Conditional Independence for Iterated Belief Revision

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Abstract

Conditional independence is a crucial concept for efficient probabilistic reasoning. For symbolic and qualitative reasoning, however, it has played only a minor role. Recently, Lynn, Delgrande, and Peppas have considered conditional independence in terms of syntactic multivalued dependencies. In this paper, we define conditional independence as a semantic property of epistemic states and present axioms for iterated belief revision operators to obey conditional independence in general. We show that c-revisions for ranking functions satisfy these axioms, and exploit the relevance of these results for iterated belief revision in general.

1 Introduction

Over the last decades, conditional independence was shown to be a crucial concept supporting adequate modelling and efficient reasoning in probabilistics. For example, in medical diagnosis, symptoms are often assumed to be conditionally independent given the disease [Pearl, 1988]. It is the fundamental concept underlying network-based reasoning in probabilistics, which has been arguably one of the most important factors in the rise of contemporary artificial intelligence. Even though many reasoning tasks on the basis of probabilistic information have a high worst-case complexity due to their semantic nature, network-based models allow an efficient computation for many concrete instances of these reasoning tasks thanks to local reasoning techniques.

On the other hand, for the modelling of intelligent agents, it is important to give a formal account of the dynamics of knowledge and belief. This is done in the field of belief change. Traditionally, researchers have been mainly interested in studying the dynamics of sets of propositional formulas (so-called AGM theory, based on [Alchourrón *et al.*, 1985]). One of the drawbacks of this perspective is that such formal models do not give an account of *iterated belief revision*, i.e. of how to revise a previously revised set of formulas. It became clear that for iterated revision, information about the state of mind of an agent that goes beyond the merely propositional level is needed. Such information can be represented by *epistemic states*, and iterated belief revision can then be viewed as the revision of these epistemic states. In the AGM framework, total preorders over propositional interpretations that represent the respective (im)plausibility of these interpretations, provide the basic semantic structures for iterated revision [Darwiche and Pearl, 1997].

In this paper, we present a concept of conditional independence for total preorders that provides a novel axiomatic foundation for iterated belief revision, allowing for local reasoning in the revised epistemic state. The basic idea of such axioms is simple: If we have conditional independence in the prior epistemic state, and the new information is compatible with that, then we also should have conditional independence in the posterior epistemic state. The challenge is to define conditional independence for total preorders that is, on the one hand, compatible with (iterated) AGM revision, and, on the other hand, shows basic characteristics of probabilistic conditional independence. However, conditional independence in probabilistics makes heavy use of the rich arithmetic properties of probabilities which are not available for total preorders. Spohn's ranking functions [Spohn, 1988; Spohn, 2012] are a good mediator between these different semantic frameworks. We use the notion of conditional independence for ranking functions to fully elaborate our approach to conditional independence in belief revision by providing c-revisions [Kern-Isberner and Huvermann, 2017] as a proof of concept. Moreover, we show how many of the features of that framework can be made relevant also for total preorders, and hence for iterated revision in general.

This paper is organized as follows: In the next section, we compile basics on propositional logic, total preoders and ranking functions, and iterated AGM belief revision. In Section 3, we recall related work on conditional independence which is relevant to this paper. Then Section 4 presents our approach to conditional independence for total preorders, and Section 5 makes it usable for iterated revision. Section 6 concludes and puts out future work.

2 Formal Preliminaries

Propositional Logic. Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet Σ with atoms $a, b, c, \ldots$, and with formulas $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connector, writing AB instead

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of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of *possible worlds* over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $Mod(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $Mod(A) \subseteq Mod(B)$, as usual. For a proposition $A \in \mathcal{L}$, $Cn(A) = \{B \in \mathcal{L} \mid A \models B\}$ is the *set of consequences of A in $\mathcal{L}$*. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to use ω both as an interpretation and a proposition, which will ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

For subsets Θ of Σ, let $\mathcal{L}(\Theta)$ denote the propositional language defined by Θ, with associated set of interpretations $\Omega(\Theta)$. Note that while each sentence of $\mathcal{L}(\Theta)$ can also be considered as a sentence of $\mathcal{L}$, the interpretations $\omega^\Theta \in \Omega(\Theta)$ are not elements of $\Omega(\Sigma)$ if $\Theta \neq \Sigma$. But each interpretation $\omega \in \Omega$ can be written uniquely in the form $\omega = \omega^\Theta \omega^{\Sigma \setminus \Theta}$ with concatenated $\omega^\Theta \in \Omega(\Theta)$ and $\omega^{\Sigma\setminus\Theta} \in \Omega(\Sigma \setminus \Theta)$. Here, ω^Θ is called the *reduct* of ω to Θ [Delgrande, 2017]. If $\Omega' \subseteq \Omega$ is a subset of models, then $\Omega'|_\Theta = \{\omega^\Theta | \omega \in \Omega'\} \subseteq \Omega(\Theta)$ restricts Ω' to a subset of $\Omega(\Theta)$. In the following, we will often consider the case that Σ is the union of three disjoint subsignatures $\Sigma_1, \Sigma_2, \Sigma_3$, i.e., $\Sigma = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3$ where $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$, and $\dot\cup$ denotes the exclusive union. Then we write ω^i instead of ω^{Σ_i} for the reducts to ease notation, i.e., in this case, each $\omega \in \Omega$ can be written in the form $\omega = \omega^1\omega^2\omega^3$ with $\omega^i \in \Omega(\Sigma_i), i \in \{1,2,3\}$. For $\Theta \subseteq \Sigma$ and $A \in \mathcal{L}(\Sigma)$, with $A^\Theta \in \mathcal{L}(\Theta)$, called the *reduct of A to Θ*, we denote any formula with $Mod(A^\Theta) = \{\omega^\Theta \mid \omega \in Mod(A)\}$. Note that A^Θ is uniquely defined up to logical equivalence. For instance, for $\Sigma = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3$ and $A \in \mathcal{L}(\Sigma)$, we get $A^{\Sigma_i} \in \mathcal{L}(\Sigma_i)$ for $i \in \{1,2,3\}$, and, e.g., $A^{\Sigma_1 \dot\cup \Sigma_3} \in \mathcal{L}(\Sigma_1 \dot\cup \Sigma_3)$. Just like ω^Θ with $\omega \in \Omega(\Sigma)$, we will consider A^Θ also as a sentence in $\mathcal{L}(\Sigma)$ and thus having models over the larger signature Σ; using this view, A^Θ is called the projection of A onto Θ in [Lynn *et al.*, 2022]. Instead of $A^{\Sigma_i \dot\cup \Sigma_j}$, we will often just write A^{Σ_i, Σ_j}.

TPOs and OCFs. *Total preorders (TPO)* $\preceq$ over possible worlds are total, reflexive, and transitive relations over Ω. As usual, $\omega \prec \omega'$ iff $\omega \preceq \omega'$ and not $\omega' \preceq \omega$, and $\omega \approx_\Psi \omega'$ iff both $\omega \preceq \omega'$ and $\omega' \preceq \omega$. Such TPOs can be lifted to total preorders on the set of propositions via $A \preceq B$ iff there is a (minimal) $\omega \in Mod(A)$ such that $\omega \preceq \omega'$ for all $\omega' \in Mod(B)$. If $\Omega' \subseteq \Omega$, then $\min_\preceq(\Omega') = \{\omega' \in \Omega' \mid \omega' \preceq \omega''$ for all $\omega'' \in \Omega'\}$ denotes the set of $\preceq$-minimal models in Ω'. If $\Omega' = \Omega$, then we simply write $\min(\preceq)$ instead of $\min_\preceq(\Omega)$. If $A \in \mathcal{L}$, then $\min_\preceq(A) = \min_\preceq(Mod(A))$. The minimal models of a TPO form its associated belief set: $Bel(\preceq) = \mathcal{T}(\min(\preceq))$, where $\mathcal{T}(\Omega')$ denotes the set of formulas which are true in all elements of $\Omega' \subseteq \Omega$. I.e., the agent believes exactly the propositions that are valid in all most plausible models. With $K_\preceq \in \mathcal{L}$, we denote the *core* of $Bel(\preceq)$, i.e., the unique (up to logical equivalence) formula that respresents $Bel(\preceq)$, in the sense that $Bel(\preceq) = Cn(K_\preceq)$.

For $\Theta \subseteq \Sigma$, any TPO $\preceq$ on $\Omega(\Sigma)$ induces uniquely a *marginalized TPO* $\preceq_{|\Theta}$ on $\Omega(\Theta)$ by

$$\omega_1^\Theta \preceq_{|\Theta} \omega_2^\Theta \text{ iff } \omega_1^\Theta \preceq \omega_2^\Theta. \qquad (1)$$

Note that on the right hand side of the *iff* condition above $\omega_1^\Theta, \omega_2^\Theta$ are considered as propositions in $\mathcal{L}(\Omega)$, hence $\omega_1^\Theta \preceq \omega_2^\Theta$ is well defined [Kern-Isberner and Brewka, 2017].

Ordinal conditional functions (OCFs), (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, were introduced (in a more general form) first by [Spohn, 1988]. They express degrees of plausibility of propositional formulas A by specifying degrees of disbeliefs of their negations $\overline{A}$. More formally, we have $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. Note that also OCFs κ induce total preorders on Ω via $\omega_1 \preceq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$. Analogously to TPOs, $Bel(\kappa) = \mathcal{T}(\{\omega \mid \kappa(\omega) = 0\}$ is the *set of beliefs of an OCF κ*. An OCF κ is called *convex* if it does not have empty layers, i.e., if $\kappa^{-1}(i) \neq \emptyset$ implies $\kappa^{-1}(j) \neq \emptyset$ for all $j < i$.

The *marginalization of κ on* $\Theta \subseteq \Sigma$, denoted by $\kappa|_\Theta$, is defined by $\kappa|_\Theta(\omega^\Theta) = \kappa(\omega^\Theta)$ for any $\omega^\Theta \in \Omega(\Theta)$. OCFs can also be conditionalized with respect to propositional formulas: for any $A \in \mathcal{L}$ with $\kappa(A) < \infty$, $\kappa|A$ is an OCF on the models of A defined by $\kappa|A(\omega) = \kappa(\omega) - \kappa(A)$.

Note that the marginalization for TPOs and OCFs as given above are special cases of the forgetful functor $Mod(\sigma)$ from Σ-models to Θ-models in [Beierle and Kern-Isberner, 2012] where $\Theta \subseteq \Sigma$ and $\sigma : \Theta \to \Sigma$ is the inclusion from Θ to Σ.

Revision of Beliefs and Epistemic States. The following theorem that generalizes results from [Katsuno and Mendelzon, 1991] is fundamental to (iterated) AGM belief revision [Alchourrón *et al.*, 1985]. In particular, it reveals that total preorders are the basic semantic structures that are needed for iterated belief revision.

Theorem 1 ([Darwiche and Pearl, 1997])**.** *A revision operator $*$ that assigns a posterior epistemic state $\Psi * A$ to a prior state Ψ and a proposition A is an AGM revision operator for epistemic states iff there exists a total preorder $\preceq_\Psi$ for an epistemic state Ψ with associated belief set $K = Bel(\Psi)$, such that for any proposition A it holds that*

$$K * A = Bel(\Psi * A) = \mathcal{T}(\min(\Psi, A)). \qquad (2)$$

This theorem also allows us to use the same symbol $*$ for revisions of belief sets and revision of epistemic states, where implicitly the validity of (2) is presupposed.

C-revisions provide a highly general framework for revising OCFs (as specific implementations of total preorders) by (sets of) conditionals resp. propositions. For the purposes of this paper it will be sufficient to recall c-revisions by a single proposition [Kern-Isberner and Huvermann, 2017].

Definition 1 (Propositional c-revisions for OCFs)**.** *Let κ be an OCF specifying a prior epistemic state, and let $A \in \mathcal{L}$. A* (propositional) c-revision of κ by *A is given by the OCF*

$$\kappa * A(\omega) = -\kappa(A) + \kappa(\omega) + \begin{cases} \eta & \text{if } \omega \models \overline{A}, \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

with an integer η satisfying $\eta > \kappa(A) - \kappa(\overline{A})$.

$\kappa(A)$ in (3) is a normalization factor, and $\eta > \kappa(A) - \kappa(\overline{A})$ ensures $\kappa * A \models A$. Each c-revision is an iterated revision in the sense of [Darwiche and Pearl, 1997].

3 Related Work on Conditional Independence

Conditional independence is the crucial ingredient to make computations efficient in probabilistic networks [Pearl, 1988]. Thanks to the close relationship between rankings and probabilities, there is a straightforward adaptation of conditional independence for OCFs [Spohn, 2012, Chapter 7].

Definition 2. *Let $\Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3 \subseteq \Sigma$ and let κ be an OCF. Σ_2, Σ_3 are* conditionally independent given Σ_1 with respect to κ, *in symbols $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3 | \Sigma_1$, if for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, and $\omega^3 \in \Omega(\Sigma_3)$, $\kappa(\omega^2 | \omega^1 \omega^3) = \kappa(\omega^2 | \omega^1)$ holds.*

As for probabilities, conditional independence for OCFs expresses that information on Σ_3 is redundant for Σ_2 if full information on Σ_1 is available and used. The following lemma is straightforward and helpful for technical proofs.

Lemma 1. *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be disjoint subsignatures of Σ, let κ be an OCF. Then Σ_2, Σ_3 are conditionally independent given Σ_1 with respect to κ iff for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, and $\omega^3 \in \Omega(\Sigma_3)$, we have $\kappa(\omega^1\omega^2\omega^3) = \kappa(\omega^1\omega^2) + \kappa(\omega^1\omega^3) - \kappa(\omega^1)$.*

As in probabilistics, conditional independence with respect to an OCF κ generalizes κ-independence, as defined in [Kern-Isberner and Huvermann, 2017; Kern-Isberner and Brewka, 2017]; more precisely, Σ_1, Σ_2 are κ-independent iff they are conditionally independent given the empty set.

Conditional independence with respect to databases resp. propositional formulas have been considered in [Darwiche, 1997; Lang and Marquis, 1998; Lang *et al.*, 2002; Lynn *et al.*, 2022]. We recall the definition of [Lynn *et al.*, 2022] here.

Definition 3 ([Lynn *et al.*, 2022])**.** *Let $K \in \mathcal{L}$ be a propositional formula. Let $\Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3 \subseteq \Sigma$. Σ_2, Σ_3 are* conditionally independent given Σ_1 modulo K *if for all $\omega \in \Omega$, for all $A_2 \in \mathcal{L}(\Sigma_2)$, $A_3 \in \mathcal{L}(\Sigma_3)$ such that $K \wedge \omega^1 \models A_2 \vee A_3$, we have $K \wedge \omega^1 \models A_2$ or $K \wedge \omega^1 \models A_3$.*

In their paper [Lynn *et al.*, 2022], Lynn, Delgrande, and Peppas considered conditional independence for (basic) AGM revision on the level of belief sets via syntactical representations called multivalued dependencies, and consider TPO-based AGM revision operators that comply with such dependencies. Our approach to conditional independence is a semantic one directly for TPOs, and hence is applicable to (more general) iterated revision tasks. For a more detailed comparison with that work, please see Section 5.

4 Conditional Independence for TPOs

The basic semantic structure for iterated belief revision are total preorders (TPOs) on possible worlds, according to the seminal works by Katsuno and Mendelzon [Katsuno and Mendelzon, 1991], and Darwiche and Pearl [Darwiche and Pearl, 1997]. We first present a semantic notion of conditional independence for total preorders.

Definition 4. *Let $\Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3 \subseteq \Sigma$ let $\preceq$ be a TPO on Ω. Σ_2, Σ_3 are* conditionally independent given Σ_1 with respect to $\preceq$, *in symbols $\Sigma_2 \perp\!\!\!\perp_\preceq \Sigma_3 | \Sigma_1$, if for all $\omega^1 \in \Omega(\Sigma_1), \omega_1^2, \omega_2^2 \in \Omega(\Sigma_2)$, and $\omega_1^3, \omega_2^3 \in \Omega(\Sigma_3)$ holds that for all $i, j \in \{2, 3\}, i \neq j$,*

$$\omega^1 \omega_1^i \omega_1^j \preceq \omega^1 \omega_2^i \omega_1^j \text{ iff } \omega^1 \omega_1^i \preceq \omega^1 \omega_2^i. \tag{4}$$

Again, conditional independence expresses that in the context of given information from Σ_1, information on Σ_j is irrelevant for the ordering of worlds on Σ_i. Practically speaking, ω_1^j can be "cancelled out". We illustrate this by examples.

Example 1. *Consider the following TPO over the signature $\Sigma = \{p, q, r\}$:*

$$\overline{p}\,\overline{q}r, p\overline{q}r \prec \overline{p}qr, pqr, p\overline{q}\,\overline{r} \prec pq\overline{r}, \overline{p}\,\overline{q}\,\overline{r} \prec \overline{p}q\overline{r}.$$

Then $\{q\} \perp\!\!\!\perp_\preceq \{r\} | \{p\}$. A witness of this is the fact that $\overline{p}\,\overline{q} \prec \overline{p}q$ and $\overline{p}\,\overline{q}r \prec \overline{p}qr$ and $\overline{p}\,\overline{qr} \prec \overline{p}q\overline{r}$.

Notice that $\{q\} \perp\!\!\!\perp_\preceq \{r\} | \{p\}$ yet it does not *hold that $\{q, p\} \perp\!\!\!\perp_\preceq \{r\} | \emptyset$. To see this, it suffices to observe that $\overline{p}\,\overline{q} \prec pq$ (in view of $\overline{p}\,\overline{q}r \prec pqr$) yet $\overline{p}\,\overline{q}\,\overline{r} \not\prec pq\overline{r}$. Likewise, it does* not *hold that $\{p\} \perp\!\!\!\perp_\preceq \{r\} | \{q\}$. This can be seen by observing that $\overline{p}\,\overline{q} \preceq p\overline{q}$ yet $\overline{p}\,\overline{q}\,\overline{r} \not\preceq p\overline{q}\,\overline{r}$.*

The next example uses slightly increased signatures.

Example 2. *Let $\Sigma = \Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3$ with $\Sigma = \{a, b, c, d\}, \Sigma_1 = \{a, b\}, \Sigma_2 = \{c\}, \Sigma_3 = \{d\}$, and let $\preceq$ be the TPO over Σ given by:*

$$\begin{aligned} & abc\overline{d},\ \overline{a}bcd,\ \overline{a}bc\overline{d} \prec abcd,\ \overline{a}b\overline{c}d,\ \overline{a}b\overline{c}\overline{d} \\ & \prec ab\overline{c}\overline{d},\ \overline{a}\overline{b}cd \prec ab\overline{c}d,\ a\overline{b}cd,\ \overline{a}\overline{b}c\overline{d},\ \overline{a}\overline{b}\overline{c}d \\ & \prec a\overline{b}c\overline{d},\ a\overline{b}\overline{c}d,\ \overline{a}\overline{b}\overline{c}\overline{d} \prec a\overline{b}\overline{c}\overline{d} \end{aligned}$$

For checking that Σ_2, Σ_3 are conditionally independent given Σ_1 with respect to $\preceq$, we have to ensure that for all $\omega^1 \in \{ab, a\overline{b}, \overline{a}b, \overline{a}\overline{b}\}$, all $\omega_1^2, \omega_2^2 \in \{c, \overline{c}\}$, and all $\omega_1^3, \omega_2^3 \in \{d, \overline{d}\}$ the condition in (4) *holds. For $\omega^1 = ab$ we get, e.g., $abc \preceq ab\overline{c}$ and $abcd \preceq ab\overline{c}d$ and $abc\overline{d} \preceq ab\overline{c}\overline{d}$, and furthermore $ab\overline{d} \preceq abd$ and $abc\overline{d} \preceq abcd$ and $ab\overline{c}\overline{d} \preceq ab\overline{c}d$. Checking also the remaining instances of* (4) *reveals that $\Sigma_2 \perp\!\!\!\perp_\preceq \Sigma_3 | \Sigma_1$ holds.*

This definition of conditional independence for TPOs is compatible with the definition for OCFs, in the sense that conditional independence with respect to an OCF κ implies conditional independence with respect to the corresponding TPO $\preceq_\kappa$:

Proposition 1. *Let $\Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3 \subseteq \Sigma$, let κ be an OCF. Then $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3 | \Sigma_1$ implies $\Sigma_2 \perp\!\!\!\perp_{\preceq_\kappa} \Sigma_3 | \Sigma_1$.*

Proof. Let $\Sigma_1 \dot{\cup} \Sigma_2 \dot{\cup} \Sigma_3 \subseteq \Sigma$, and let κ be an OCF such that $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3 | \Sigma_1$ holds. For the proof of $\Sigma_2 \perp\!\!\!\perp_{\preceq_\kappa} \Sigma_3 | \Sigma_1$, we focus on Σ_2, the proof for Σ_3 is analogous. Let $\omega^1 \in \Omega(\Sigma_1), \omega_1^2, \omega_2^2 \in \Omega(\Sigma_2)$, and $\omega^3 \in \Omega(\Sigma_3)$. $\omega^1\omega_1^2\omega^3 \preceq_\kappa \omega^1\omega_2^2\omega^3$ means $\kappa(\omega^1\omega_1^2\omega^3) \leqslant \kappa(\omega^1\omega_2^2\omega^3)$, which is equivalent to $\kappa(\omega_1^2|\omega^1\omega^3) \leqslant \kappa(\omega_2^2|\omega^1\omega^3)$. Because of $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3 | \Sigma_1$, this is equivalent to $\kappa(\omega_1^2|\omega^1) \leqslant \kappa(\omega_2^2|\omega^1)$, and hence to $\kappa(\omega^1\omega_1^2) \leqslant \kappa(\omega^1\omega_2^2)$, i.e., $\omega^1\omega_1^2 \preceq_\kappa \omega^1\omega_2^2$, which was to be shown. □

The converse of Proposition 1, i.e., that $\Sigma_2 \perp\!\!\!\perp_\preceq \Sigma_3 | \Sigma_1$ implies $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3 | \Sigma_1$ for some κ with $\preceq_\kappa = \preceq$, cannot be expected in general because TPOs are too weak to mimick the arithmetic features of OCFs faithfully. However, in some cases, we might indeed lift conditional independence with respect to TPOs to conditional independence with respect to suitable OCFs.

Proposition 2. *Let $\Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3 \subseteq \Sigma$, and let $\preceq$ be a TPO on Ω such that $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3 | \Sigma_1$ holds. Let κ be an OCF such that $\preceq_\kappa = \preceq$. Furthermore presuppose that all conditional OCFs in the sets $\{\kappa|\omega^1\omega^3 \mid \omega^1 \in \Omega(\Sigma_1), \omega^3 \in \Omega(\Sigma_3)\}$ and $\{\kappa|\omega^1 \mid \omega^1 \in \Omega(\Sigma_1)\}$ are convex. Then we also have $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3|\Sigma_1$.*

Proof. We have to show that $\kappa(\omega^2|\omega^1\omega^3) = \kappa(\omega^2|\omega^1)$ for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2), \omega^3 \in \Omega(\Sigma_3)$. Since all conditional OCFs $\kappa|\omega^1\omega^3, \kappa|\omega^1$ define convex OCFs on $\Omega(\Sigma_2)$ and start with rank 0, this is equivalent to $(\kappa(\omega_1^2|\omega^1\omega^3) \leqslant \kappa(\omega_2^2|\omega^1\omega^3)$ iff $\kappa(\omega_1^2|\omega^1) \leqslant \kappa(\omega_2^2|\omega^1))$, for all $\omega^1 \in \Omega(\Sigma_1), \omega_1^2, \omega_2^2 \in \Omega(\Sigma_2), \omega^3 \in \Omega(\Sigma_3)$. But this is straightforward: we have $\kappa(\omega_1^2|\omega^1\omega^3) \leqslant \kappa(\omega_2^2|\omega^1\omega^3)$ iff $\kappa(\omega^1\omega_1^2\omega^3) \leqslant \kappa(\omega^1\omega_2^2\omega^3)$. Due to $\preceq_\kappa = \preceq$, this means $\omega^1\omega_1^2\omega^3 \preceq \omega^1\omega_2^2\omega^3$. Because of $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3|\Sigma_1$, this is equivalent to $\omega^1\omega_1^2 \preceq \omega^1\omega_2^2$, and hence to $\kappa(\omega^1\omega_1^2) \leqslant \kappa(\omega^1\omega_2^2)$, i.e., to $\kappa(\omega_1^2|\omega^1) \leqslant \kappa(\omega_2^2|\omega^1)$. ◻

We illustrate this by continuing Example 1.

Example 3. *The minimal OCF implementing the TPO from Example 1 is given by $\kappa(\overline{p}\,\overline{q}r) = \kappa(p\overline{q}r) = 0, \kappa(\overline{p}qr) = \kappa(pqr) = \kappa(p\overline{q}\,\overline{r}) = 1, \kappa(pq\overline{r}) = \kappa(\overline{p}\,\overline{q}\,\overline{r}) = 2, \kappa(\overline{p}q\overline{r}) = 3$. It is straightforward to check that all conditional OCFs $\kappa|\omega^1\omega^3, \kappa|\omega^1$ are convex (they all take on only the ranks 0 and 1), so we also have $\{q\} \perp\!\!\!\perp_\kappa \{r\}|\{p\}$ according to Prop. 2 which can easily be verified.*

Next, we show that conditional independence of total preorders is compatible with the notion of conditional independence modulo propositional formulas (see Definition 3):

Proposition 3. *Let $\Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3 \subseteq \Sigma$. If Σ_2, Σ_3 are conditionally independent given Σ_1 with respect to $\preceq$, then Σ_2, Σ_3 are also conditionally independent given Σ_1 modulo the core $K_\preceq$.*

The proof of this proposition is straightforward, but omitted due to lack of space. The following example illustrates Proposition 3.

Example 4. *For $\Sigma = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3$ and $\preceq$ as in Example 2 we have $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3|\Sigma_1$ and $K_\preceq \equiv abc\overline{d} \vee \overline{a}bc$. For checking whether Σ_2, Σ_3 are conditionally independent given Σ_1 modulo $K_\preceq$, we have to consider all $\omega \in \Omega$, all $A_2 \in \mathcal{L}(\Sigma_2)$, and all $A_3 \in \mathcal{L}(\Sigma_3)$. If $K_\preceq \wedge \omega^1 \equiv \bot$ or if any of A_2, A_3 is equivalent to $\top$ or equivalent to $\bot$, the conditional independence requirement holds trivially. Thus, we are left to check the requirement for $\omega \in \{abc\overline{d}, \overline{a}bcd, \overline{a}bc\overline{d}\}$, $A_2 \in \{c, \overline{c}\}$, and $A_3 \in \{d, \overline{d}\}$. First, let $\omega = abc\overline{d}$ and thus $\omega^1 = ab$. Since $K_\preceq \wedge ab \equiv abc\overline{d}$, we get $K_\preceq \wedge \omega^1 \models A_2 \vee A_3$ only for the three cases (i) $A_2 \equiv c$ and $A_3 \equiv d$, (ii) $A_2 \equiv c$ and $A_3 \equiv \overline{d}$, or (iii) $A_2 \equiv \overline{c}$ and $A_3 \equiv \overline{d}$. In each of these cases, $K_\preceq \wedge \omega^1 \models A_2$ or $K_\preceq \wedge \omega^1 \models A_3$ holds. Otherwise, let $\omega \in \{\overline{a}bcd, \overline{a}bc\overline{d}\}$ and thus $\omega^1 = \overline{a}b$ in both cases. Since $K_\preceq \wedge \overline{a}b \equiv \overline{a}bc$, we get $K_\preceq \wedge \omega^1 \models A_2 \vee A_3$ only for the two cases (i) $A_2 \equiv c$ and $A_3 \equiv d$, or (ii) $A_2 \equiv c$ and $A_3 \equiv \overline{d}$. In both cases, $K_\preceq \wedge \omega^1 \models A_2$ holds. In summary, we observe that Σ_2, Σ_3 are conditionally independent given Σ_1 modulo $K_\preceq$.*

Among the most important properties of independence relations is that of being a (semi-)graphoid:

Definition 5 ([Pearl and Paz, 1985])**.** *A ternary relation $\cdot \perp\!\!\!\perp \cdot|\cdot$ among subsets of some signature Σ is a* semi-graphoid *over Σ iff for all pairwise disjoint $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \subseteq \Sigma$ the following properties hold:*

Symmetry *if $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1$ then $\Sigma_3 \perp\!\!\!\perp \Sigma_2|\Sigma_1$.*

Decomposition *if $\Sigma_2 \perp\!\!\!\perp \Sigma_3 \cup \Sigma_4|\Sigma_1$ then $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1$.*

Weak Union *if $\Sigma_2 \perp\!\!\!\perp \Sigma_3 \cup \Sigma_4|\Sigma_1$ then $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1 \cup \Sigma_4$.*

Contraction *if $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1$ and $\Sigma_2 \perp\!\!\!\perp \Sigma_4|\Sigma_3 \cup \Sigma_1$ then $\Sigma_2 \perp\!\!\!\perp \Sigma_3 \cup \Sigma_4|\Sigma_1$.*

$\cdot \perp\!\!\!\perp \cdot|\cdot$ is a graphoid *(over Σ), if the following additional property is satisfied::*

Intersection *If $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1 \cup \Sigma_4$ and $\Sigma_2 \perp\!\!\!\perp \Sigma_4|\Sigma_3 \cup \Sigma_1$ then $\Sigma_2 \perp\!\!\!\perp \Sigma_3 \cup \Sigma_4|\Sigma_1$.*

Spohn already showed that conditional independence in OCFs is a graphoid [Spohn, 2012, Theorem 7.10]. Since TPOs cannot make use of any of the arithmetic features that ranking functions and probabilities rely on, we cannot expect a similarly strong result here. Nevertheless, we are able to prove some of these properties, at least in a weakened form:

Proposition 4. *Let a TPO $\preceq$ be given. Then conditional independence $. \perp\!\!\!\perp_{\preceq} .|.$ over Σ satisfies symmetry, weak union, and the following variant of decomposition:*

Symmetric decomposition *if $\Sigma_2 \perp\!\!\!\perp \Sigma_3 \cup \Sigma_4|\Sigma_1$ and $\Sigma_2 \cup \Sigma_4 \perp\!\!\!\perp \Sigma_3|\Sigma_1$ then $\Sigma_2 \perp\!\!\!\perp \Sigma_3|\Sigma_1$.*

The proof of this proposition is a lengthy technical exercise and left out due to lack of space. It is an open question whether conditional independence for TPOs is, in general, a graphoid.

For OCFs, and hence also regarding the appertaining TPOs according to Proposition 1, ranking functions respecting specific conditional independence statements can be easily constructed under mild conditions of compatibility, as shown next:

Proposition 5. *Let $\Sigma = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3$, let κ_2, κ_3 be OCFs defined over $\Sigma_1 \dot\cup \Sigma_2$ and $\Sigma_1 \dot\cup \Sigma_3$, respectively, such that $\kappa_2(\omega^1) = \kappa_3(\omega^1)$ for all $\omega^1 \in \Omega(\Sigma_1)$. Define κ over Σ via $\kappa(\omega) = \kappa_2(\omega^1\omega^2) + \kappa_3(\omega^1\omega^3) - \kappa_2(\omega^1)$. Then κ is an OCF such that $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3|\Sigma_1$ holds.*

Proof. First, κ is an OCF, i.e., there are worlds ω with $\kappa(\omega) = 0$. Due to the prerequisite $\kappa_2(\omega^1) = \kappa_3(\omega^1)$ for all $\omega^1 \in \Omega(\Sigma_1)$, $\kappa_2|_{\Sigma_1}, \kappa_3|_{\Sigma_1}$ are marginalized OCFs such that $\kappa_2|_{\Sigma_1} = \kappa_3|_{\Sigma_1}$. So, there is $\omega_0^1 \in \Omega(\Sigma_1)$ such that $\kappa_2(\omega_0^1) = \kappa_3(\omega_0^1) = 0$. Then there are $\omega_0^2 \in \Omega(\Sigma_2)$ and $\omega_0^3 \in \Omega(\Sigma_3)$ such that $\kappa_2(\omega_0^1\omega_0^2) = \kappa_2(\omega_0^1) = 0 = \kappa_3(\omega_0^1) = \kappa_3(\omega_0^1\omega_0^3)$. For $\omega_0 = \omega_0^1\omega_0^2\omega_0^3$, we have $\kappa(\omega_0) = \kappa_2(\omega_0^1\omega_0^2) + \kappa_3(\omega_0^1\omega_0^3) - \kappa_2(\omega_0^1) = 0$.

Furthermore, we simply have to show that $\kappa(\omega^1\omega^2) = \kappa_2(\omega^1\omega^2), \kappa(\omega^1\omega^3) = \kappa_3(\omega^1\omega^3)$, and $\kappa(\omega^1) = \kappa_2(\omega^1)(= \kappa_3(\omega^1))$, then the statement is immediate from the definition of κ. We show the first statement, the others are anologous:

$$\begin{aligned}\kappa(\omega^1\omega^2) &= \min_{\omega^3 \in \Omega(\Sigma_3)} \kappa(\omega^1\omega^2\omega^3)\\ &= \min_{\omega^3 \in \Omega(\Sigma_3)} (\kappa_2(\omega^1\omega^2) + \kappa_3(\omega^1\omega^3) - \kappa_2(\omega^1))\end{aligned}$$

ω^{12}	$\kappa_2(\omega^{12})$	$\kappa_2(\omega^1)$	ω^{13}	$\kappa_3(\omega^{13})$	$\kappa_3(\omega^1)$
abc	1	1	abd	2	1
$ab\overline{c}$	2		$ab\overline{d}$	1	
$a\overline{b}c$	3	3	$a\overline{b}d$	3	3
$a\overline{b}\overline{c}$	4		$a\overline{b}\,\overline{d}$	4	
$\overline{a}bc$	1	0	$\overline{a}bd$	0	0
$\overline{a}b\overline{c}$	0		$\overline{a}b\overline{d}$	0	
$\overline{a}\overline{b}c$	2	2	$\overline{a}\overline{b}d$	2	2
$\overline{a}\overline{b}\overline{c}$	3		$\overline{a}\overline{b}\,\overline{d}$	3	

Table 1: OCFs κ_2, κ_3 for Example 5

ω	$\kappa(\omega)$	ω	$\kappa(\omega)$	ω	$\kappa(\omega)$	ω	$\kappa(\omega)$
$abcd$	2	$abc\overline{d}$	1	$ab\overline{c}d$	3	$ab\overline{c}\overline{d}$	2
$a\overline{b}cd$	3	$a\overline{b}c\overline{d}$	4	$a\overline{b}\overline{c}d$	4	$a\overline{b}\overline{c}\overline{d}$	5
$\overline{a}bcd$	1	$\overline{a}bc\overline{d}$	1	$\overline{a}b\overline{c}d$	0	$\overline{a}b\overline{c}\overline{d}$	0
$\overline{a}\overline{b}cd$	2	$\overline{a}\overline{b}c\overline{d}$	3	$\overline{a}\overline{b}\overline{c}d$	3	$\overline{a}\overline{b}\overline{c}\overline{d}$	4

Table 2: The constructed OCF κ for Example 5

$$= \kappa_2(\omega^1\omega^2) + \min_{\omega^3}(\kappa_3(\omega^1\omega^3)) - \kappa_2(\omega^1)$$
$$= \kappa_2(\omega^1\omega^2) + \kappa_3(\omega^1) - \kappa_3(\omega^1) = \kappa_2(\omega^1\omega^2). \qquad \square$$

We illustrate this construction by an example.

Example 5. *Let $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$ with $\Sigma = \{a,b,c,d\}, \Sigma_1 = \{a,b\}, \Sigma_2 = \{c\}, \Sigma_3 = \{d\}$. With ω^{12}, ω^{13}, we denote worlds from $\Omega(\Sigma_1\dot{\cup}\Sigma_2)$ resp. $\Omega(\Sigma_1\dot{\cup}\Sigma_3)$, where ω^1 is their common part over Σ_1. Let κ_2, κ_3 over $\Sigma_1\dot{\cup}\Sigma_2$ resp. $\Sigma_1\dot{\cup}\Sigma_3$ be defined as in Table 1. We can easily observe that the compatibility condition $\kappa_2(\omega^1) = \kappa_3(\omega^1)$ is satisfied. The constructed OCF $\kappa(\omega) = \kappa_2(\omega^1\omega^2) + \kappa_3(\omega^1\omega^3) - \kappa_2(\omega^1)$ is shown in Table 2.*

We might also use this construction for TPOs that are sufficiently compatible. More precisely, suppose we have $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$ and two TPOs $\preceq_2, \preceq_3$ on $\Sigma_1\dot{\cup}\Sigma_2$ resp. $\Sigma_1\dot{\cup}\Sigma_3$ are given such that we can associate with them suitable OCFs κ_2, κ_3 over the respective subsignatures satisfying $\preceq_{\kappa_2}=\preceq_2$, $\preceq_{\kappa_3}=\preceq_3$ and obeying the compatibility condition $\kappa_2(\omega^1) = \kappa_3(\omega^1)$ for any $\omega^1 \in \Omega(\Sigma_1)$. Then κ constructed as in Proposition 5 is an OCF, and Propositions 5 and 1 ensure that $\Sigma_2 \perp\!\!\!\perp_{\preceq_\kappa} \Sigma_3|\Sigma_1$ holds.

At this point, the reader might wonder where conditional independencies come from. There are at least three sources where a knowledge engineer might look for conditional independencies. The first is background knowledge about the domain that being modelled, not unlike Bayesian networks, which are often manually constructed on the basis of background assumptions about the domain being modelled. A second source of conditional independencies is combining a set of component OCFs over different (possibly overlapping) sub-signatures as outlined in Proposition 5. Thirdly, we plan to develop algorithms for the extraction of conditional independencies between sub-signatures, taking inspiration of the learning of the structure of Bayesian networks [Pearl, 1988, Ch. 8].

5 Conditional Independence Axioms for Iterated Revision

The basic idea of axioms of conditional independence for iterated belief revision would be that conditional independencies should be respected under revision, as far as possible. This means that if a conditional independence prevails in the prior epistemic state, and the new information is compatible with this independence in the sense that it does not contain atoms from both conditionally independent sets (see also [Lynn *et al.*, 2022]), then also the posterior epistemic state should show that same conditional independence.

(CI^{tpo}) Let $*$ be a revision operator for TPOs, let $\preceq$ be a TPO over Ω. Let $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$. If $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3|\Sigma_1$ and $A \in \mathcal{L}(\Sigma_1\dot{\cup}\Sigma_2)$, then $\Sigma_2 \perp\!\!\!\perp_{\preceq * A} \Sigma_3|\Sigma_1$.

(CI^{ocf}) Let $*$ be a revision operator for OCFs, let κ be an OCF over Ω. Let $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$. If $\Sigma_2 \perp\!\!\!\perp_{\kappa} \Sigma_3|\Sigma_1$ and $A \in \mathcal{L}(\Sigma_1\dot{\cup}\Sigma_2)$, then $\Sigma_2 \perp\!\!\!\perp_{\kappa * A} \Sigma_3|\Sigma_1$.

Proposition 6. *C-revisions satisfy (CI^{ocf}).*

Proof Sketch. Let $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$, and let κ be an OCF over Ω such that $\Sigma_2 \perp\!\!\!\perp_{\kappa} \Sigma_3|\Sigma_1$ holds. Let $A \in \mathcal{L}(\Sigma_1\dot{\cup}\Sigma_2)$. We have to show that then $\Sigma_2 \perp\!\!\!\perp_{\kappa^*} \Sigma_3|\Sigma_1$ also holds, where $\kappa^* = \kappa * A$ is a c-revision of κ by A according to Definition 1. In other words, we have to show that $\kappa^*(\omega^1\omega^2\omega^3) = \kappa^*(\omega^1\omega^2) + \kappa^*(\omega^1\omega^3) - \kappa^*(\omega^1)$.

We outline the proof of the case for $\omega \models A$, the case for $\omega \models \overline{A}$ is similar. The claim is shown by first deriving the following equalities: $\kappa^*(\omega) = -\kappa(A) + \kappa(\omega^1\omega^2) + \kappa(\omega^1\omega^3) - \kappa(\omega^1)$ (I), $\kappa^*(\omega^1\omega^3) = -\kappa(A) + \min\{\kappa(A\omega^1), \kappa(\overline{A}\omega^1) + \eta\} + \kappa(\omega^1\omega^3) - \kappa(\omega^1)$ (II), and $\kappa^*(\omega^1) = -\kappa(A) + \min\{\kappa(A\omega^1), \kappa(\overline{A}\omega^1) + \eta\}$ (III). Combining (I), (II) and (III) then yields $\kappa^*(\omega^1\omega^2\omega^3) = \kappa^*(\omega^1\omega^2) + \kappa^*(\omega^1\omega^3) - \kappa^*(\omega^1)$ as desired. $\square$

We illustrate this result by an example.

Example 6. *We have $\Sigma = \Sigma_1\dot{\cup}\Sigma_2\dot{\cup}\Sigma_3$ with $\Sigma = \{a,b,c,d\}, \Sigma_1 = \{a,b\}, \Sigma_2 = \{c\}, \Sigma_3 = \{d\}$, and κ is given by Table. 3 with $\Sigma_2 \perp\!\!\!\perp_{\kappa} \Sigma_3|\Sigma_1$ (see also Example 5) and Bel $(\kappa) = Cn(\overline{a}b\overline{c})$, i.e., $K_\kappa = \overline{a}b\overline{c}$. We revise by $A = bc$. We have $\kappa(A) = 1$, so any c-revision κ^* of κ by A has the form*

$$\kappa * A(\omega) = -1 + \kappa(\omega) + \begin{cases} \eta & \text{if } \omega \models \overline{b} \vee \overline{c}, \\ 0 & \text{otherwise.} \end{cases}$$

We illustrate this with κ and a generic c-revision κ^* with $\eta > \kappa(A) - \kappa(\overline{A}) = 1$ in Table 3. It is straightforward to verify that indeed, Σ_2, Σ_3 are also conditionally independent given Σ_1 in κ^*. Choosing $\eta = 2$ yields an OCF κ^* whose induced TPO $\preceq_{\kappa^*}$ coincides with $\preceq$ given in Example 2.

It is also interesting to note that in Example 6, all c-revisions as shown in Table 3 with $\eta > 2$ would yield the same total preorder. This means that in spite of the numerical nature of OCFs, the purely qualitative result of the revision,

ω	$\kappa(\omega)$	$\kappa^*(\omega)$	ω	$\kappa(\omega)$	$\kappa^*(\omega)$
$abcd$	2	1	$\overline{a}bcd$	1	0
$abc\overline{d}$	1	0	$\overline{a}bc\overline{d}$	1	0
$ab\overline{c}d$	3	$2+\eta$	$\overline{a}b\overline{c}d$	0	$-1+\eta$
$ab\overline{c}\overline{d}$	2	$1+\eta$	$\overline{a}b\overline{c}\overline{d}$	0	$-1+\eta$
$a\overline{b}cd$	3	$2+\eta$	$\overline{a}\overline{b}cd$	2	$1+\eta$
$a\overline{b}c\overline{d}$	4	$3+\eta$	$\overline{a}\overline{b}c\overline{d}$	3	$2+\eta$
$a\overline{b}\overline{c}d$	4	$3+\eta$	$\overline{a}\overline{b}\overline{c}d$	3	$2+\eta$
$a\overline{b}\overline{c}\overline{d}$	5	$4+\eta$	$\overline{a}\overline{b}\overline{c}\overline{d}$	4	$3+\eta$

Table 3: The (revised) OCF κ for Example 6

ω	$\kappa(\omega)$	$\kappa^*(\omega)$	ω	$\kappa(\omega)$	$\kappa^*(\omega)$
pqr	1	0	$\overline{p}qr$	1	2
$pq\overline{r}$	2	1	$\overline{p}q\overline{r}$	3	4
$p\overline{q}r$	0	1	$\overline{p}\,\overline{q}r$	0	1
$p\overline{q}\,\overline{r}$	1	2	$\overline{p}\,\overline{q}\,\overline{r}$	2	3

Table 4: The (revised) OCF κ for Example 7

i.e., the TPO $\preceq_{\kappa * A}$, does not depend so much on the chosen impact factor η, but more on the qualitative structure in the prior epistemic state. Therefore, we can very well use the power of OCF-based c-revisions also for TPOs to obtain revisions that satisfy (CI^{tpo}), if suitable transformations into OCFs are possible. More precisely, if $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3|\Sigma_1$ holds in a TPO $\preceq$, and we can find an OCF κ with $\preceq_\kappa=\preceq$ and $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3|\Sigma_1$, then we can apply c-revisions to this κ such that also $\Sigma_2 \perp\!\!\!\perp_{\kappa^*} \Sigma_3|\Sigma_1$ holds, and via $\preceq_{\kappa^*}$ we obtain a revision of $\preceq$ that satisfies (CI^{tpo}) thanks to Propositions 6 and 1. We illustrate this by continuing Examples 1 and 3.

Example 7. *The TPO $\preceq$ from Example 1 is defined over the signature $\Sigma = \{p, q, r\}$, has the belief set $Bel(\preceq) = Cn(\overline{q}r)$, and satisfies $\{q\} \perp\!\!\!\perp_{\preceq} \{r\}|\{p\}$, i.e., $\Sigma_1 = \{p\}, \Sigma_2 = \{q\}, \Sigma_3 = \{r\}$. We want to revise this TPO by $A = pq \in \mathcal{L}(\Sigma_1\dot\cup\Sigma_2)$ with the help of c-revisions. The minimal OCF κ implementing this TPO has been computed in Example 3 and is shown in Table 4. According to Proposition 2 and Example 3, we have $\{q\} \perp\!\!\!\perp_\kappa \{r\}|\{p\}$, and revising κ by A via c-revisions yields $\kappa^* = \kappa * A$ with*

$$\kappa * A(\omega)(\omega) = -1 + \kappa(\omega) + \begin{cases} \eta & \text{if } \omega \models \overline{A}, \\ 0 & \text{otherwise} \end{cases}$$

with $\eta > 1 - 0$. We choose $\eta = 2$ and obtain the revised κ^* shown in Table 4 satisfying $\{q\} \perp\!\!\!\perp_{\kappa^*} \{r\}|\{p\}$ according to Proposition 6. Then a revision of $\preceq$ by A can be defined via $\preceq * A =\preceq_{\kappa * A}$ and is given by

$$\begin{aligned} \preceq^* = \preceq *A: \quad pqr \quad &\prec^* \quad pq\overline{r}, p\overline{q}r, \overline{p}\,\overline{q}r \prec^* p\overline{q}\,\overline{r}, \overline{p}qr \\ &\prec^* \quad \overline{p}\,\overline{q}\,\overline{r} \prec^* \overline{p}q\overline{r} \end{aligned}$$

Thanks to Proposition 1, we also have $\{q\} \perp\!\!\!\perp_{\preceq * A} \{r\}|\{p\}$.

Finally, we show some partial agreement with the results from the paper [Lynn *et al.*, 2022] that focuses on propositional AGM revision; the proof is straightforward and left out due to lack of space.

Proposition 7. *Let $*$ be a revision operator for TPOs, let $\Sigma = \Sigma_1\dot\cup\Sigma_2\dot\cup\Sigma_3$. Let $\preceq$ be a TPO over Ω such that $\Sigma_2 \perp\!\!\!\perp_{\preceq} \Sigma_3|\Sigma_1$ holds. Let $A \in \mathcal{L}(\Sigma_1\dot\cup\Sigma_2)$ such that $A \wedge K_{\preceq}^{\Sigma_1,\Sigma_3}$ is consistent. If $*$ satisfies (CI^{tpo}), then $K_{\preceq * A}^{\Sigma_1,\Sigma_2} \wedge K_{\preceq}^{\Sigma_1,\Sigma_3} \models K_{\preceq * A}$.*

Note that full compliance of a (propositional) revision operator $*$ with a conditional independence statement modulo formulas in [Lynn *et al.*, 2022] postulates $K_{\preceq * A}^{\Sigma_1,\Sigma_2} \wedge K_{\preceq}^{\Sigma_1,\Sigma_3} \equiv K_{\preceq * A}$. This seems to be a bit too strong for iterated revision, as also Example 6 illustrates. Here we have $K_\kappa \equiv \overline{a}b\overline{c}$ and $A = bc$, so $A \wedge K_\kappa^{\Sigma_1,\Sigma_3} \equiv bc \wedge \overline{a}b \equiv \overline{a}bc$ is consistent, and $\Sigma_2 \perp\!\!\!\perp_\kappa \Sigma_3|\Sigma_1$ and $\Sigma_2 \perp\!\!\!\perp_{\kappa^*} \Sigma_3|\Sigma_1$ imply conditional independence modulo the respective belief sets according to Prop. 3. But $K_{\kappa^*} \equiv abc\overline{d} \vee \overline{a}bc$, hence we only have $K_{\kappa^*}^{\Sigma_1,\Sigma_2} \wedge K_\kappa^{\Sigma_1,\Sigma_3} \equiv (abc \vee \overline{a}bc) \wedge \overline{a}b \equiv \overline{a}bc \models K_{\kappa^*}$, verifying Prop. 7.

6 Conclusion

We presented a semantic approach to conditional independence for total preorders which are the basic structure for revision of epistemic states resp. iterated revision according to the AGM and Darwiche-Pearl frameworks [Alchourrón *et al.*, 1985; Darwiche and Pearl, 1997]. We showed that this concept of conditional independence is compatible to respective notions for purely propositional frameworks, Spohn's ranking functions, and some of Pearl's basic characteristics of conditional independence coming from probabilistics. Moreover, we made precise by axioms what it means for an iterated revision operator to respect conditional independence, both for total preorders and ranking functions, which basically say that conditional independencies should be preserved under revision if the revision input does not violate it. We proposed c-revision operators [Kern-Isberner and Huvermann, 2017] as a proof of concept to satisfy these axioms. Therefore, for the framework of ranking functions, conditional independence in the revised epistemic state can be easily implemented via c-revisions: once we have conditional independence in the prior ranking function, any c-revision (with a suitable input) will preserve it. We also provided a constructive method to build up a prior ranking function from local ranking functions (on subsignatures) that satisfies a conditional independence statement, thereby bringing in ideas of local reasoning. Via the close relationships between total preorders and ranking functions, many of these results can be used for the purely qualitative framework of total preorders. This paper complements the work [Kern-Isberner and Brewka, 2017] that deals with *independence over marginalizations*, whereas this paper deals with *independence over conditionalization*.

As part of our future work, we plan to elaborate on the construction of total preorders from local ones that are compatible with conditional independence statements, and we will extend our considerations to also include revisions by conditionals. Furthermore, we plan to take advantage of the notion of conditional independence to design efficient network-based models for iterated revision, and to show how complexity of revision tasks can be reduced effectively in these models.

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