A Multivariate Complexity Analysis of Qualitative Reasoning Problems

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Abstract

Qualitative reasoning is an important subfield of artificial intelligence where one describes relationships with qualitative, rather than numerical, relations. Many such reasoning tasks, e.g., ALLEN'S INTERVAL ALGEBRA, can be solved in $2^{\mathcal{O}(n \cdot \log n)}$ time, but single-exponential running times $2^{\mathcal{O}(n)}$ are currently far out of reach. In this paper we consider single-exponential algorithms via a *multivariate* analysis consisting of a *fine-grained* parameter n (e.g., the number of variables) and a *coarse-grained* parameter k expected to be relatively small. We introduce the classes $\mathbf{FPE}$ and $\mathbf{XE}$ of problems solvable in $f(k) \cdot 2^{\mathcal{O}(n)}$, respectively $f(k)^n$, time, and prove several fundamental properties of these classes. We proceed by studying temporal reasoning problems and (1) show that the PARTIALLY ORDERED TIME problem of effective width k is solvable in 16^{kn} time and is thus included in $\mathbf{XE}$, and (2) that the network consistency problem for ALLEN'S INTERVAL ALGEBRA with no interval overlapping with more than k others is solvable in $(2nk)^{2k} \cdot 2^n$ time and is included in $\mathbf{FPE}$. Our multivariate approach is in no way limited to these to specific problems and may be a generally useful approach for obtaining single-exponential algorithms.

1 Introduction

Qualitative reasoning is an important formalism in artificial intelligence where the objective is to reason about continuous properties given certain relations between the unknown entities, expressed qualitatively rather than quantitatively (e.g., numerically). Two important subfields are *spatial* reasoning, where basic objects e.g. may be regions in space defined in terms of other regions, and *temporal* reasoning, where one wishes to describe the relationship between time points, and more generally, between intervals of time points. Examples of the former include the REGION CONNECTION CALCULUS and the CARDINAL DIRECTION CALCULUS, and to exemplify the latter we may e.g. mention ALLEN'S INTERVAL ALGEBRA, the BRANCHING TIME problem, and the PARTIALLY ORDERED TIME problem. For many more examples and practical applications of these formalisms, cf. the survey by Dylla et al. [2017]. The classical complexity of qualitative reasoning is well understood: they are generally NP-hard but form non-trivial tractable fragments by restricting the types of basic relations. However, the tractable cases have limited strength, and in practice we therefore also need to solve NP-hard reasoning tasks. Hence, is there any hope of solving these intractable problems, or do we have to be contempt with heuristics? Here, modern complexity theory tells a more nuanced story than the classical theory, where NP-hardness is the beginning rather than the end. There are two prominent views:

1. Fine-grained complexity: for a *fine-grained* parameter n, e.g., the number of variables, how fast can the problem be solved, and are existing algorithms close to being optimal (given complexity theoretical assumptions such as *the (strong) exponential-time hypothesis* ((S)ETH))?
2. Parameterized complexity: for a *coarse-grained* parameter k, e.g., tree-width, which problems are *fixed-parameter tractable* ($\mathbf{FPT}$) — are in P if one is allowed unlimited computational resources (with respect to k) to preprocess the instance?

Crucially, n is expected to grow with the size of the instance while k is expected to stay small. Hence, fine-grained and parameterized complexity are not competing methods but tackle different aspects of intractability, with diverging algorithmic toolboxes. Unfortunately, neither approach seem fit for qualitative reasoning problems. On the one hand, they are solvable in $2^{\mathcal{O}(n^2)}$ time or $2^{\mathcal{O}(n \cdot \log n)}$ time in certain cases [Jonsson and Lagerkvist, 2017], but we can currently only rule out subexponential $2^{o(n)}$ algorithms under ETH [Jonsson *et al.*, 2021]. On the other hand, despite the immense success of parameterized complexity, there is a lack of natural $\mathbf{FPT}$ algorithms for qualitative reasoning, and we are only aware of a handful of less surprising examples such as tree-width [Bodirsky and Dalmau, 2013; Dabrowski *et al.*, 2021].

Hence, are we asking the right questions by attacking these problems with parameterized and fine-grained complexity, or are current attempts misguided? Could it be the case that they simply are too hard to admit natural $\mathbf{FPT}$ algorithms? Similarly, could it be the case that (e.g.) ALLEN'S INTERVAL ALGEBRA is not solvable in single-exponential time while still being fundamentally too different from 3-SAT to admit stronger lower bounds than $2^{o(n)}$? In this article we bridge the gap between fine-grained and parameterized complexity and attack this question by a *multivariate* complexity analysis of single-exponential time solvability: which NP-hard qualitative reasoning problems admit single-exponential time algorithms (with respect to a fine-grained parameter n) once preprocessed with respect to a coarse-grained parameter k? This approach is a natural continuation of Bringmann & Künnemann [2018] who studied the complexity of the LONGEST COMMON SUBSEQUENCE problem in order to investigate which natural parameters could possibly break the SETH barrier. However, the exponential world of NP-hard qualitative reasoning problems is very different from the tractable LONGEST COMMON SUBSEQUENCE problem and requires new tools and techniques. Our paper has two main contributions. First, in Section 3 we initiate a systematic investigation of complexity classes taking both a coarse-grained parameter k and a fine-grained parameter n into consideration. We identify two natural classes: $\mathbf{FPE}$, problems solvable in $f(k) \cdot 2^{\mathcal{O}(n)}$ time, and $\mathbf{XE}$, problems solvable in $f(k)^n$ time. Naturally, $\mathbf{FPE}$ is more desirable due to the complete separation of the parameter n and k and should be viewed as the exponential analogue of $\mathbf{FPT}$, while $\mathbf{XE}$ corresponds to the less desirable class $\mathbf{XP}$ where the two parameters are intertwined. Second, in Section 4 we begin the multivariate analysis of single-exponential time solvability by classifying parameterized problems as belonging to $\mathbf{FPE}$, $\mathbf{XE}$, or neither (under the ETH). We first consider the finite-domain *constraint satisfaction problem* (CSP) and manage to rule out inclusion in both $\mathbf{FPE}$ and $\mathbf{XE}$ under the ETH for several natural parameterizations. We then turn to two well-known problems from temporal reasoning: the PARTIALLY ORDERED TIME problem [Anger *et al.*, 1999], which is the problem of determining whether there exists a partial order subjected to a set of constraints over a variable set, and ALLEN'S INTERVAL ALGEBRA [Allen, 1983], the problem of determining whether a set of 2-dimensional intervals described by 1-dimensional temporal constraints over the start- and end-points, is consistent or not. ALLEN'S INTERVAL ALGEBRA has seen applications in e.g. planning [Allen and Koomen, 1983; Dorn, 1995; Mudrová and Hawes, 2015], natural language processing [Denis and Muller, 2011; Song and Cohen, 1988], and molecular biology [Golumbic and Shamir, 1993], while the PARTIALLY ORDERED TIME problem occurs naturally in distributed systems where time is not totally ordered, cf. Lamport's classical interprocessor communication model [Lamport, 1986; Anger, 1989]. Both of these problems can be solved in $2^{\mathcal{O}(n \cdot \log n)}$ time by enumeration, and ALLEN'S INTERVAL ALGEBRA is additionally known to admit an improved $\mathcal{O}(1.0615n^n)$ algorithm [Eriksson and Lagerkvist, 2021]. For PARTIALLY ORDERED TIME we consider *effective width* as parameter, which in strength lies between the well-known properties of *width* and *dimension*. Here, we construct an $\mathbf{XE}$-algorithm with a running time dominated by 16^{kn} where k is the effective width of the partial order. The algorithm is non-trivial and attempts to construct a partial order of effective width k by guessing a suitable partition of the variable set and the ordering among all variables, which can be efficiently implemented via a recursive strategy making use of dynamic programming. For ALLEN'S INTERVAL ALGEBRA we define a parameter k based on the maximum number of overlapping intervals and construct a $(2nk)^{2k} \cdot 2^n$ time algorithm, and thus prove membership in $\mathbf{FPE}$. Hence, for instances where k stays reasonably small our algorithm is effectively as good as a 2^n-time algorithm, which is a major improvement over the aforementioned $\mathcal{O}(1.0615n^n)$ time algorithm. It is worth mentioning that both of these algorithms also solve the more involved problem of *counting* the number of solutions.

Our work opens up several directions for future research, which we describe in greater detail in Section 5. Most importantly, which parameterized problems in qualitative reasoning, and artificial intelligence as a whole, belong to $\mathbf{FPE}$, and which belong to $\mathbf{XE}$?

The proofs of statements marked with an asterisk ($*$) have been omitted due to space constraints.

2 Preliminaries

Throughout, let Σ be an alphabet, i.e., a finite set of symbols. For $x \in \Sigma^*$ we let $|x|$ be its length.

Definition 1. *A* parameterized problem *is a subset of* $\Sigma^* \times \mathbb{N}$ *where* $k \in \mathbb{N}$ *is called the* parameter.

The goal of parameterized complexity is then to describe the complexity in terms of the parameter, and, ideally, design algorithms which decouples the parameter to $|x|$.

Definition 2. *We introduce the following running times and the corresponding classes from parameterized complexity:*

1. $f(k) \cdot |x|^{\mathcal{O}(1)}$ *time by a deterministic algorithm (*$\mathbf{FPT}$*),*
2. $|x|^{f(k)}$ *time by a deterministic algorithm (*$\mathbf{XP}$*),*

where $f : \mathbb{N} \to \mathbb{N}$ *is a computable function.*

$\mathbf{FPT}$ can be seen as the parameterized version of P where the parameter is completely decoupled from the rest of the instance via the factor $f(k)$. The class $\mathbf{XP}$, in comparison, still yields a polynomial-time algorithm for every fixed k but the dependency on k is much worse.

If a *fine-grained* parameter such as the number of variable n is used as parameter then the objective is typically to obtain precise upper and lower bounds on running times of the form $f(n) \cdot |x|^{\mathcal{O}(1)}$, and the resulting field is sometimes called *fine-grained complexity*. If we let $\mathbf{SE}$ be the subclass of $\mathbf{FPT}$ allowing subexponential running times of the form $2^{o(n)} \cdot |x|^{\mathcal{O}(1)}$ then the conjecture that 3-SAT is not included in $\mathbf{SE}$ is known as the *exponential-time hypothesis* (ETH).

Before introducing the qualitative reasoning problems under consideration in this paper we introduce the more general class of *constraint satisfaction problems* (CSPs).

CSP

Instance: $I = (V, C)$ where V is a set of variables and C a set of constraints over a domain D.
Question: Does there exist a function $f : V \to D$ which satisfies all constraints, i.e., $(f(x_1), \ldots, f(x_m)) \in R$ for every $R(x_1, \ldots, x_m) \in C$?

A set of relations Γ naturally induces a problem CSP(Γ) where constraints only uses relations from Γ. For a set $\mathcal{B}$ of relations of the same arity we write $\mathcal{B}^{\vee=}$ for the set of all relations that can be obtained by unions of the relations in $\mathcal{B}$.

Definition 3. *A pair $(S, \leq)$ is a* partial order *if $\leq$ is reflexive, asymmetric ($\forall x, y \in S$, $x \leq y \Rightarrow$ not $y \leq x$), and transitive.*

If $\odot \in \{<, >, ||, =\}$ and $P = (S, \leq_P)$ is a partial order then we write $\odot_P$ for the relation induced by P: $x <_P y$ if $x \leq_P y$ and $y \leq_P x$ does not hold, conversely for $>_P$, $||_P$ if neither $x \leq_P y$ nor $y \leq_P x$, and $x =_P y$ if $x \leq_P y$ and $y \leq_P x$. We will differ between relations in P and the relations induced by P in the same manner, i.e. $x \leq_P y$ is a relation in P, while $x <_P y$ is a relation induced by P.

PARTIALLY ORDERED TIME
Instance: A set of variables V and a set of binary constraints C where $c \subseteq \{<, >, ||, =\}$ for each $c(x, y) \in C$. **Question:** Is there a partial order $P = (S, \leq)$ with $\lvert S\rvert \leq \lvert V\rvert$ and a function $f\colon V \to S$ such that for every constraint $c(x, y) \in C$, $f(x) \odot_P f(y)$ for some $\odot \in c$?

For a PARTIALLY ORDERED TIME instance $I = (V, C)$ and $X \subseteq V$ we write $I[X] = (X, C_X)$ for the sub-instance where $C_X = \{c(x, y) \in C \mid x, y \in X\}$. Similarly, if $P = (S, \leq)$ is a partial order and $S' \subseteq S$ then we by $P[S']$ denote the partial order induced by P and S'.

The second major problem analysed in the paper is ALLEN'S INTERVAL ALGEBRA: define a as the thirteen possible atomic relations between two intervals on the same line (see the technical appendix for precise definitions of these relations). The consistency problem for ALLEN'S INTERVAL ALGEBRA can then be seen as CSP($\mathcal{A}$) where $\mathcal{A} = a^{\vee=}$, i.e., each constraint can be expressed as a union of basic constraints. We represent intervals as pairs of start- and end-points $X = (x^-, x^+)$ such that $x^- < x^+$. In the same manner, for any CSP($\mathcal{A}$) (V, C) we introduce the two auxiliary sets $V^- = \{x^- \mid (x^-, x^+) \in V\}$ and $V^+ = \{x^+ \mid (x^-, x^+) \in V\}$.

3 Multivariate Complexity Classes

Our main interest are problems solvable in $2^{f(n)}$ time for $f \in \mathcal{O}(n)$ (henceforth written $2^{\mathcal{O}(n)}$). This naturally leads to the following parameterized variant of the classical complexity class $\mathbf{E}$, which is typically defined via the parameter $|x|$.

Definition 4. $\mathbf{E}$ *is the class of parameterized problems solvable in $2^{\mathcal{O}(n)} \cdot |x|^{\mathcal{O}(1)}$ time.*

The choice of parameter n greatly influences the existence of a $2^{\mathcal{O}(n)}$ algorithm. For example, if we use the number of constraints m as the parameter then CSP($\mathcal{A}$) is in $\mathbf{E}$ via a trivial backtracking algorithm, but it is only known to be solvable in $2^{\mathcal{O}(n \cdot \log n)}$ time. Similarly, a $c^n \cdot |x|^{\mathcal{O}(1)}$ algorithm for $c < 2$ for CNF-SAT would constitute a major break through in complexity theory, but if we instead use the number of clauses m as the complexity parameter then the problem can be solved in $\mathcal{O}(1.2226^m)$ time [Chu *et al.*, 2021].

Example 5. *The* TRAVELLING SALESMAN *problem is trivially solvable in $n! \cdot |x|^{\mathcal{O}(1)}$ time but can be solved in $2^n \cdot |x|^{\mathcal{O}(1)}$ time by the Held-Karp algorithm. Similarly,* GRAPH COLORING *can be solved in $k^n \cdot |x|^{\mathcal{O}(1)}$ time, where k is the number of colors, or $n^n \cdot |x|^{\mathcal{O}(1)}$ time if analyzed only by the number of vertices n, but can be solved in $2^n \cdot |x|^{\mathcal{O}(1)}$ time via an inclusion-exclusion based algorithm [Björklund* et al.*, 2009].*

*For infinite-domain examples, let Γ^k be the set of all k-ary first-order definable relations over $(\mathbb{Q}; <)$ (*temporal *CSPs). Then CSP(Γ^3) is solvable in $3^n \cdot |x|^{\mathcal{O}(1)}$ time [Eriksson and Lagerkvist, 2021] but CSP(Γ^4) is not in $\mathbf{E}$ under the ETH [Jonsson and Lagerkvist, 2018]. In contrast, if Δ^k is the set of all k-ary first-order definable relations over $(\mathbb{N}; =)$ (*equality *CSPs) then CSP(Δ^k) is solvable in $(\frac{k(k-1)}{2})^n \cdot |x|^{\mathcal{O}(1)}$ time [Jonsson and Lagerkvist, 2020] and is thus in $\mathbf{E}$.*

Importantly, no NP-hard qualitative reasoning tasks are known to be solvable in single-exponential time, necessitating a multivariate analysis where the complexity is measured with respect to several parameters.

Definition 6. *A* multi-parameterized problem *is a subset of $\Sigma^* \times \mathbb{N}^c$ where $n, k_1, \ldots, k_{c-1} \in \mathbb{N}$ are called the* parameters*. A* bi-parameterized problem *is a multi-parameterized problem with only two parameters: n and k.*

For simplicity we, given an instance (x, n, k) of a bi-parameterized problem, will always view the first parameter n as the fine-grained parameter, and the second parameter k as the coarse-grained parameter. Inspired by $\mathbf{FPT}$ and $\mathbf{XP}$ we then define the following bi-parameterized analogues.

Definition 7. *We introduce the following two classes.*

1. $\mathbf{FPE}$*: problems solvable in $f(k) \cdot 2^{\mathcal{O}(n)} \cdot |x|^{\mathcal{O}(1)}$ time, and*
2. $\mathbf{XE}$*: problems solvable in $f(k)^n \cdot |x|^{\mathcal{O}(1)}$ time,*

where $f\colon \mathbb{N} \to \mathbb{N}$ is a computable function.

Note that these running times are not symmetric with respect to the two parameters since we expect the parameter k to stay reasonably small, while n, naturally, grows rapidly with the size of the instance. Hence, the common trick in parameterized complexity to collapse two parameters into a single parameter $n + k$ is not suitable for defining these classes. Additionally, it is possible to establish several fundamental properties of these classes and (1) derive $\mathbf{FPE}$ as a class of the form para-$\mathbf{E}$ using standard constructions in parameterised complexity, (2) define reductions preserving membership in these classes, and (3) prove that $\mathbf{FPE}$ can equivalently be defined via running times of the following form.

Theorem 8. *($*$) A bi-parameterized problem is in $\mathbf{FPE}$ if and only if it is solvable in $n^{f(k)} \cdot 2^{\mathcal{O}(n)} \cdot |x|^{\mathcal{O}(1)}$ time for computable $f\colon \mathbb{N} \to \mathbb{N}$.*

Example 9. *The CSP problem is in $\mathbf{XE}$ if k is the size of the domain and n the number of variables but is* not *in $\mathbf{FPE}$ unless the ETH fails (see Theorem 10). For a less trivial $\mathbf{XE}$ example, consider the problem of finding a homomorphism between two graphs G and H over vertices V_G and V_H. If we let n be $|V_G|$, and k be the* tree-width *of H, then the problem is solvable in $(k + 3)^n \cdot |x|^{\mathcal{O}(1)}$ time, and if $n = \max\{V_G, V_H\}$ and k is the* clique-width *of H then it is solvable in $(2k + 1)^n \cdot |x|^{\mathcal{O}(1)}$ time [Fomin* et al.*, 2007; Wahlström, 2011]. From Example 5, equality CSPs are in $\mathbf{XE}$ when parameterized by maximum arity while temporal CSPs* *are not in* $\mathbf{XE}$ *with the same parameterization. More generally, if* $\mathcal{B}$ *is a so-called* partition scheme*, then the CSP problem where constraints are formed by disjunctions of arity at most* k*, using relations from* $\mathcal{B}$*, is in* $\mathbf{XE}$ *when parameterized by the maximum variable degree [Jonsson* et al.*, 2021]. A non-trivial, related example can also be found in the* CHANNEL ASSIGNMENT PROBLEM *which is in* $\mathbf{XE}$ *when parameterized by the edge-length* k *of the input graph [McDiarmid, 2003].*

Curiously, while $\mathbf{XE}$-algorithms appear to be abundant in the literature, $\mathbf{FPE}$-algorithms are much rarer and we are not aware of any non-trivial examples.

4 Parameterized Problems in FPE and XE

We now analyse $\mathbf{FPE}$ and $\mathbf{XE}$ in greater detail and begin in Section 4.1 by considering finite-domain CSPs with parameters such as *domain size* and *degree*. In Section 4.2 we consider the *effective width* of a partial order as a coarse-grained parameter for PARTIALLY ORDERED TIME, and construct a non-trivial $\mathbf{XE}$-algorithm. Last, in Section 4.3, we use the maximum number of possible interval *overlaps* as a parameter for ALLEN'S INTERVAL ALGEBRA, resulting in the first non-trivial example of a problem in $\mathbf{FPE}$.

4.1 Finite-Domain CSPs

For a finite-domain CSP instance (V, C) we consider the following parameters ($\text{ar}(R)$ is the arity of a relation R).

1. $\text{dom}((V,C)) = \bigcup_{(d_1,\ldots,d_{\text{ar}(R)})\in R, R(\mathbf{x})\in C}\{d_1,\ldots,d_{\text{ar}(R)}\}$,
2. $\text{max-arity}((V,C)) = \max\{\text{ar}(R) \mid R(\mathbf{x}) \in C\}$,
3. $\text{max-degree}((V,C)) = \max\{\text{degree}(x,C) \mid x \in V\}$ where $\text{degree}(x,C) = |\{R(\mathbf{x}) \in C \mid x \text{ occurs in } \mathbf{x}\}|$,
4. $\text{max-cardinality}((V,C)) = \max\{|R| \mid R(\mathbf{x}) \in C\}$,

For each such parameter f we write p-f-CSP for the bi-parametrized problem obtained by letting the coarse-grained parameter k equal $f((V,C))$ and the fine-grained parameter n equal the number of variables $|V|$ (for a CSP instance (V,C)).

Theorem 10. *($*$) Assume that the ETH is true. Then:*

1. *p-dom-CSP is in* $\mathbf{XE}$ *but not in* $\mathbf{FPE}$*,*
2. *p-max-arity-CSP is not in* $\mathbf{XE}$*,*
3. *p-max-degree-CSP is not in* $\mathbf{FPE}$*, and*
4. *p-max-cardinality-CSP is in* $\mathbf{XE}$ *but not in* $\mathbf{FPE}$*.*

It may be interesting to note that all these bound are tight, with the possible exception of p-max-degree-CSP, where we currently do not know whether it is included in $\mathbf{XE}$.

4.2 Partially Ordered Time

The first major temporal reasoning problem we consider is the PARTIALLY ORDERED TIME problem where we use a form of *width* as the coarse-grained parameter.

Definition 11. *A partial order $P = (S, \leq_P)$ is said to have* effective width k *if S can be partitioned into at most $k+2$ disjoint subsets $(S_1,\ldots,S_k,S_<,S_>)$ such that:*

1. *at least two of the sets $(S_1,\ldots,S_k,S_<,S_>)$ are non-empty, unless all elements are equal,*
2. *for all $x \in S_<$ there is a S_i such that for all $y \in S_i$ then $x \leq_P y$ and not $y \leq_P x$,*
3. *for all $x \in S_>$ there is a S_i such that for all $y \in S_i$ then $x \geq_P y$ and not $y \geq_P x$,*
4. *for all pairs $x \in S_<$, $y \in S_>$ with $x \leq_P y$, then there is a S_i such that for all $z \in S_i$ then $x \leq_P z \leq_P y$,*
5. *all partial orders $P[S_1],\ldots,P[S_k],P[S_<],P[S_>]$ have effective width k, and*
6. *for every pair $S_i, S_j \in \{S_1,\ldots,S_k\}$, $i \neq j$, all $x \in S_i$ and all $y \in S_j$ are incomparable in P.*

The collection of sets $S_1,\ldots,S_k$ is called the waist *of P.*

Effective width is a weaker constraint than width (the size of the largest anti-chain), but is likely a stronger condition than the *dimension* of a partial order, i.e., the least number of total orders whose intersection equals the partial order.

PARTIALLY ORDERED TIME OF EFFECTIVE WIDTH k
Instance: A partially ordered time instance (V, C). **Parameters:** $n = |V|$, $k \in \mathbb{N}$. **Question:** Is there a partial order P with effective width k which satisfies the instance?

We introduce the following two definitions to simplify the notation in the forthcoming algorithm.

Definition 12. *Let $\mathbf{B}(n,k)$ be the value of the k-th bit of the bit-string representing the integer n.*

Definition 13. *Given a binary relation $\odot$, a variable x and a set V, we say that $x \odot V$ if $x \odot v$ for all $v \in V$.*

This notation easily extends to pairs of sets. The basic idea is then to recursively construct a partial order P of effective width k by enumerating all possible waists and relations between the other variables and said waist. This enumeration is then combined with dynamic programming to keep track of already solved subproblems. First, observe that each element not in a waist can (independently) be either $<$ (or $>$) or incomparable to each of the k sets of said waist. If we are relating to two waists at once we then obtain 4^k possibilities.

Theorem 14. *Partially Ordered Time of effective width k is solvable in $16^{kn} \cdot |x|^{\mathcal{O}(1)}$ time, and is hence in* $\mathbf{XE}$*.*

Proof. For a PARTIALLY ORDERED TIME instance (V, C) of effective width k we define a recursive function $\mathcal{X}$, taking as input 4^k pairwise disjoint subsets of V, $S_1,\ldots,S_{4^k}$ as follows. If only one $S_i \neq \emptyset$, check if the assumption that all variables of that set are equal is consistent with C. This can easily be done in polynomial time. If yes, the instance is accepted, otherwise we continue as follows. We introduce two temporary waists of size k: $\{T_1^<,\ldots,T_k^<\}$ and $\{T_1^>,\ldots,T_k^>\}$. Assume that $T_i^< <_P S_j \Leftrightarrow \mathbf{B}(j, 2i-1) = 1$ and $T_i^> <_P S_j \Leftrightarrow \mathbf{B}(j, 2i) = 1$. Partition $\bigcup S_i$ into the new sets N_i^W for $i \in \{1,\ldots,k\}$, and $N_i^<$, $N_i^>$ for $i \in \{1,\ldots,4^k\}$, such that no constraint is broken under the following assumptions:

1. if $\mathbf{B}(i, 2j) = 1$ then $N_i^< <_P N_j^W$ and $N_i^> <_P T_j^>$,
2. if $\mathbf{B}(i, 2j-1) = 1$ then $T_j^< <_P N_i^<$ and $N_j^W <_P N_i^>$,

3. all pairs N_i^W and N_j^W are incomparable if $i \neq j$.

This means that for each i there is a j such that $N_i^W \subseteq S_j$, and since every variable not in a waist must relate to the waist, some sets must always be empty. For each partition, recursively check that $I[N_i^W]$, $I[\bigcup N_i^<]$ and $I[\bigcup N_i^>]$ are 'yes'-instances for all i (when they exist) under the assumptions just made. If one partition is found that is consistent with C and $\mathcal{X}(N_1^<, \ldots, N_{4^k}^<)$, $\mathcal{X}(N_1^W, \emptyset, \ldots)$, $\ldots$, $\mathcal{X}(N_k^W, \emptyset, \ldots)$ and $\mathcal{X}(N_1^>, \ldots, N_{4^k}^>)$ all accept, then accept $\mathcal{X}(S_1, \ldots, S_{4^k})$.

For $|\bigcup S_i| \leq 1$ the algorithm is clearly correct. Assume that the algorithm is correct for all $|\bigcup S_i| < n$. Now, for correctness of $|\bigcup S_i| = n$, we start with soundness. We brute-force enumerate all partitions and reject a partition if there is any constraint not consistent with said partition. For every constraint $c(x, y) \in C$, there will at some point be a partition where x and y are not part of the same recursive call to X. At that point we will know the exact relation between x and y, and hence we can check if this relation is accepted by the constraint $c(x, y)$. Hence, if we answer 'yes', all constraints will be satisfied. To see that we find an partial order, the question boils down to knowing transitivity holds correctly. Since we reject all partitions that breaks our assumptions, we will never accept $x <_P y$ if we have earlier assumed that $z <_P x$ and that y and z are incomparable. The same is true for the symmetric case when $z <_P x$ is assumed by some earlier partition and $x <_P y$ by the current partition. Also $x \in N_i^W$ and $y \in N_j^W$ would only be accepted it x being incomparable to y is accepted by our constraint $c(x, y)$. Hence, these assumptions are enough to ensure that we actually obtain a partial order. For effective width we follow the definition and in each step make sure that: everything not part of the waist is related to some part of the waist, all parts of the waist are incomparable to every other part of the waist, and that all of $I[\bigcup N_i^<]$, $I[N_1^W]$, $\ldots$, $I[\bigcup N_k^W]$, and $I[\bigcup N_i^>]$ have effective width k. Hence, we will only answer 'yes' if we are given a 'yes'-instance. For completeness, assume there is some partial order P_A of effective width k satisfying I. Since P_A exists, the waist $N_1^W, \ldots, N_k^W$ exists and we will find it and the correct partitions to give as inputs to the subproblems, when brute-force testing all possible partitions. Since the algorithm is assumed to be correct for every problem smaller than n, it must be correct (and thus complete) for these subproblems. The complexity analysis is straightforward but is omitted due to space constraints. □

4.3 Allen's Interval Algebra

The second major qualitative reasoning problem we study is a variant of ALLEN'S INTERVAL ALGEBRA where we restrict the number of intervals any interval may *overlap* with.

Definition 15. *We say that two intervals I and J overlap if there is a point x such that $I^- < x < I^+$ and $J^- < x < J^+$.*

k-CSP($\mathcal{A}$)
Instance: A CSP($\mathcal{A}$) instance $I = (V, C)$. **Parameters:** $n = \lvert V \rvert$, $k \in \mathbb{N}$. **Question:** Is there a satisfying assignment where no interval overlaps with k or more intervals?

We will prove that k-CSP($\mathcal{A}$) is solvable in roughly $(2kn)^{2k} \cdot 2^n$ time and space, and thus prove membership in $\mathbf{FPE}$. Our algorithm makes use of a relationship between satisfying assignments and so-called *ordered partitions*.

Definition 16. *A finite sequence of non-empty finite sets $(S_1, \ldots, S_\ell)$ is an* ordered partition *of a set S if $S_1, \ldots, S_\ell$ is a partitioning of S. The* ranking function *r is the function $r\colon S \to \{1, \ldots, \ell\}$ such that $r(x) = i$ for every $x \in S_i$ and every $i \in \{1, \ldots, \ell\}$. The number of unique ordered partitions for a set of size n is given by the* Ordered Bell Number *$\mathrm{OBN}(n)$ and is strictly less than the number of possible ways to arrange n items into n different sets if $n > 1$, n^n.*

It is known that CSP($\mathcal{A}$) over V is satisfiable if and only if there exists an ordered partition of $V^- \cup V^+$ and a ranking function r, such that, for any relation $\odot \in \{<, >, =\}$, $r(x) \odot r(y)$ if and only if $f(x) \odot f(y)$ for all $x, y \in V^- \cup V^+$ [Eriksson and Lagerkvist, 2021]. To simplify the notation we allow ordered partitions to (temporally) contain empty sets S_i and we then assume that $(S_1, \ldots, S_i, \ldots, S_\ell) = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_\ell)$. We will explicitly make sure this step is taken, but allowing this slight bending of the definition lets us avoid some unnecessarily complex steps.

Definition 17. *Given a sequence (such as e.g. an ordered partition) $(x_1, \ldots, x_i)$ let the notation $(x_1 : x_i)$ denote this sequence. Let $(x_1 : x_i - x_j) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i)$, $j \in \{1, \ldots, i\}$.*

Theorem 18. *k-CSP($\mathcal{A}$) is solvable in $(2nk)^{2k} \cdot 2^n \cdot |x|^{\mathcal{O}(1)}$ time, and is hence in $\mathbf{FPE}$.*

Proof. The algorithm is similar to [Eriksson and Lagerkvist, 2021] and works by the following principle: by introducing intervals from smallest starting-point, to largest end-point, we only need to keep track of intervals already passed, current open intervals, how many other intervals these has already overlapped with, and intervals not yet opened. Given an k-CSP($\mathcal{A}$) instance $I = (V, C)$, for inputs V_1^- (a set), $X = (X_1 : X_i)$ (a partial order), $v = (v_1 : v_i)$ (a collection of values) and V_2^- (a set) we then define a recurrence relation $R(V_1^-, X, v, V_2^-)$ as follows:

- $R(V_1^-, (X_1 : X_i - X_j), (v_1 : v_i - v_j), V_2^-)$, if $X_j = \emptyset$,
- 0, if $v_j < 0$ for any $j \in \{1, \ldots, i\}$,
- 1, if $V_1^- = V^-$, and
- $\sum_{x^- \subseteq \bigcup X_j, y \subseteq V_2^-} f(V_1^-, X, v, V_2^-, x^-, y)$ otherwise.

If two cases are applicable for the same input, the first, as ordered here above, have the highest priority. Given x^- and $X = (X_1 : X_i)$ define $x^+ = \{x_j^+ \mid (x_j^-, x_j^+) \in V \wedge x_j^- \in x^-\}$ and $X^+ = \{x_j^+ \mid (x_j^-, x_j^+) \in V \wedge x_j^- \in \bigcup X_l \setminus x^-\}$, the sequences $X' = (X_1 \setminus x^- : X_i \setminus x^-, y)$ and $v' = (v_1 - |x^-| : v_i - |x^-|, k - |\bigcup X_j \setminus x^-|)$ and the function f as

$$f(V_1^-, X, v, V_2^-, x^-, y) = R(V_1^- \cup x^-, X', v', V_2^- \setminus y)$$

if and only if the assumptions $(X_1 : X_i, y \cup x^+)$ and $u < v$ for all $u \in \bigcup X_j \cup y \cup x^+$, $v \in V_2^- \cup X^+$ is consistent with C. I.e. for any constraint $c^+(u,v) = c^-(u,v) \in C$ ($\mathcal{A}$ is closed under symmetry, so the constraints are equal) with $v^- \in \bigcup X_j \cup V_2^-$ and $u^+ \in x^+$ then the assumption satisfies $c^+(u,v)$. Otherwise f outputs 0.

Our algorithm for the decision problem is simply to check if $R(\emptyset, (), (), V^-) > 0$. To see that this approach is sound we follow a sequence of R-recurrences from $R(\emptyset, (), (), V^-)$ to $R(V^-, (), (), \emptyset)$ such that in each step we have that $R > 0$. Take the ordered partitions that are part of this sequence, and merge them into a single ordered partition $(Y_1 : Y_n)$ such that if $x < y$ in some $(X_1 : X_l, V_2^-)$ for some $R(V_1^-, (X_1 : X_l), v, V_2^-)$, then $x < y$ in $(Y_1 : Y_n)$. Since R works by moving starting-points from the set V_2^- to X and then further to V_1^-, while keeping track of relevant previous intervals $(Y_1 : Y_n)$ exists and is uniquely defined for each such sequence. The positions of the end-points in $(Y_1 : Y_n)$ is implicit from the relation between x^- and x^+ in our definition of f. Let $u^- \in Y_{i^-}$, $u^+ \in Y_{i^+}$, $u'^- \in Y_{j^-}$ and $u'^+ \in Y_{j^+}$. Assume $i^+ \leq j^+$, then for each constraint $c(u, u') \in C$ in our sequence of R-recurrences, there is an $R(V_1^-, X, v, V_2^-)$ followed by $R(V_1'^-, X', v', V_2'^-)$ such that $u^- \in V_1'^-$ and $u^- \notin V_1^-$. From these two inputs, we know the relations between i^-, i^+, j^- and j^+ (since $j^- < j^+$ must hold). Since we know these relations, and since the sequence exists, f must have accepted that $c(u, u')$ is holds. This is true for all $c \in C$, so $(Y_1 : Y_n)$ must be a solution for I. To verify that this solution also fulfills the overlap requirement, we at each step have v which is handled such that no value in V ever drops below 0. Hence we have a 'yes'-instance if $R(\emptyset, (), (), V^-) > 0$.

For completeness, take an ordered partition $Y = (Y_1 : Y_i)$ over $V^- \cup V^+$ satisfying our arbitrary instance I. Given this ordered partition, iterative construct a sequence of $R(V_1^-, X, v, V_2^-)$ and and choose x^- and y such that $x^+ = Y_j \cap V^+$ and $y = Y_j \cap V^-$, starting from $R(\emptyset, (), (), V^-)$ and Y_1, working towards Y_i and $R(V^-, (), (), \emptyset)$. Now, observe that for $X = (X_1 : X_{i'})$ we have that $X_1 \subseteq Y_l, \ldots, X_j \subseteq Y_{l+i'-1}$, and $V_2^- = V^- \cap \bigcup_{u=l+i'}^{i} Y_u$, which means that when placing Y_l, we will know the relations between all intervals ending in Y_l, and all intervals ending later. Since these relations are the same as those in Y, f will always return a value greater than 0. Similarly, v will never contain a value less than 0, since the numbers in v simply counts how many overlaps have occurred for corresponding intervals. The ordering $(Y_1 : Y_n)$ will also be tested as every sub-ordering of size at most k will be found by brute-force enumeration. So, if given a 'yes'-instance, we will reach $R(V^-, (), (), \emptyset) = 1$ and so we have $R(V_1^-, X, v, V_2^-) > 0$ for every other R in our sequence (including $R(\emptyset, (), (), V^-)$). Hence, this approach is correct. The complexity analysis is straightforward and the proof is omitted due to space constraints. □

Let us also remark that this algorithm, as well as the one in Section 4.2, solve the more general problem of counting the number of solutions (i.e., the number of satisfying partial orders or ordered partitions)

5 Concluding Remarks

We explored how a multivariate complexity analysis can generate interesting cases with significant complexity improvements compared to the classical case of single parameters. This led to the introduction of the classes **FPE** and **XE** which turned out to be the natural exponential-time analogoues of the well-known classes **FPT** and **XP**. We proved several fundamental properties of these classes and gave examples of **XE** and **FPE** from qualitative reasoning. These algorithms are significantly faster than existing methods (provided the coarse-grained parameters stay relatively small) and constitute an important breakthrough for single-exponential time algorithms in qualitative reasoning.

Systematically classifying XE and FPE. The overarching open question is to classify problems and parameters in terms of **XE** or **FPE** membership (or neither, under assumptions such as the ETH). Here, qualitative spatial reasoning problems (such as the REGION CONNECTION CALCULUS) seem to be a promising continuation since the NP-hard cases in general are not known to be solvable in single-exponential time. More generally, is it possible to find canonical parameters which result in **FPE** algorithms, similarly to how parameters such as tree-width almost always results in **FPT** algorithms? Identifying such parameters could open up entirely new algorithmic approaches for qualitative reasoning, in particular, and infinite-domain CSPs, in general.

Limited versus unlimited equality. The algorithm in Section 4.3 for k-CSP($\mathcal{A}$) has similarities to the one in [McDiarmid, 2003] for the CHANNEL ASSIGNMENT PROBLEM with bounded edge-length, making the two problems interesting to compare. One interpretation is that k-CSP($\mathcal{A}$) belongs to **FPE** because the problem bounds the number of intervals that may be equal (since they overlap), which is not true for CHANNEL ASSIGNMENT, which seemingly prevents our approach from proving **FPE** membership. However, if one limits the size of equivalence classes in the same sense in the CHANNEL ASSIGNMENT PROBLEM, this new problem would naturally fall into **FPE**, using just a slight modification of the original algorithm. Is this a problem specific behaviour, or is it a deeper difference between **FPE** and **XE**?

Extending and improving the algorithms. The algorithm in Section 4.2 and Section 4.3 can likely be improved to handle higher-arity constraints, although it would make the presentation and the analysis significantly more complex. Another interesting direction is to consider different parameters: the algorithm from Section 4.2 can very likely be modify to use the width of the partial order as the coarse-grained parameter instead of effective width, but it is less clear whether the dimension of a partial order is sufficient for an **XE**-algorithm.

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