds, which yields \( I(K) = I(K \cup \{\phi\}) \) (MONOTONICITY). Therefore, we have \( K \vdash_2 \phi \).

For convenience, since \( \vdash_1 \), \( \vdash_2 \) and \( \vdash_3 \) coincide whenever I satisfies DISJUNCT MINIMALITY, we write \( \vdash \) in this case to refer to the three relations.

**Proposition 6.** Let I be an inconsistency measure that satisfies DISJUNCT MINIMALITY. Then, for every beliefbase K, and every formulas \( \phi, \psi \) and \( \phi \vdash \psi \), if \( K \vdash_1 \phi \) and \( K \vdash_1 \psi \) then \( K \vdash_1 \phi \lor \psi \).
Proposition 7. An inconsistency measure \( I \) satisfies Disjunction Dominance iff it satisfies Dominance.

Proof. Let us first consider the if part. In the case of \( \phi \not\vdash \bot \) and \( \psi \not\vdash \bot \), this property is a direct consequence of the fact that \( \phi \not\vdash \phi \lor \psi \) and \( \psi \not\vdash \phi \lor \psi \) for every \( \phi, \psi \in \text{Form} \). In the remaining case, it is a direct consequence of Equivalence. Consider now the only if part. Let \( K \in \text{K}_\text{Form} \) and \( \phi, \psi \in \text{Form} \) s.t. \( \phi \not\vdash \bot \) and \( \psi \not\vdash \bot \). Using Equivalence, we have \( I(K \{ \phi \}) = I(K \{ \phi \lor \psi \}) \). It follows from Disjunction Dominance that \( I(K \{ \phi \}) \leq \min(I(K \{ \phi \}), I(K \{ \psi \})) \). Thus, \( I(K \{ \phi \}) \geq I(K \{ \phi \lor \psi \}) \).

In the following theorem, we show that Dominance allows characterizing IM-based consequence relations closed under conjunction elimination. A set of formula \( S \) is said to be closed under conjunction elimination if for every \( \phi \land \psi \in S \), \( \phi \in S \) and \( \psi \in S \).

Theorem 5. Let \( I \) be an inconsistency measure. If \( \text{Cn}(K, I, 1) \) is closed under conjunction elimination for every \( K \in \text{K}_\text{Form} \), then \( I \) satisfies Dominance.

The proof is obtained by showing that the considered measure satisfies Disjunction Dominance and using Proposition 7.

One might think that interesting properties can be captured by considering the dual property of Disjunction Minimality defined as follows: for every \( K \in \text{K}_\text{Form} \) and every \( \phi, \psi \in \text{Form} \), \( I(K \cup \{ \phi \land \psi \}) \leq \max(I(K \cup \{ \phi \}), I(K \cup \{ \psi \})) \). However, there is no inconsistency measure that satisfies this property. Indeed, using Consistency, \( I(\{p \land \neg p \}) > 0 \) holds for every inconsistency measure \( I \). Then, using the same property, we obtain \( I(\{p\}) = 0 \) and \( I(\{-p\}) = 0 \); hence, it results that \( I(\{p \land \neg p\}) > \max(I(\{p\}), I(\{-p\})) \).

6 MCS-based Consequence Relations

In this section, we focus on relationships between our consequence relations and three well-known relations that are based on the use of maximal consistent subsets (MCS) [Rescher and Manor, 1970; Benferhat et al., 1993]. We first show that two of the considered MCS-based relations (namely credulous and argumentative relations) can be characterized using the inconsistency measure \( I_M \). Then, we establish direct connections between properties of inconsistency measures and these two relations. Finally, we introduce a new IM property that allows characterizing the remaining MCS-based relation (skeptical relation).

Let us first define the considered MCS-based consequence relations:

- **Credulous Inference**: \( K \vdash_{cr} \phi \) iff there exists \( K' \) in \( \text{MCS}(K) \) s.t. \( K' \vdash \phi \).
- **Argumentative Inference**: \( K \vdash_{arg} \phi \) iff \( K \vdash_{cr} \phi \) and for every \( K' \) in \( \text{MCS}(K) \), \( K' \vdash \neg \phi \).
- **Skeptical Inference**: \( K \vdash_{sk} \phi \) iff for every \( K' \) in \( \text{MCS}(K) \), \( K' \vdash \phi \).

Clearly, for every beliefbase \( K \), we have \( \{ \phi \in \text{Form} : K \vdash_{cr} \phi \} \subseteq \{ \phi \in \text{Form} : K \vdash_{arg} \phi \} \subseteq \{ \phi \in \text{Form} : K \vdash_{sk} \phi \} \). Moreover, \( \vdash_{sk} \) is the unique relation that always generates jointly consistent consequences.

The proposition below shows that the inconsistency measure \( I_M \) allows us to capture the credulous and argumentative consequence relations.
Proposition 8. Let $K$ be a belief base and $\phi$ a formula. Then, the following properties are satisfied:

- if $Eq(K, \neg \phi) = \emptyset$, $K \vdash_{I_M} \phi$ iff $K \vdash_{cr} \phi$;
- if $Eq(K, \neg \phi) = \emptyset$ and $Eq(K, \phi) = \emptyset$, $K \vdash_{I_M}^2 \phi$ iff $K \vdash_{arg} \phi$.

The condition $Eq(\neg \phi, K) = \emptyset$ is required because our relations cannot entail the negation of a formula equivalent to an element of the beliefbase. Regarding the condition $Eq(\phi, K) = \emptyset$, it comes from the fact that adding a formula that already belongs to the beliefbase does not change the amount of contradiction, even if the added formula is rejected by one of the consistent subsets of the beliefbase. To illustrate this point, consider the beliefbase $K = \{p \lor q, \neg p, \neg q\}$. We have $K \vdash_{I_M}^2 \neg p \lor q$ since $I_M(K) = 1$ and $I_M(K \cup \{\neg p \land \neg q\}) = 2$. However, we have $K \not\vdash_{arg} \neg p \lor q$ because of $(\neg p, \neg q) \not\vdash \neg (p \lor q)$.

Note that the same properties can be obtained if we use the inconsistency measure $I_A$ instead of $I_M$. A natural question to ask is whether there are relationships between common properties of $I_A$ and $I_M$, and the credulous and argumentative inferences. Certain relationships are stated in the theorem below.

Theorem 6. Let $I$ be an inconsistency measure, $K$ a belief base and $\phi$ a formula. The following properties hold:

1. if $I$ satisfies FREE FORMULA and $K \vdash_1 \phi$, then $K \vdash_{cr} \phi$;
2. if $I$ satisfies EPenalty, $Eq(\neg \phi, K) = \emptyset$ and $K \vdash_{cr} \phi$, then $K \vdash_{I_M}^1 \phi$;
3. if $I$ satisfies FREE FORMULA and EPenalty, $Eq(\phi, K) = \emptyset$, and $K \vdash_{I_M}^2 \phi$, then $K \vdash_{arg} \phi$;
4. if $I$ satisfies FREE FORMULA and EPenalty, $Eq(\neg \phi, K) = \emptyset$, and $K \vdash_{arg} \phi$, then $K \vdash_{I_M}^2 \phi$.

Proof. Property 1. We proceed by contradiction. Assume that $K \not\vdash_{cr} \phi$. Then, for every $K' \in MCS(K)$, $K' \not\vdash \phi$ holds $(K' \cup \{\neg \phi\} \not\subseteq K)$. Thus, we have $\neg \phi \in Free(K \cup \{\neg \phi\})$. The property FREE FORMULA yields $I(K \cup \{\neg \phi\}) = I(K)$, which means that $K \not\vdash_{I_M}^1 \phi$. Hence, we have a contradiction.

Property 2. For the sake of contradiction, suppose that $K \vdash_{I_M}^1 \phi$, so $I(K \cup \{\neg \phi\}) = I(K)$ holds. Using $Eq(\neg \phi, K) = \emptyset$ and EPenalty, we obtain $\neg \phi \in Free(K \cup \{\neg \phi\})$. Consequently, for every $K' \in MCS(K)$, $K' \not\vdash \phi$ holds. Therefore, $K \not\vdash_{cr} \phi$, and we have a contradiction.

Property 3. Suppose by way of contradiction that $K \vdash_{arg} \phi$. Then, $(i)$ for every $K' \in MCS(K)$, $K' \not\vdash \phi$ holds; or $(ii)$ there exists $K' \in MCS(K)$ s.t. $K' \not\vdash_{\neg \phi} (K' \cup \{\phi\}) \subseteq K$. We only consider the case of $(ii)$, the proof in the case of $(i)$ being similar to that of Property 1. Using the property $(ii)$, $\neg \phi$ is not free in $K \cup \{\phi\}$. As a result, we have $I(K \cup \{\phi\}) > I(K)$ because of EPenalty and $Eq(\phi, K) = \emptyset$. Thus, we obtain a contradiction.

Property 4. Assume that $K \vdash_{arg} \phi$. Then $\neg \phi$ is a problematic formula in $K \cup \{\neg \phi\}$ and $\phi$ is free in $K \cup \{\phi\}$. Using FREE FORMULA, EPenalty and $Eq(\neg \phi, K) = \emptyset$, $I(K) = I(K \cup \{\phi\})$ and $I(K) < I(K \cup \{\neg \phi\})$ hold. Consequently, we have $K \vdash_{I_M}^2 \phi$. □

Now, we aim at finding a property on inconsistency measures that allows us to capture skeptical inference. Before that, we describe a simple characterization of this MCS-based inference.

Proposition 9. For every belief base $K$ and every formula $\phi$, $K \vdash_{sk} \phi$ iff $MCS(K) \subseteq MCS(K \cup \{\neg \phi\})$.

We call MCS DEPENDANCE the following property on inconsistency measures that allows us to characterize skeptical relations:

For every $K \in K_{Form}$ and every $\phi \in Form$, we have $I(K) < I(K \cup \{\phi\})$ if $MCS(K) \subseteq MCS(K \cup \{\phi\})$. Consequently, using MCS DEPENDANCE, we have $I(K) = I(K \cup \{\neg \phi\})$.

Let us now establish a direct relationship between MCS DEPENDANCE and skeptical inference.

Proposition 10. Let $I$ be an inconsistency measure that satisfies MCS DEPENDANCE, then it satisfies FREE FORMULA.

Proof. Let $K$ be a belief base and $\phi$ a free formula in $K$. Clearly, for every $K' \in MCS(K)$, we have $\phi \in K'$. Thus, $MCS(K \setminus \{\phi\}) \subseteq MCS(K)$. Consequently, using MCS DEPENDANCE, we have $I(K) = I(K \setminus \{\phi\})$. □

Let us now establish a direct relationship between MCS DEPENDANCE and skeptical inference.

Proposition 11. Let $I$ be an inconsistency, $K$ a belief base and $\phi$ a formula s.t. $Eq(\neg \phi, K) = \emptyset$. Then, we have the property $K \vdash_{sk} \phi$ if $K \vdash_{I_M}^1 \phi$, if and only if $I$ satisfies MCS DEPENDANCE.

Proposition 12. Let $I$ be an inconsistency measure that satisfies MCS DEPENDANCE. Then, the consequence relations $\vdash_1^1, \vdash_1^2$ and $\vdash_1^3$ coincide.

Since $\vdash_{sk}$ is a relation that leads to jointly consistent consequences, the property MCS DEPENDANCE is an additional property with DISJUNCT MINIMALITY that allows obtaining consistent consequence relations. Note that none of the measures described in this work satisfy MCS DEPENDANCE.

7 Conclusion and Perspectives

We have introduced an IM-based framework for defining different types of non-monotonic paraconsistent consequence relations. We have also proposed properties of inconsistency measures that allow defining consistent consequence relations. Finally, we have described direct relationships between properties of inconsistency measures and well-known MCS-based consequence relations.

There are several perspectives for future work. Among them, we first mention the study of the impact on the proposed consequence relations of other properties of inconsistency measures from the literature. It is also interesting to examine the possible connections between our framework and that based on MCS selection [Konieczny et al., 2019]. More generally, it is worthwhile to investigate other approaches for defining consequence relations through inconsistency measurement.
References


