

A Description Logic for Analogical Reasoning

Steven Schockaert, Yazmín Ibáñez-García, Víctor Gutiérrez-Basulto

Cardiff University, UK

{schockaerts1,ibanezgarciay, gutierrezbasultov}@cardiff.ac.uk

Abstract

Ontologies formalise how the concepts from a given domain are interrelated. Despite their clear potential as a backbone for explainable AI, existing ontologies tend to be highly incomplete, which acts as a significant barrier to their more widespread adoption. To mitigate this issue, we present a mechanism to infer plausible missing knowledge, which relies on reasoning by analogy. To the best of our knowledge, this is the first paper that studies analogical reasoning within the setting of description logic ontologies. After showing that the standard formalisation of analogical proportion has important limitations in this setting, we introduce an alternative semantics based on bijective mappings between sets of features. We then analyse the properties of analogies under the proposed semantics, and show among others how it enables two plausible inference patterns: rule translation and rule extrapolation.

1 Introduction

The last decade has witnessed an increasing interest in methods for automated knowledge base completion. While most work has focused on predicting plausible missing facts in knowledge graphs [Bordes *et al.*, 2013; Yang *et al.*, 2015; Trouillon *et al.*, 2017; Balazevic *et al.*, 2019], some authors have also looked at the problem of predicting plausible missing rules in ontologies [Beltagy *et al.*, 2013; Bouraoui and Schockaert, 2019]. The underlying principle of these latter approaches is to rely on external knowledge about the similarity structure of concepts, typically in the form of a vector space embedding of concept names. The main idea is that knowledge about concepts can often be extended to concepts with a similar representation in the given vector space. The same principle also lies at the basis of rule-based frameworks with a soft unification mechanism [Medina *et al.*, 2004; Rocktäschel and Riedel, 2017]. However, these similarity based reasoning methods can clearly only provide us with knowledge that is similar to what is already in the knowledge base. Humans, on the other hand, can also infer plausible knowledge in more creative ways, where a particularly prominent role is played by the idea of reasoning by analogy. This

phenomenon has been extensively studied in cognitive science and philosophy [Gentner, 1983; Hofstadter *et al.*, 1995; Holyoak and Thagard, 1997], but to the best of our knowledge, the use of analogical reasoning for completing ontologies has not yet been considered¹

Within Artificial Intelligence, the formalisation of analogical reasoning typically builds on analogical proportions, i.e. statements of the form “*A is to B what C is to D*” [Bayouh *et al.*, 2007; Prade and Richard, 2014; Barbot *et al.*, 2019]. A key result in this area has been the development of analogical classifiers, which are based on the principle that whenever the features of four examples are in an analogical proportion, then their class labels should be in an analogical proportion as well [Bayouh *et al.*, 2007; Hug *et al.*, 2016]. The same principle can be applied to infer plausible concept inclusions, i.e. concept inclusions which are not entailed from a given TBox, but which are likely to hold given additional background knowledge that we have about analogical relationships between different concepts. The resulting inference pattern, which we call *rule extrapolation*, is illustrated in the next example.

Example 1 (Rule extrapolation). *Suppose we have an ontology with the following concept inclusions:*

$$\text{Young} \sqcap \text{Cat} \sqsubseteq \text{Cute} \quad (1)$$

$$\text{Adult} \sqcap \text{WildCat} \sqsubseteq \text{Dangerous} \quad (2)$$

$$\text{Young} \sqcap \text{Dog} \sqsubseteq \text{Cute} \quad (3)$$

Suppose we are furthermore given that “Cat is to WildCat what Dog is to Wolf”. Trivially, we also have that “Young is to Adult what Young is to Adult” and “Cute is to Dangerous what Cute is to Dangerous” Using rule extrapolation, we can then infer the following:

$$\text{Adult} \sqcap \text{Wolf} \sqsubseteq \text{Dangerous} \quad (4)$$

Analogies can also be used to infer plausible knowledge by allowing us to transfer knowledge from one domain to another. This is illustrated in the following example.

¹We note that the term “analogical reasoning” has been used in the literature to refer to a form of similarity-based ontology completion [d’Amato *et al.*, 2006]. In contrast, we reserve this term for methods that require drawing parallels between different domains.

Example 2 (Rule translation). *Suppose we are given the following knowledge:*

$$\text{Program} \sqsubseteq \exists \text{specifies. Software} \quad (5)$$

and the fact that “Program is to Plan what Software is to Building”. Then we can plausibly infer:

$$\text{Plan} \sqsubseteq \exists \text{specifies. Building} \quad (6)$$

Ontologies often use the same “templates” to encode knowledge from different domains (e.g. knowledge about different professions). The strategy from Example 2 then allows us to complete the ontology by introducing additional domains.

Using analogies for identifying plausible missing knowledge is appealing, because they can be learned from text quite effectively. For instance, Turney [2006] proposed a method for identifying similarities between pairs of words (i.e. analogical proportions) using matrix factorization and ternary co-occurrence statistics, which approximated the performance of an average US college applicant. Moreover, the GPT-3 language model is able to identify analogical word pairs with even higher accuracy [Brown *et al.*, 2020]. To a more limited extent, some types of analogical relationships can also be obtained from word embeddings [Mikolov *et al.*, 2013].

The main aim of this paper is to propose a semantics for modelling analogies within the setting of description logics, which introduces a number of unique challenges (see Section 3). We focus in particular on an extension of \mathcal{EL}_\perp . We start from the semantics proposed by Ibáñez-García *et al.* [2020], which extends the \mathcal{EL} semantics by assigning to each individual a set of features. Their aim was to support another form of plausible inference, called interpolation (see Section 2). We show that having access to these features in the semantics also allows us to formalise analogies.²

2 Background

We start by introducing the Description Logic (DL) $\mathcal{EL}_\perp^{\boxtimes}$, which is a straightforward extension of the logic $\mathcal{EL}_\perp^{\boxtimes}$ that was proposed in [Ibáñez-García *et al.*, 2020] to formalise *rule interpolation*³. Our approach to analogical reasoning in this paper will build on $\mathcal{EL}_\perp^{\boxtimes}$. Rule interpolation is another inference pattern for obtaining plausible missing concept inclusions in a DL ontology. Interpolation is based on the notion of *betweenness*, where a concept B is said to be between concepts A and C if B has all the natural properties that A and C have in common. In such a case, knowledge that holds for both A and C seems likely to hold for B as well. This inference pattern is illustrated in the next example.

Example 3 (Rule interpolation). *Suppose we have the following concept inclusions:*

$$\text{Cat} \sqsubseteq X \qquad \text{Wolf} \sqsubseteq X \quad (7)$$

²An appendix with proofs of all results, as well as counterexamples for some claims, is available at <https://arxiv.org/abs/2105.04620>

³We include \perp because disjointness will play an important role in this paper.

As long as X is a “natural concept”, it seems plausible that the following concept inclusion also holds:

$$\text{Dog} \sqsubseteq X$$

This is because all the common (natural) properties of Cat and Wolf are also satisfied by Dog. In such a case, we say that the concept Dog is between the concepts Cat and Wolf.

The notion of naturalness plays an important role in most philosophical accounts of induction. Intuitively speaking, a natural concept or property is one that admits inductive inferences. The semantics from [Ibáñez-García *et al.*, 2020] is based on the common view that natural concepts are those which can be characterised as a set of features (i.e. a conjunction of elementary properties) [Tversky, 1977].

Syntax The logic $\mathcal{EL}_\perp^{\boxtimes}$ extends the standard DL \mathcal{EL}_\perp with *in-between concepts* of the form $C \boxtimes D$, describing the set of objects that are between the concepts C and D . Further, $\mathcal{EL}_\perp^{\boxtimes}$ includes an infinite set of *natural concept names*. More precisely, consider countably infinite but disjoint sets of *concept names* \mathbb{N}_C and *role names* \mathbb{N}_R , where \mathbb{N}_C contains a distinguished infinite set of *natural concept names* $\mathbb{N}_C^{\text{Nat}}$. The syntax of $\mathcal{EL}_\perp^{\boxtimes}$ concepts C, D is defined by the following grammar, where $A \in \mathbb{N}_C$, $A' \in \mathbb{N}_C^{\text{Nat}}$ and $r \in \mathbb{N}_R$:

$$\begin{aligned} C, D &:= \top \mid \perp \mid A \mid C \sqcap D \mid \exists r.C \mid N \\ N, N' &:= A' \mid N \sqcap N' \mid N \boxtimes N' \end{aligned}$$

Concepts of the form N, N' are called *natural concepts*. An $\mathcal{EL}_\perp^{\boxtimes}$ *TBox* is a finite set of concept inclusions $C \sqsubseteq D$, where C, D are $\mathcal{EL}_\perp^{\boxtimes}$ concepts.

Semantics The semantics of $\mathcal{EL}_\perp^{\boxtimes}$ is given in terms of feature-enriched interpretations, which extend standard first-order interpretations by also specifying a mapping π from individuals to sets of *features*. Formally, a *feature-enriched* interpretation is a tuple $\mathcal{I} = (\mathcal{I}, \mathcal{F}, \pi)$ in which $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a classical DL interpretation, \mathcal{F} is a finite set of features, and $\pi : \Delta^{\mathcal{I}} \rightarrow 2^{\mathcal{F}}$, such that the following hold:

1. For each $d \in \Delta^{\mathcal{I}}$ it holds that $\pi(d) \subset \mathcal{F}$;
2. for each $F \subset \mathcal{F}$ there exists some individual $d \in \Delta^{\mathcal{I}}$ such that $\pi(d) = F$.

For an $\mathcal{EL}_\perp^{\boxtimes}$ concept C , $C^{\mathcal{I}}$ is defined as a pair $\langle C^{\mathcal{I}}, \varphi(C) \rangle$ where $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and $\varphi(C)$ is the set of all features associated with a concept C , defined as:

$$\varphi(C) = \bigcap \{ \pi(d) \mid d \in C^{\mathcal{I}} \}$$

Intuitively, the set of features $\varphi(C)$ describes the concept C at a finer-grained level that what may be possible in the language. This makes it possible to capture knowledge about what different concepts have in common, which is needed in $\mathcal{EL}_\perp^{\boxtimes}$ to model the semantics of in-between concepts.

For a standard \mathcal{EL}_\perp concept C , $C^{\mathcal{I}}$ is defined as usual [Baader *et al.*, 2017]. For in-between concepts, $\cdot^{\mathcal{I}}$ is defined as follows.

$$(N \boxtimes N')^{\mathcal{I}} = \{ d \in \Delta^{\mathcal{I}} \mid \varphi(N) \cap \varphi(N') \subseteq \pi(d) \}.$$

Intuitively, $(N \boxtimes N')^{\mathcal{I}}$ contains all elements from the domain that have all the features that are common to both N and N' .

A feature-enriched interpretation $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ satisfies a concept inclusion $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. \mathfrak{I} is a *model* of an \mathcal{EL}^{\bowtie} TBox \mathcal{T} if it satisfies all CIs in \mathcal{T} and for every natural concept N in \mathcal{T} , it holds that

$$N^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \varphi(N) \subseteq \pi(d)\} \quad (8)$$

i.e. N is fully specified by its features. It is easy to verify that (8) is satisfied for a complex natural concept, as soon as it is satisfied for its constituent natural concept names. A concept C is *satisfiable w.r.t. a TBox \mathcal{T}* , if there is a model \mathfrak{I} of \mathcal{T} such that $C^{\mathfrak{I}} \neq \{\emptyset, \mathcal{F}\}$.

The purpose of introducing features in the semantics of \mathcal{EL}^{\bowtie} is to make explicit what different concepts have in common, and to use this as the basis for enabling a particular kind of inductive inference (i.e. interpolation). For instance, in the case of Example 3, if the concept inclusions in (7) are complemented with $Dog \sqsubseteq Cat \bowtie Wolf$, it can be verified that $Dog \sqsubseteq X$ can indeed be inferred (assuming all concept names are natural). In applications, knowledge about in-between concepts would typically be induced from a vector space embedding. See, for instance, [Bouraoui and Schockaert, 2019] for a practical application of rule interpolation based on pre-trained word embeddings. In this paper, we will build on the feature-enriched semantics to encode correspondences between analogous domains.

3 Analogical Concepts

Analogical proportions are a central notion in the formalisation of analogical reasoning, going back to Aristotle (see [Barbot *et al.*, 2019] for a historical perspective). While they can be defined more generally, here we will focus on analogical proportions between sets. In particular, the sets S_1, S_2, S_3, S_4 are said to be in an analogical proportion, denoted as $S_1:S_2 :: S_3:S_4$ if S_1 and S_2 differ in the same way that S_3 and S_4 differ. Formally, $S_1:S_2 :: S_3:S_4$ is satisfied if:

$$S_1 \setminus S_2 = S_3 \setminus S_4 \quad S_2 \setminus S_1 = S_4 \setminus S_3 \quad (9)$$

Some key properties of analogical proportions are as follows:

Reflexivity $A:B :: A:B$

Symmetry $(A:B :: C:D) \Leftrightarrow (C:D :: A:B)$

Exchange of means $(A:B :: C:D) \Leftrightarrow (A:C :: B:D)$

S-transitivity $(A:B :: C:D) \wedge (A:B :: E:F) \Rightarrow (C:D :: E:F)$

C-transitivity $(A:C :: D:B) \wedge (A:E :: F:B) \Rightarrow (C:E :: F:D)$

3.1 Analogical Proportions between DL Concepts

In this paper, we are concerned with analogies between description logic concepts. We can define analogical proportions between \mathcal{EL}^{\bowtie} concepts A, B, C, D as $A^{\mathcal{I}}:B^{\mathcal{I}} :: C^{\mathcal{I}}:D^{\mathcal{I}}$ or as $\varphi(A):\varphi(B) :: \varphi(C):\varphi(D)$. In general these two expressions are not equivalent, and there are advantages in requiring that both of them are satisfied at the same time, which has been studied in detail in [Barbot *et al.*, 2019] within the setting of Formal Concept Analysis. However, if A, B, C, D are natural, then $A^{\mathcal{I}}:B^{\mathcal{I}} :: C^{\mathcal{I}}:D^{\mathcal{I}}$ and $\varphi(A):\varphi(B) :: \varphi(C):\varphi(D)$ are equivalent. Moreover, for natural concepts, the semantic

constraint $\varphi(A):\varphi(B) :: \varphi(C):\varphi(D)$ can be modelled syntactically in \mathcal{EL}^{\bowtie} . Indeed it holds that $\varphi(A):\varphi(B) :: \varphi(C):\varphi(D)$ is satisfied iff the following conditions are satisfied⁴:

$$\varphi(A) \cap \varphi(D) = \varphi(B) \cap \varphi(C) \quad (10)$$

$$\varphi(A) \cup \varphi(D) = \varphi(B) \cup \varphi(C) \quad (11)$$

The analogical proportion $\varphi(A):\varphi(B) :: \varphi(C):\varphi(D)$ is thus satisfied if the following concept inclusions are satisfied:

$$\begin{array}{ll} A \sqcap D \sqsubseteq B \sqcap C & B \sqcap C \sqsubseteq A \sqcap D \\ A \bowtie D \sqsubseteq B \bowtie C & B \bowtie C \sqsubseteq A \bowtie D \end{array}$$

In the following, we will write $A:B :: C:D$ as an abbreviation for these four concept inclusions. The fact that we can model $A:B :: C:D$ within \mathcal{EL}^{\bowtie} is an advantage, but as we will see below, modelling analogies between DL concepts in this way has a number of important limitations.

3.2 Desiderata for Modelling Analogies in DLs

Our motivation for studying analogies is to enable plausible inferences. A clear requirement is thus that we want some form of the *rule extrapolation* and *rule translation* inference patterns, as illustrated in Examples 1 and 2, to be satisfied. Another important requirement comes from the fact that we usually only have access to information about analogical relationships between concept names (e.g. obtained from a language model). However, the aforementioned inference pattern may rely on analogical relationships between complex concepts. To enable non-trivial inferences in practice, it is thus important that analogies between concept names can be *lifted* to analogies between complex concepts.

Let us now consider the suitability of analogical proportions, in light of these desiderata. First, rule translation is satisfied for analogical proportions.

Proposition 1. *Let $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ be a feature-enriched interpretation, and let A_1, A_2, B_1, B_2 be natural concepts in \mathfrak{I} . If $\varphi(A_1):\varphi(A_2) :: \varphi(B_1):\varphi(B_2)$ holds and \mathfrak{I} satisfies $A_1 \sqsubseteq B_1$, then \mathfrak{I} also satisfies $A_2 \sqsubseteq B_2$.*

However, rule extrapolation is not valid for analogical proportions. Furthermore, analogical proportions between concept names cannot be lifted to complex concepts. For instance, from $A_1:B_1 :: C_1:D_1$ and $A_2:B_2 :: C_2:D_2$, in general we cannot infer $(A_1 \sqcap A_2):(B_1 \sqcap B_2) :: (C_1 \sqcap C_2):(D_1 \sqcap D_2)$. Counterexamples are provided in the appendix.

4 The Logic $\mathcal{EL}^{\bowtie}_{\perp}$

Given the limitations of analogical proportions that were highlighted in Section 3.2, we propose an alternative approach for modelling analogies between description logic concepts. This approach is based on the common view that analogies are mappings from one domain into another, which lies among others at the basis of the seminal Structure Mapping framework [Gentner, 1983]. In particular, we propose the logic $\mathcal{EL}^{\bowtie}_{\perp}$, which extends \mathcal{EL}^{\bowtie} with two novel elements: analogy assertions and intra-domain roles. Analogy

⁴This follows immediately from the characterisation of Boolean analogical proportions in terms of conjunction and disjunction; see [Prade and Richard, 2013] for details.

assertions are similar to analogical proportions, in that they encode a relationship of the form “ A is to B what C is to D ”, but their semantics is defined in terms of mappings between different domains, where domains will be identified with sets of features. Intuitively, intra-domain roles are roles which preserve the structure of analogous domains.

4.1 Syntax

Let N_C , N_C^{Nat} and N_R be defined as before. We assume that N_R contains an infinite set N_R^{Int} of distinguished *intra-domain role names*. The syntax of $\mathcal{EL}_{\perp}^{\text{ana}}$ concepts C, D is defined by the following grammar, where $A \in N_C$, $A' \in N_C^{\text{Nat}}$, $r \in N_R$ and $r' \in N_R^{\text{Int}}$:

$$\begin{aligned} C, D &:= \top \mid \perp \mid A \mid C \sqcap D \mid \exists r.C \mid N \\ N, N' &:= A' \mid N \sqcap N' \mid N \bowtie N' \mid \exists r'.N \end{aligned}$$

$\mathcal{EL}_{\perp}^{\text{ana}}$ concepts extend $\mathcal{EL}_{\perp}^{\text{box}}$ concepts by allowing existential restrictions over intra-domain roles as natural concepts. An $\mathcal{EL}_{\perp}^{\text{ana}}$ TBox is a finite set containing two types of assertions: (i) $\mathcal{EL}_{\perp}^{\text{ana}}$ concept inclusions, and (ii) *analogy assertions* of the form $C_1 \triangleright D_1 :: C_2 \triangleright D_2$, where C_1, C_2, D_1, D_2 are natural $\mathcal{EL}_{\perp}^{\text{ana}}$ concepts.

4.2 Semantics

Analogies intuitively involve transferring knowledge from one domain to another domain⁵, e.g. from software engineering to architecture in the case of Example 2. In our framework, these domains will be associated with subsets of \mathcal{F} . In particular, we will require that interpretations specify a partition $[\mathcal{F}_1, \dots, \mathcal{F}_k]$ of \mathcal{F} , defining the different domains of interest. Some of the partition classes will furthermore be viewed as being analogous, in the sense that there is some kind of structure-preserving mapping between them.

Another extension of the feature-enriched semantics is aimed at improving how disjointness can be modelled. In the semantics from [Ibáñez-García *et al.*, 2020], no individual $d \in \Delta^{\mathcal{I}}$ is allowed to have all the features from \mathcal{F} , but all proper subsets $F \subset \mathcal{F}$ are witnessed in the sense that there is some d such that $\pi(d) = F$. This limits how disjoint concepts can be modelled. For instance, B cannot be satisfied w.r.t. $\{B \sqsubseteq A \bowtie C, A \sqcap B \sqsubseteq \perp, A \sqcap C \sqsubseteq \perp, B \sqcap C \sqsubseteq \perp\}$ using a feature-enriched interpretation. Disjointness will play an important role in our semantics, as concepts from different domains will be required to be disjoint (see below). For this reason, we extend the feature-enriched semantics with sets of *forbidden feature combinations*. In particular, interpretations will specify a set $\mathcal{X} \subseteq 2^{\mathcal{F}}$ such that for $X \in \mathcal{X}$, it holds that no individual can have all the features from X . For the ease of presentation, we write \mathcal{C} for the set of all *consistent sets of features*, i.e. $F \in \mathcal{C}$ iff $F \not\supseteq X$ for all $X \in \mathcal{X}$. We also write \mathcal{C}^i for the restriction of \mathcal{C} to subsets of \mathcal{F}_i .

Definition 1. Let $[\mathcal{F}_1, \dots, \mathcal{F}_k]$ be a partition of a non-empty finite set \mathcal{F} . We call $\mathfrak{I} = (\mathcal{I}, [\mathcal{F}_1, \dots, \mathcal{F}_k], \mathcal{X}, \pi, \sim, \mathcal{S})$ a domain constrained interpretation if $\mathcal{I} = (\Delta^{\mathcal{I}}, \mathcal{I})$ is a classical DL interpretation, $\mathcal{X} \subseteq 2^{\mathcal{F}}$, $\mathcal{F} \in \mathcal{X}$, $\pi : \Delta^{\mathcal{I}} \rightarrow$

⁵In this paper, we use the term domain to refer to a particular thematic area. This should not be confused with the set $\Delta^{\mathcal{I}}$, which is often referred to as the domain of the interpretation \mathcal{I} .

$2^{\mathcal{F}}$, \sim is an equivalence relation over $\{1, \dots, k\}$ and $\mathcal{S} = \{\sigma_{(s,t)} \mid (s,t) \in \sim\}$, with each $\sigma_{s,t}$ a bijection from \mathcal{F}_s to \mathcal{F}_t , and we have:

1. for every $d \in \Delta^{\mathcal{I}}$ and $X \in \mathcal{X}$ it holds that $X \not\subseteq \pi(d)$;
2. if $X \in \mathcal{C}$ then $\pi(d) = X$ for some $d \in \Delta^{\mathcal{I}}$;
3. we have $\sigma_{(s,t)}^{-1} = \sigma_{(t,s)}$ and $\sigma_{(t,u)} \circ \sigma_{(s,t)} = \sigma_{(s,u)}$ for any $(s,t), (t,u) \in \sim$;
4. for $F \in \mathcal{C}^i$ and $(i,j) \in \sim$, we have $\{\sigma_{(i,j)}(f) \mid f \in F\} \in \mathcal{C}$;
5. if $f \in \mathcal{F}_i$ and $g \in \mathcal{F}_j$ then $\{f, g\} \in \mathcal{X}$, for all $(i,j) \in \sim$ with $i \neq j$.

Intuitively, each of the sets \mathcal{F}_i corresponds to a different domain. If $(s,t) \in \sim$, it means that there is an analogy between the *source domain* \mathcal{F}_s and the *target domain* \mathcal{F}_t . In that case, there is a one-to-one mapping $\sigma_{(s,t)}$ between the features from \mathcal{F}_s and those from \mathcal{F}_t . The first two conditions from Definition 1 capture the fact that a set of features $X \subseteq \mathcal{F}$ is witnessed by some individual iff it is consistent, i.e. $X \in \mathcal{C}$. The third condition ensures that the mappings $\sigma_{(i,j)}$ can be composed and reversed. The fourth condition encodes that the mapping $\sigma_{(i,j)}$ maps consistent feature combinations to consistent feature combinations. This is a natural requirement, given the intuition that analogous domains should have the same structure. The last condition captures the requirement that individuals cannot have features from two analogous domains. While analogies are normally indeed defined between distinct domains, the reader may wonder at this point whether this restriction is necessary. We will come back to this question in Section 4.4.

Domain Translations Before presenting the semantics of analogy assertions, we first study how the bijections $\sigma_{(i,j)}$ can be combined to define mappings between the sets of features $\varphi(C), \varphi(D)$ associated with two concepts. First, we define a *domain assignment mapping* δ , which maps each concept C to the set of domains on which it depends:

$$\delta(C) = \{i \mid \mathcal{F}_i \cap \varphi(C) \neq \emptyset\}$$

Next, we extend the mappings $\sigma_{(i,j)}$ to mappings between sets of domains. Let $U = \{(s_1, t_1), \dots, (s_l, t_l)\}$, with s_1, \dots, s_l all distinct and t_1, \dots, t_l all distinct. The mapping σ_U is defined as follows:

$$\sigma_U(f) = \begin{cases} \sigma_{(s_i, t_i)}(f) & \text{if } f \in \mathcal{F}_{s_i} \\ f & \text{otherwise} \end{cases}$$

We call σ_U a *domain translation*, and write $\text{src}(U)$ for the set $\{s_1, \dots, s_l\}$ of source domains and $\text{tgt}(U)$ for the set $\{t_1, \dots, t_l\}$ of target domains. The source domains need to be distinct to ensure that the domain translation is uniquely defined. Target domains are required to be distinct to allow domain translations to be reversible. For the ease of presentation, we will write $\sigma_U(F)$ to denote the set $\{\sigma_U(f) \mid f \in F\}$.

For a pair of concepts C and D , we write $\mu(C, D)$ for the set of domain translations σ_U such that:

$$\varphi(D) = \sigma_U(\varphi(C)) \quad (12)$$

$$\text{src}(U) \subseteq \delta(C) \quad (13)$$

$$tgt(U) \cap (\delta(C) \setminus src(U)) = \emptyset \quad (14)$$

The first condition states that the domain translations in $\mu(C, D)$ essentially “translate” the concept C to the concept D . The second condition ensures that $\mu(C, D)$ contains minimal domain translations only, in the sense that U should not contain any redundant pairs. The third condition is needed to ensure that domain translations are reversible. To see why this is needed, let $\varphi(C) = \{f_1, g_2\}$, $\varphi(D) = \{g_1, g_2\}$, $\mathcal{F}_1 = \{f_1, f_2\}$, $\mathcal{F}_2 = \{g_1, g_2\}$, $\sigma_{(1,2)}(f_i) = g_i$. Then $\varphi(D) = \sigma_{(1,2)}(\varphi(C))$, but there is no domain translation σ_U s.t. $\varphi(C) = \sigma_U(\varphi(D))$. As the following result shows, imposing (14) is enough to ensure reversibility.

Proposition 2. *Let U be a domain translation, and let $U^\sim = \{(t, s) \mid (s, t) \in U\}$. It holds that $\sigma_{U^\sim}^{-1} = \sigma_U$.*

As the next result shows, the composition of two domain translations is also a valid domain translation. In particular, if there is some domain translation σ_U that maps C to D and some domain translation σ_V that maps D to E , then σ_U and σ_V can be composed to define a domain translation from C to E . Moreover, in such a case, any domain translation from C to E can be defined as such a composition.

Proposition 3. *If $\mu(C, D) \neq \emptyset$ and $\mu(D, E) \neq \emptyset$ we have:*

$$\mu(C, E) = \{\sigma_{U \oplus V} \mid \sigma_U \in \mu(C, D), \sigma_V \in \mu(D, E)\}$$

where

$$\begin{aligned} U \oplus V = & \{(i, k) \mid (i, j) \in U, (j, k) \in V, i \neq k\} \\ & \cup \{(i, j) \mid (i, j) \in U, j \notin src(V)\} \\ & \cup \{(j, k) \mid (j, k) \in V, j \notin tgt(U)\} \end{aligned}$$

Semantics of Intra-Domain Roles We will need to put additional constraints on the interpretation of a role r to be able to infer $(\exists r.A) \triangleright (\exists r.B) :: (\exists r.C) \triangleright (\exists r.D)$ or $A \triangleright B :: (\exists r.C) \triangleright (\exists r.D)$ from $A \triangleright B :: C \triangleright D$. To allow such lifting of analogy assertions, we will associate with each intra-domain role r a mapping κ_r between sets of features, which satisfies a number of conditions. In particular, we introduce the following notion of intra-domain relation.

Definition 2. *Let $\mathcal{I} = (\mathcal{I}, [\mathcal{F}_1, \dots, \mathcal{F}_k], \mathcal{X}, \pi, \sim, \mathcal{S})$ be a domain constrained interpretation and let $r \in \mathbb{N}_R$. We say that r is interpreted as an intra-domain relation if for every concept C , we have $(\exists r.C)^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid \pi(d) \supseteq \kappa_r(\varphi(C))\}$, for a mapping κ_r satisfying:*

1. $\kappa_r(F) = \kappa_r(F \cap \mathcal{F}_1) \cup \dots \cup \kappa_r(F \cap \mathcal{F}_k)$, for all $F \in \mathcal{C}$;
2. $\kappa_r(F) \subseteq \mathcal{F}_i$, for all $i \in \{1, \dots, k\}$ and $F \in \mathcal{C}^i$;
3. $\kappa_r(\sigma_{\{(i,j)\}}(F)) = \sigma_{\{(i,j)\}}(\kappa_r(F))$, for all $(i, j) \in \sim$ and $F \in \mathcal{C}^i$;
4. $\kappa_r(F) \neq \emptyset$, for all $i \in \{1, \dots, k\}$ and $F \in \mathcal{C}^i \setminus \{\emptyset\}$.

The first two conditions in Definition 2 state that the features in $\kappa_r(F)$ are determined per domain. The third condition captures the intuition that analogous domains should have the same structure. The last condition essentially encodes that whenever C depends on some domain i then $\exists r.C$ should also depend on domain i , i.e. if $\varphi(C)$ contains at least one feature from \mathcal{F}_i then the same should be true for

$\kappa_r(\varphi(C))$. Note that if r is interpreted as an intra-domain relation and C is a natural concept, then $\exists r.C$ is a natural concept, whose features are determined by the features in $\varphi(C)$. We then have $\varphi(\exists r.C) = \kappa_r(\varphi(C))$.

The semantics of \mathcal{EL}_\perp^{ana} concepts can now be defined similarly to Section 2, but we additionally require that every $r \in \mathbb{N}_R^{Int}$ is interpreted as an intra-domain relation.

Semantics of TBoxes We now define the semantics of \mathcal{EL}_\perp^{ana} TBoxes. We start with that of analogy assertions. We say that a domain constrained interpretation \mathcal{I} satisfies the analogy assertion $C_1 \triangleright C_2 :: D_1 \triangleright D_2$ if:

$$\mu(C_1, C_2) \cap \mu(D_1, D_2) \neq \emptyset \quad (15)$$

Clearly, $C_1 \triangleright C_2 :: D_1 \triangleright D_2$ is equivalent to $D_1 \triangleright D_2 :: C_1 \triangleright C_2$. One may wonder whether (15) is sufficient, i.e. whether we should not require $\mu(C_1, C_2) = \mu(D_1, D_2)$. However, as the following result shows, for non-empty concepts, $\mu(C_1, C_2)$ and $\mu(D_1, D_2)$ can have at most one element.

Proposition 4. *Let $\mathcal{I} = (\mathcal{I}, \mathcal{F}, \mathcal{X}, \pi, \sim, \mathcal{S})$ be a domain-constrained interpretation. If $C^\mathcal{I} \neq \emptyset$, $D^\mathcal{I} \neq \emptyset$ and $\mu(C, D) \neq \emptyset$, then $|\mu(C, D)| = 1$.*

We define the semantics of \mathcal{EL}_\perp^{ana} TBoxes similarly to Section 2, but now including analogy assertions. A domain constrained interpretation \mathcal{I} is a model of an \mathcal{EL}_\perp^{ana} TBox \mathcal{T} if \mathcal{I} satisfies all CIs and analogy assertions in \mathcal{T} ; every natural concept $N \in \mathcal{T}$ is fully specified by its features; and every intra-domain role is interpreted as an intra-domain relation. For a TBox \mathcal{T} and CI or analogy assertion ϕ we write $\mathcal{T} \models \phi$ to denote that every model of \mathcal{T} satisfies ϕ . If \mathcal{T} is a singleton of the form $\{\psi\}$, we also write this as $\psi \models \phi$.

4.3 Properties of Analogy Assertions

Basic Properties Before returning to the desiderata from Section 3.2, we briefly look at the properties of analogical proportions that were listed in Section 3. First, reflexivity is trivially satisfied. The symmetry property also holds for analogy assertions, thanks to the reversibility of domain translations.

Proposition 5. *We have $C_1 \triangleright C_2 :: D_1 \triangleright D_2 \models C_2 \triangleright C_1 :: D_2 \triangleright D_1$.*

Exchange of means is not satisfied. This can easily be seen from the fact that whenever $C_1 \triangleright C_2 :: D_1 \triangleright D_2$ is satisfied, we have $|\varphi(C_1)| = |\varphi(C_2)|$ and $|\varphi(D_1)| = |\varphi(D_2)|$ but not necessarily $|\varphi(C_1)| = |\varphi(D_1)|$. As a result of this, there are two variants of S-transitivity that can be considered. As the next proposition shows, both of these variants are satisfied.

Proposition 6. *It holds that:*

$$\{C_1 \triangleright C_2 :: D_1 \triangleright D_2, D_1 \triangleright D_2 :: E_1 \triangleright E_2\} \models C_1 \triangleright C_2 :: E_1 \triangleright E_2 \quad (16)$$

$$\{C_1 \triangleright C_2 :: D_1 \triangleright D_2, C_2 \triangleright C_3 :: D_2 \triangleright D_3\} \models C_1 \triangleright C_3 :: D_1 \triangleright D_3 \quad (17)$$

We also have that C -transitivity is satisfied.

Proposition 7. *It holds that:*

$$\{C_1 \triangleright D_1 :: D_2 \triangleright C_2, C_1 \triangleright E_1 :: E_2 \triangleright C_2\} \models D_1 \triangleright E_1 :: E_2 \triangleright D_2$$

Lifting analogy assertions As the next two results show, analogy assertions can indeed be lifted to (non-empty) conjunctions and existentially quantified concepts.

Proposition 8. *Let $\mathcal{I} = (\mathcal{I}, [\mathcal{F}_1, \dots, \mathcal{F}_k], \mathcal{X}, \pi, \sim, \mathcal{S})$ be a domain-constrained interpretation satisfying $(C_i \sqcap D_i)^{\mathcal{I}} \neq \emptyset$ for $i \in \{1..4\}$, $C_1 \triangleright C_2 :: C_3 \triangleright C_4$ and $D_1 \triangleright D_2 :: D_3 \triangleright D_4$. Then \mathcal{I} also satisfies $(C_1 \sqcap D_1) \triangleright (C_2 \sqcap D_2) :: (C_3 \sqcap D_3) \triangleright (C_4 \sqcap D_4)$.*

Proposition 9. *Let r be an intra-domain role. It holds that:*

$$C \triangleright D :: E \triangleright F \models (\exists r.C) \triangleright (\exists r.D) :: (\exists r.E) \triangleright (\exists r.F) \quad (18)$$

$$C \triangleright D :: E \triangleright F \models C \triangleright D :: (\exists r.E) \triangleright (\exists r.F) \quad (19)$$

Analogy Based Inference Patterns We now return to the two considered analogy based inference patterns: rule translation and rule extrapolation. First, similar as for analogical proportions, we find that rule translation is supported.

Proposition 10. *Let \mathcal{I} be a domain constrained interpretation. If \mathcal{I} satisfies $\{C_1 \triangleright D_1 :: C_2 \triangleright D_2, C_1 \sqsubseteq C_2\}$ then \mathcal{I} also satisfies $D_1 \sqsubseteq D_2$.*

Example 4. *Suppose $\text{Program} \triangleright \text{Plan} :: \text{Software} \triangleright \text{Building}$ holds and assume that specifies is an intra-domain role. Using Proposition 9 we can then infer:*

$$\text{Program} \triangleright \text{Plan} :: (\exists \text{specifies}.\text{Software}) \triangleright (\exists \text{specifies}.\text{Building})$$

If we are additionally given that the concept inclusion (5) is satisfied, we can infer (6) using Proposition 10.

A version of rule extrapolation is also supported.

Proposition 11. *Let $\mathcal{I} = (\mathcal{I}, [\mathcal{F}_1, \dots, \mathcal{F}_k], \mathcal{X}, \pi, \sim, \mathcal{S})$ be a domain constrained interpretation. Suppose that $C_1^{\mathcal{I}} \neq \emptyset$ and that \mathcal{I} satisfies the following TBox:*

$$\mathcal{T} = \{C_1 \triangleright C_2 :: C_3 \triangleright C_4, D_1 \triangleright D_2 :: D_3 \triangleright D_4, \\ D_1 \triangleright D_3 :: D_2 \triangleright D_4, C_1 \sqsubseteq D_1, C_2 \sqsubseteq D_2, C_3 \sqsubseteq D_3\}$$

Then \mathcal{I} also satisfies the assertion $C_4 \sqsubseteq D_4$.

Example 5. *Suppose the concept inclusions (1)–(3) are satisfied, as well as the following analogy assertions⁶:*

$$\text{Young} \triangleright \text{Adult} :: \text{Young} \triangleright \text{Adult}$$

$$\text{Cat} \triangleright \text{WildCat} :: \text{Dog} \triangleright \text{Wolf}$$

$$\text{Cute} \triangleright \text{Dangerous} :: \text{Cute} \triangleright \text{Dangerous}$$

$$\text{Cute} \triangleright \text{Cute} :: \text{Dangerous} \triangleright \text{Dangerous}$$

Assuming the intersections involved are all non-empty, using Proposition 8 we can infer $(\text{Young} \sqcap \text{Cat}) \triangleright (\text{Adult} \sqcap \text{WildCat}) :: (\text{Young} \sqcap \text{Dog}) \triangleright (\text{Adult} \sqcap \text{Wolf})$. Finally, using Proposition 11 we can infer that (4) holds.

Note that both $D_1 \triangleright D_2 :: D_3 \triangleright D_4$ and $D_1 \triangleright D_3 :: D_2 \triangleright D_4$ are required for the above proposition to hold (see the appendix for a counterexample that shows this). For analogical proportions, adding both conditions makes no difference, as $D_1 : D_2 :: D_3 : D_4$ and $D_1 : D_3 :: D_2 : D_4$ are equivalent.

⁶Note that $\text{Cute} \triangleright \text{Cute} :: \text{Dangerous} \triangleright \text{Dangerous}$ is trivially satisfied, since $\sigma_\emptyset \in \mu(\text{Cute}, \text{Cute}) \cap \mu(\text{Dangerous}, \text{Dangerous})$.

4.4 Alternative Semantics

As shown in Section 4.3, the proposed semantics for analogy assertions satisfies the main desiderata from Section 3.2. We may wonder, however, whether all aspects of the semantics are necessary for this to hold. We return in particular to Condition 5 from Definition 1, which is perhaps the most restrictive condition. In particular, let us define the notion of *weak domain constrained interpretation* as a tuple $(\mathcal{I}, [\mathcal{F}_1, \dots, \mathcal{F}_k], \mathcal{X}, \pi, \sim, \mathcal{S})$ that satisfies all conditions from Definition 1, apart from Condition 5. One immediate consequence of dropping Condition 5 is that Proposition 4 is no longer valid. As a result, in addition to analogy assertions of the form $A_1 \triangleright A_2 :: B_1 \triangleright B_2$, with the semantics defined in (15), we can now also consider *strong analogy assertions*, denoted as $A_1 \blacktriangleright A_2 :: B_1 \blacktriangleright B_2$, which are satisfied if (15) is satisfied and moreover:

$$\mu(A_1, A_2) = \mu(B_1, B_2)$$

Under this weak semantics, C -transitivity (i.e. Proposition 7) is no longer satisfied, neither for the standard analogy assertions nor for strong analogy assertions (see the appendix for counterexamples). Regarding S -transitivity, the variant in (17) remains valid and is furthermore also valid for strong analogy assertions. However, the variant in (16) is no longer valid for standard analogy assertions, although it is satisfied for strong analogy assertions. In contrast, Proposition 9 remains valid for standard analogy assertions, but it is not satisfied for strong analogy assertions. Proposition 8 is neither satisfied for standard analogy assertions nor for strong analogy assertions. Finally, in terms of inference patterns, Proposition 10 remains valid, but Proposition 11 does not, neither for standard nor strong analogy assertions.

5 Conclusions

We have proposed a framework for analogy assertions of the form “ A is to B what C is to D ” that is suitable for analogical reasoning in description logics. The underlying assumption is that analogy assertions between concept names can be learned from text, and that we can then lift these to obtain analogy assertions between complex DL concepts. We have shown how the resulting semantics allows us to infer concept inclusions using two analogy based inference patterns. This complements other types of plausible inference patterns, such as interpolation and similarity based reasoning.

There are two important lines for immediate future work. First, we plan to study the computational complexity of reasoning in $\mathcal{EL}_\perp^{\text{ana}}$. Results from [Ibáñez-García *et al.*, 2020] provide a CONP lower bound for concept subsumption w.r.t. $\mathcal{EL}_\perp^{\text{ana}}$ TBoxes. For the upper bound, one key issue would be to relate the number of domains and features of concepts occurring in analogy assertions, as well as those of existential restrictions over intra-domain roles of such concepts. One would also need to establish a bound on the number of features and domains. From the practical side we need mechanisms to deal with the noisy nature of the available knowledge about betweenness and analogy assertions (which typically would be learned from data) and the inconsistencies that may introduce. To this end, we plan to study probabilistic or non-monotonic extensions of our framework.

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