Flaws of Termination and Optimality in ADOPT-based Algorithms

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Abstract

A distributed constraint optimization problem (DCOP) is a framework to model multi-agent coordination problems. Asynchronous distributed optimization (ADOPT) is a well-known complete DCOP algorithm, and owing to its superior characteristics, many variants have been proposed over the last decade. It is considered proven that ADOPT-based algorithms have the key properties of termination and optimality, which guarantee that the algorithms terminate in a finite time and obtain an optimal solution, respectively. In this paper, we present counterexamples to the termination and optimality of ADOPT-based algorithms. The flaws are classified into three types, at least one of which exists in each of ADOPT and seven of its variants that we analyzed. In other words, the algorithms may potentially not terminate or terminate with a suboptimal solution. We also propose an amended version of ADOPT that avoids the flaws in existing algorithms and prove that it has the properties of termination and optimality.

1 Introduction

Distributed constraint optimization problems (DCOPs) [Modi *et al.*, 2005; Weiss, 2013; Fioretto *et al.*, 2018] involve agents in a network that must coordinate the values of their variables. Many multi-agent coordination problems, such as meeting scheduling [Maheswaran *et al.*, 2004], sensor network operation [Jain *et al.*, 2009], and disaster management problems [Lass *et al.*, 2008], are modeled as DCOPs. Owing to this wide range of applications, various algorithms to solve DCOPs have been proposed over the decades.

Asynchronous distributed optimization (ADOPT) [Modi *et al.*, 2005] is a well-known DCOP algorithm. The major advantages of ADOPT are asynchrony and completeness. Asynchrony allows agents to process their computations without waiting for other agents, which enables the algorithm to use agent resources efficiently, while completeness guarantees that the algorithm finds an optimal solution. Although complete algorithms require exponential time or space, they are still desirable, especially from a theoretical point of view.

Many researchers have worked on improving the performance of ADOPT. While the original version could handle only binary constraints, the extension ADOPT-N [Pecora *et al.*, 2006] is able to deal with n-ary ones. ADOPT$^+$ [Gutierrez and Meseguer, 2010] saves redundant messages exchanged in ADOPT. IDB-ADOPT [Yeoh *et al.*, 2009] executes ADOPT iteratively to change the search strategy from best-first to depth-first. The same authors of IDB-ADOPT proposed BnB-ADOPT [Yeoh *et al.*, 2010], which employs depth-first branch-and-bound search. BnB-ADOPT$^+$ [Gutierrez and Meseguer, 2010] reduces the number of redundant messages in BnB-ADOPT. Other variants include ADOPT(k) [Gutierrez *et al.*, 2011], which generalizes ADOPT and BnB-ADOPT, and BD-ADOPT [Chen *et al.*, 2018], which combines best-first and depth-first search. ADOPT-ing [Silaghi and Yokoo, 2009] introduces nogood, which is used in asynchronous backtrack (ABT) [Yokoo *et al.*, 1998] to solve distributed constraint satisfaction problems.

For a DCOP algorithm to be complete, it must have two important properties: termination and optimality. Termination guarantees that an algorithm terminates in a finite time, while optimality guarantees that an optimal solution is obtained at termination. Each study of ADOPT and its variant algorithms claims that the algorithm has both of these properties. However, many of the theorems in these studies are invalid or inappropriate to derive termination or optimality.

The objectives of this paper are threefold: to show the flaws in ADOPT and its variant algorithms, to propose an amended version of ADOPT, and to prove its termination and optimality. First, we provide counterexamples to the termination and optimality of ADOPT and show that similar flaws also exist in many of the variants. Table 1 summarizes the flaws in ADOPT and seven of its variants. They are classified into three types by their properties and causes: failure of termination, failure of optimality caused by algorithm initialization, and failure of optimality caused by messages sent at termination. As the table shows, at least one flaw exists in each of the ADOPT-based algorithms that we analyzed. In particular, ADOPT itself has all three flaws, that is, it may potentially not terminate or terminate with a suboptimal solution. Moreover, most of the variants have the third type of flaw.

Second, we propose three amendments to ADOPT, each of which corresponds to a type of flaw. The first amendment

Flaw	ADOPT	N	+	IDB	BnB	BnB+	(k)	BD
(1)	✓	✓	✓	-	-	-	-	-
(2)	✓	✓	✓	✓	-	-	-	-
(3)	✓	✓	-	✓	✓	✓	✓	✓

Table 1: Flaws in ADOPT and seven variant algorithms, where N, +, IDB, BnB, BnB+, (k), and BD stand for ADOPT-N, ADOPT$^+$, IDB-ADOPT, BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT, respectively. The flaws are classified into three types: (1) failure of termination; (2) failure of optimality caused by initialization; (3) failure of optimality caused by TERMINATE messages.

modifies the update rules for lower and upper bounds since the counterexample to termination is caused by the nonmonotonicity of the bounds. The second deals with the initialization of a current context. Additionally, the third modifies termination messages and their receiving procedure.

Finally, we prove the termination and optimality properties of our amended version of ADOPT.

This paper is organized as follows. Section 2 defines DCOPs and summarizes ADOPT. Section 3 presents the flaws of ADOPT and its variants in detail. Section 4 describes the amended version of ADOPT, and Section 5 proves its termination and optimality. Finally, Section 6 concludes this work.

2 Preliminaries

In this section, we define DCOPs and provide a summary of ADOPT.

2.1 DCOP

A DCOP is defined as a tuple $\langle A, X, D, F\rangle$. $A = \{a_1, \ldots, a_n\}$ is a finite set of agents, and $X = \{x_1, \ldots, x_n\}$ is a finite set of variables, where x_i is the variable assigned to agent a_i. An agent can control only the value of the variable assigned to it. Following the studies of ADOPT and its variants, we assume that each agent a_i has only one variable, denoted by x_i. Additionally, we use the terms "agent" and "variable" interchangeably. Here, $D = \{D_1, \ldots, D_n\}$ is a set of domains of variables, where D_i is the finite domain of variable x_i. Furthermore, F is a finite set of binary cost functions. A cost function for two variables x_i and x_j is defined as $f_{i,j} : D_i \times D_j \to \mathbb{N}$. Each cost function is known only to the agents involved. Agents aim to find an assignment $\boldsymbol{d}^*$ for all variables such that it minimizes the global objective function, which is the sum of the cost functions in F. This assignment is a solution to a DCOP and is expressed as follows:

$$\boldsymbol{d}^* := \arg \min_{\boldsymbol{d} \in \prod_k D_k} \sum_{f_{i,j} \in F} f_{i,j}(d_i, d_j).$$

A DCOP can be expressed as a constraint graph. The nodes correspond to variables in a DCOP, and each edge indicates that the two connected variables share a cost function. A depth-first search (DFS) pseudo-tree extends a spanning tree of a constraint graph. Here, the edges are classified into tree edges and back edges. Tree edges exist in the spanning tree and establish parent/child relationships, while back edges do not exist in the spanning tree but exist in the constraint graph and establish pseudo-parent/pseudo-child relationships.

2.2 ADOPT

ADOPT [Modi *et al.*, 2005] is an asynchronous and complete algorithm for solving DCOPs. In ADOPT, agents are arranged in a DFS pseudo-tree and exchange four types of messages, namely VALUE, COST, THRESHOLD, and TERMINATE. Agents update their variable values and compute the bounds of costs through messages. Here, we only introduce some notations used in subsequent sections and review the properties of termination and optimality in the original study. A detailed description of the algorithm is presented in Appendix A[1].

The notations concerning the states of agent x_i in this study are as follows: d_i denotes the current value of x_i; CX_i denotes the current context of x_i; $\delta_i(d, CX)$ denotes the local cost of x_i when x_i takes value d and its higher neighbors (i.e., its parent and pseudo-parents) take the values given in context CX; $LB_i(d)$ and $UB_i(d)$ denote the lower and upper bounds of x_i for value d; LB_i and UB_i denote the lower and upper bounds of x_i; TH_i denotes the threshold of x_i; $cx_i(d, x_c), lb_i(d, x_c)$, and $ub_i(d, x_c)$ denote the context, the lower bound, and the upper bound, respectively, stored by x_i when it receives a COST message in which the context contains assignment (x_i, d) from child x_c; $th_i(d, x_c)$ denotes the threshold allocated to child x_c when x_i takes value d. Additionally, $\gamma_i(d, CX)$ and $\gamma_i(CX)$ denote the optimal costs for the subtree rooted at x_i given context CX.

The original study of ADOPT provides three properties (as theorems in the paper) in terms of its termination and optimality; Property 2 implies the termination, while Property 3 implies the optimality.

Property 1. $\forall x_i \in X, LB_i \leq \gamma_i(CX_i) \leq UB_i$.

Property 2. *$\forall x_i \in X$, if the current context CX_i is fixed, then $TH_i = UB_i$ will eventually occur.*

Property 3. *$\forall x_i \in X$, x_i's final threshold value TH_i is equal to $\gamma_i(CX_i)$.*

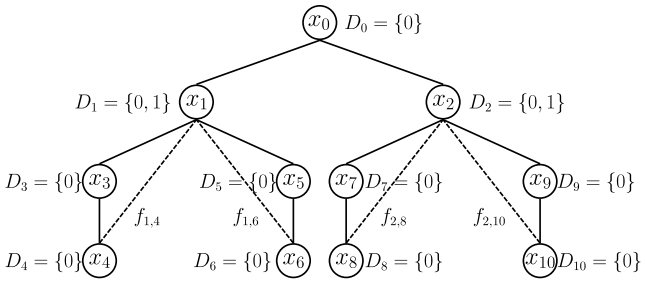
There are counterexamples to these properties. We describe them in the next section.

3 Flaws in ADOPT-based Algorithms

We present counterexamples to the theorems for termination and optimality in the original study of ADOPT, which means that the proofs presented in the study are incorrect. We also show that similar flaws exist in seven variant algorithms: ADOPT-N, ADOPT$^+$, IDB-ADOPT, BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT. These flaws exist because the studies attempt to prove the desired properties relying on ADOPT's invalid theorems or introduce theorems that are not strong enough to derive the properties.

As mentioned above, counterexamples are classified into three types by their properties and causes. The first is the counterexample to termination, described in Section 3.1,

[1]Appendices and other supplemental materials are available at https://mas.cs.tsukuba.ac.jp/~noshiro.



(a) DFS pseudo-tree

d_1	d_4	$f_{1,4}$
0	0	5
1	0	6

d_1	d_6	$f_{1,6}$
0	0	2
1	0	3

d_2	d_8	$f_{2,8}$
0	0	5
1	0	6

d_2	d_{10}	$f_{2,10}$
0	0	2
1	0	3

(b) Cost functions

Figure 1: DFS pseudo-tree and cost functions of a DCOP for the counterexample to termination. The solid lines indicate parent/child relationships, and the dashed lines indicate pseudo-parent/pseudo-child relationships.

which is caused by the nonmonotonicity of the bounds. The second and third are counterexamples to optimality, described in Sections 3.2 and 3.3, respectively; the second is due to initialization, while the third is due to TERMINATE messages.

In these counterexamples, we make some assumptions in terms of message transfer. First, a finite random delay exists between sending a message and receiving it. Second, messages exchanged between a pair of agents are received in the order they were sent, while an agent receives messages from different agents in any order. Finally, agents send messages after processing all received messages, which is called a cycle [Modi *et al.*, 2005]. These assumptions are adopted in the studies of ADOPT and its variants.

In this section, we show the counterexamples with several figures that indicate crucial points. More detailed traces are presented in Appendix B.

3.1 Counterexample to Termination

Figure 1 shows the DFS pseudo-tree and cost functions of a DCOP in which the counterexample to termination occurs. The cost functions not specified in Figure 1(b) are defined as the constant functions whose values are 0. In the DCOP, only two agents x_1 and x_2 have the domain $\{0, 1\}$, and the other agents always take the same value, 0.

Trace of the Counterexample

In the trace of this counterexample, agents repeat the transition of their states; in particular, agents x_1 and x_2 change their variable values infinitely, which results in nontermination and contradicts Property 2 in the original study of ADOPT. Here, the states of some agents at the initial step of the iteration (and the trace itself) are shown in Figure 2.

In the following description, we only focus on the variable values, contexts, lower bounds, and thresholds, which are directly related to this trace, and ignore the rest unless otherwise noted.

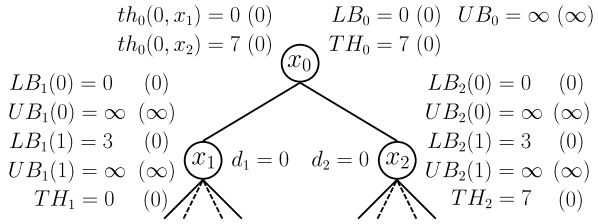


Figure 2: States of agents at the initial step of the iteration in the counterexample to termination. The states in Step 0 are shown inside parentheses.

Step 0. As the initial state, we assume that agents x_1 and x_2 set their values (i.e., d_1 and d_2) to 0. In the initialization, the thresholds allocated to the children of x_0 (i.e., $th_0(0, x_1)$ and $th_0(0, x_2)$) are set to 0; the lower bounds of x_1 and x_2 (i.e., $LB_1(0), LB_1(1), LB_2(0)$, and $LB_2(1)$) are set to 0; and the thresholds of x_1 and x_2 (i.e., TH_1 and TH_2) are set to 0. The states of agents at this point are presented inside the parentheses in Figure 2.

Step 1. Let us begin with the process in the subtree rooted at x_1. First, x_1 sends VALUE messages to its children and pseudo-children x_3, x_4, x_5, and x_6. Afterwards, x_4 receives the VALUE message and computes the lower bound using the updated context $CX_4 \ni (x_1, 0)$, and thus $LB_4 = 5$. Then x_4 reports the lower bound to x_3 by sending a COST message. Subsequently, x_3 receives the VALUE message from x_1 and the COST message from x_4 and computes the lower bound as $LB_3 = 5$. Then x_3 sends a COST message with $CX_3 \ni (x_1, 0)$ and $LB_3 = 5$ to x_1. After x_1 receives the COST message, x_1 updates $LB_1(0)$ to 5. Since $LB_1(0) = 5 > TH_1 = 0$, x_1 changes its value d_1 to 1 and sends VALUE messages to its children and pseudo-children.

Step 2. Next, x_5 receives the VALUE messages from x_1 and sends a COST message to x_1, but this COST message does not affect the bounds of x_1 because $LB_5 = 0$ and $UB_5 = \infty$, which are the initial bounds. This message is necessary for sending the reinitialized bounds of x_5 to x_1 in the second or subsequent iteration. Similar to the procedure in Step 1, x_1, x_3, and x_4 receive and send messages, and then x_1 updates the lower bound as $LB_1(1) = 6$. Additionally, the threshold of x_1 increases as $TH_1 = LB_1 = \min\{LB_1(0), LB_1(1)\} = 5$ because of Threshold-Invariant (for more information on the invariants concerning thresholds, see Appendix A or [Modi *et al.*, 2005]). Since $LB_1(1) = 6 > TH_1 = 5$, x_1 changes its value d_1 back to 0.

Step 3. Subsequently, similar cost calculations and value changes are performed between x_1, x_5, and x_6, and the results are summarized as follows. First, x_1 computes the lower bound as $LB_1(0) = lb_1(0, x_3) + lb_1(0, x_5) = 5 + 2 = 7$ and then changes its value d_1 from 0 to 1. Next, x_1 also computes the lower bound as $LB_1(1) = lb_1(1, x_3) + lb_1(1, x_5) = 6 + 3 = 9$, and then the value d_1 returns to 0. After the cost calculation, x_1 sends VALUE messages with the current value $d_1 = 0$ to the lower neighbors and a COST message with $LB_1 = 7$ to x_0. Afterwards, x_0 receives the COST message and then increases LB_0, TH_0, and $th_0(0, x_1)$

to 7 (the thresholds change due to ThresholdInvariant and ChildThresholdInvariant). As the result of processes from Step 0 to Step 3, the current contexts of the lower neighbors of x_1 (i.e., x_3, x_4, x_5, and x_6) contain assignment $(x_1, 1)$. Note that the lower neighbors of x_1 still do not receive the last VALUE messages with $d_1 = 0$ from x_1; in particular, x_3 and x_4 still do not receive the VALUE messages that x_1 sent during the processes of x_5 and x_6 after x_3 had sent a COST message to x_1 in Step 2.

Step 4. Here, x_3 receives the three VALUE messages from x_1, in which the values are $d_1 = 0, d_1 = 1$, and $d_1 = 0$, in the order of sending. When x_3 processes the first VALUE message, the lower bound of x_3 is reinitialized as $LB_3 = LB_3(0) = lb_3(0, x_4) = 0$ since context $cx_3(0, x_4) \ni (x_1, 1)$, received from x_4 through the COST message, is incompatible with the updated context $CX_3 \ni (x_1, 0)$. After x_3 processes the remaining messages, x_3 sends two types of COST messages to x_1, i.e., the message with $LB_3 = 0$ and $CX_3 = \{(x_1, 0)\}$ and the message with $LB_3 = 0$ and $CX_3 = \{(x_1, 1)\}$. Similarly, x_5 receives the VALUE message with $d_1 = 0$ from x_1 and reinitializes the bounds. Additionally, x_5 sends a COST message with $LB_5 = 0$ and $CX_5 = \{(x_1, 0)\}$ to x_1.

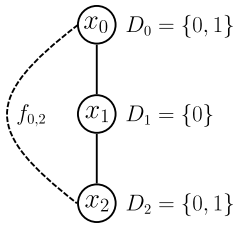
Step 5. Next, x_1 receives the COST messages from x_3 and x_5 and updates the lower bounds as $LB_1(0) = lb_1(0, x_3) + lb_1(0, x_5) = 0$, $LB_1(1) = lb_1(1, x_3) + lb_1(1, x_5) = 3$. Thus, x_1 obtains the lower bound as $LB_1 = 0$ and keeps the value $d_1 = 0$. After x_1 sends a COST message with $LB_1 = 0$ to x_0, x_0 receives it and updates the lower bound as $LB_0 = LB_0(0) = lb_0(0, x_1) + lb_0(0, x_2) = 0$. Although LB_0 decreases, the thresholds for the children are kept as $th_0(0, x_1) = 7$ and $th_0(0, x_2) = 0$.

Step 6. After the procedures are completed in the subtree rooted at x_1, the same process is performed in the subtree rooted at x_2. The states of agents after performing the process are presented in Figure 2 (outside the parentheses). In this process, x_2 changes its value d_2 repeatedly, and eventually the lower bounds of x_2 are obtained as $LB_2(0) = 0$ and $LB_2(1) = 3$, and thus $LB_2 = 0$. Here, the thresholds of x_0 are crucial. Since x_2 once increased LB_2 to 7, x_0 also increased $th_0(0, x_2)$ to 7. However, the threshold TH_0 has not been changed from 7 because $LB_0 = lb_0(0, x_1) + lb_0(0, x_2) = 0 + 7 = 7$. Thus, x_0 decreases $th_0(0, x_1)$ to 0 by AllocationInvariant. Hence, x_0 does not satisfy the termination condition since $TH_0 = 7 < UB_0 = \infty$ (as x_0 received the reinitialized upper bound $UB_1 = \infty$ from x_1). Thus, no agent terminates at this point.

Subsequent Steps. Afterwards, the subtrees rooted at x_1 and x_2 alternately repeat the same process as described from Step 1 to Step 6. This causes x_1 and x_2 to change their values (i.e., d_1 and d_2) infinitely; ADOPT never terminates because the root agent x_0 never satisfies the termination condition. This result contradicts Property 2.

Cause of the Counterexample

The cause of this counterexample is the nonmonotonicity of lower bounds. In the trace demonstrated above, the lower bounds of x_1 and x_2 (i.e., LB_1 and LB_2) increase from 0 to 7 and then decrease to 0. This transition causes x_0 to repeatedly change the thresholds allocated to x_1 and x_2 (i.e., $th_0(0, x_1)$ and $th_0(0, x_2)$) without altering TH_0. Thus, x_1 and x_2 keep changing their values, and x_0 does not satisfy the termination condition.



(a) DFS pseudo-tree

d_0	d_2	$f_{0,2}$
0	0	100
0	1	100
1	0	100
1	1	0

(b) Cost function

Figure 3: DFS pseudo-tree and the cost function of a DCOP for the counterexample to optimality caused by initialization.

Occurrence in Variants

Similar counterexamples to termination appear in the variants ADOPT-N and ADOPT$^+$. ADOPT-N is equivalent to the original ADOPT if the given DCOP does not contain n-ary $(n > 2)$ cost functions. Since the problem consists of only binary functions, ADOPT-N has this flaw. ADOPT$^+$ is a variant of ADOPT that saves its redundant messages. Although the trace above contains redundant messages, a similar trace can be performed without such messages. Therefore, this flaw can appear in ADOPT$^+$.

On the other hand, the counterexample does not occur in IDB-ADOPT. The algorithm repeats the execution of ADOPT, decreasing the initial threshold of the root agent, and eventually, the initial threshold becomes less than the optimal cost by 1. In the DCOP shown in Figure 1, the initial threshold of the root agent at the final execution is 13, and hence the threshold of x_1 or x_2 must be equal to or greater than 7. Therefore, the agents cannot change their variable values repeatedly, that is, the counterexample does not appear in IDB-ADOPT. Additionally, BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT do not have the flaw because the bounds are updated monotonically in these algorithms.

3.2 Counterexample to Optimality Caused by Initialization

The DFS pseudo-tree and cost function of a DCOP for the second counterexample are shown in Figure 3. The domains of agents x_0 and x_2 are $\{0, 1\}$, and that of agent x_1 is $\{0\}$. The cost between x_0 and x_2 is 0 when the two variable values are 1, otherwise, it is 100. Similar to the DCOP for the counterexample to termination, the other cost functions are the constant functions whose values are 0. Thus, the optimal solution of this DCOP is $(x_0, x_1, x_2) = (1, 0, 1)$, whose cost is 0.

Trace of the Counterexample

In this counterexample, a delay of a VALUE message causes termination with a suboptimal solution. The states of agents when x_0 terminates (in Step 2) are shown in Figure 4.

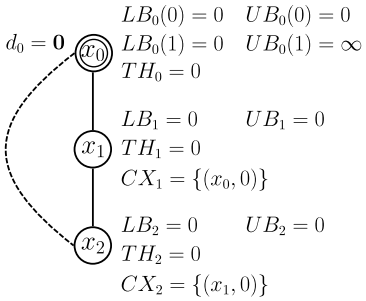


Figure 4: States of agents in Step 2 in the counterexample to optimality caused by initialization. The terminated agent is indicated with a double circle.

Step 0. We assume that all agents set their variable values to 0 in the initialization. At this point, their current contexts are initialized to be empty.

Step 1. First, agents send VALUE messages to their children and pseudo-children. Then x_2 receives the VALUE message only from its parent x_1, which means that the messages from the pseudo-parent x_0 are delayed. Here, x_2 updates the current context as $CX_2 = \{(x_1, 0)\}$. From the definition of the local cost $\delta_i(d, CX)$ (see Appendix A), x_2 computes the local costs as $\delta_2(0, \{(x_1, 0)\}) = f_{1,2}(0, 0) = 0$ and $\delta_2(1, \{(x_1, 0)\}) = f_{1,2}(0, 1) = 0$. Thus, the bounds of x_2 are obtained as $LB_2 = UB_2 = 0$. Then x_2 sends a COST message with $LB_2 = UB_2 = 0$ to x_1.

Step 2. After x_1 receives the VALUE message from x_0 and the COST message from x_2, x_1 computes the bounds as $LB_1 = UB_1 = 0$. Similarly, x_0 updates the bounds as $LB_0(0) = UB_0(0) = 0$ through the COST message sent from x_1. Since $LB_0 = TH_0 = UB_0 = UB_0(0) = 0$ due to ThresholdInvariant, x_0 keeps its value as $d_0 = 0$ and satisfies the termination condition. However, the variable value $d_0 = 0$ is suboptimal. This is a contradiction to Property 3.

Cause of the Counterexample

This counterexample occurs due to the initialization of the current context CX_i of agent x_i. Since CX_i is initialized as an empty set, it may not contain the assignments of some higher neighbors, resulting in an underestimation of the local cost $\delta_i(d, CX)$ and the bounds of x_i's ancestors. This fact contradicts Property 1. Moreover, the underestimation results in premature termination.

Occurrence in Variants

This counterexample can occur in ADOPT-N and ADOPT$^+$ since the DCOP contains only binary cost functions and no redundant messages exist in the trace. IDB-ADOPT also encounters this flaw because the initial threshold of the root agent at the final iteration is less than (or essentially equal to) that in ADOPT.

In BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT, the counterexample does not occur since a current context is initialized to contain the assignments of all of the higher neighbors.

3.3 Counterexample to Optimality Caused by TERMINATE Messages

The third counterexample occurs in a DCOP whose DFS pseudo-tree and cost functions are shown in Figure 5. In this DCOP, three agents x_0, x_2, and x_3 have the domain $\{0, 1\}$, while the other agents only take values 0. As shown in Figure 5(c), the cost functions $f_{0,3}$ and $f_{2,3}$ involved by x_3 can be considered as one cost function $f_{0,2}$ between x_0 and x_2 since x_3 chooses its variable value to minimize the sum of the cost functions, that is, x_3 chooses the same value as x_0. Note that $f_{0,2}$ is calculated by x_3 and depends on the assignments of x_0 and x_2 in the current context of x_3. Similar to the counterexamples previously shown in this section, the cost functions not specified in Figure 5(b) are the constant functions whose values are 0. The optimal solution of the DCOP is $(x_0, x_1, x_2, x_3, x_4) = (0, 0, 0, 0, 0)$, whose cost is 1.

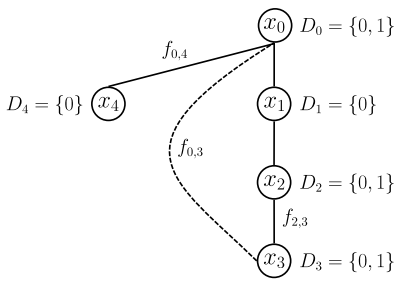
Trace of the Counterexample

In the DCOP, agent x_2 can terminate with a suboptimal value because the termination condition is satisfied despite an incorrect threshold. The states of agents in Step 6, where x_2 terminates, are shown in Figure 6.

Step 0. Similar to the other counterexamples, let us assume that all agents set the variable values to 0 as the initial states. First, agents calculate the cost in the case where x_0 takes the value 0. We omit this procedure, which is presented in Appendix B. The calculation of the cost results in the following: $LB_0(0) = UB_0(0) = 1$; $LB_0(1) = 0$; $UB_0(1) = \infty$; $TH_0 = 0$; $TH_1 = 1$; $LB_2 = UB_2 = 1$; $TH_2 = 1$; and $d_2 = 0$. Furthermore, the contexts of the agents except x_0 contain assignment $(x_0, 0)$. After the calculation, since $LB_0(0) = 1 > TH_0 = 0$, x_0 changes its value d_0 to 1 and sends VALUE messages to x_1, x_3, and x_4.

Step 1. Next, x_3 and x_4 receive the VALUE messages from x_0. At this moment, x_3 updates the context and bounds as $CX_3 = \{(x_0, 1), (x_2, 0)\}$ and $LB_3 = UB_3 = 200$, and x_4 also updates them as $CX_4 = \{(x_0, 1)\}$ and $LB_4 = UB_4 = 1000$. Then they send COST messages to their parents: from x_3 to x_2 and from x_4 to x_0. After x_0 receives the COST message from x_4, x_0 changes its value d_0 back to 0 since the bounds are obtained as $LB_0(1) = 1000$ and $LB_0 = TH_0 = UB_0 = UB_0(0) = 1$ due to ThresholdInvariant. Therefore, x_0 satisfies the termination condition and then terminates. Here, x_0 sends VALUE messages with $d_0 = 0$ to its lower neighbors (i.e., x_1, x_3, and x_4) and THRESHOLD and TERMINATE messages to its children (i.e., x_1 and x_4) in this order, and then x_0 executes termination.

Step 2. Afterwards, x_1 receives the VALUE messages from x_0, including the message that x_0 sent when $d_0 = 1$, in the order of sending. Then x_1 changes CX_1 from $\{(x_0, 0)\}$ into $\{(x_0, 1)\}$ and returns it to $\{(x_0, 0)\}$. Since $\{(x_0, 1)\}$ is incompatible with $cx_1(0, x_2) = \{(x_0, 0)\}$, x_1 reinitializes the bounds as $LB_1 = 0$ and $UB_1 = \infty$. By contrast, TH_1 is not changed from 1. Furthermore, x_1 receives the THRESHOLD and TERMINATE messages from x_0, and then x_1 records receiving the TERMINATE message but does not terminate



(a) DFS pseudo-tree

d_0	d_3	$f_{0,3}$
0	0	1
0	1	1000
1	0	1000
1	1	100

d_2	d_3	$f_{2,3}$
0	0	0
0	1	100
1	0	10
1	1	0

d_0	d_4	$f_{0,4}$
0	0	0
1	0	1000

(b) Cost functions

d_0	d_2	$f_{0,2}$
0	0	1
0	1	11
1	0	200
1	1	100

(c) Cost function induced by $f_{0,3}, f_{2,3}$

Figure 5: DFS pseudo-tree and the cost functions of a DCOP for the counterexample to optimality caused by TERMINATE messages.

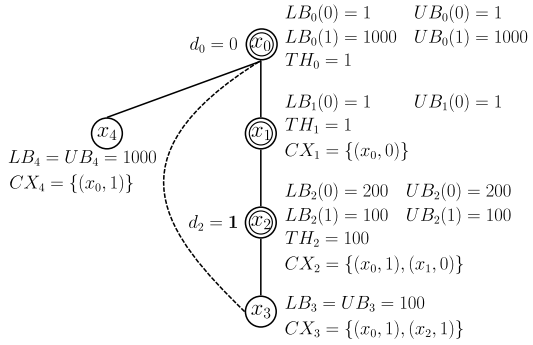


Figure 6: States of agents in Step 6 in the counterexample to optimality caused by TERMINATE message.

because $TH_1 = 1 < UB_1 = \infty$. Subsequently, x_1 sends a VALUE message to x_2.

Step 3. Afterwards, x_2 receives only the message from x_1, not from x_3, which means that the messages from x_3 are delayed. Then x_2 sends a COST message to x_1 with $CX_2 = \{(x_0, 0), (x_1, 0)\}$ and $LB_2 = UB_2 = 1$, which are the same states as in Step 0.

Step 4. Next, x_2 receives the COST message from x_3 with $CX_3 = \{(x_0, 1), (x_2, 0)\}$ and $LB_3 = UB_3 = 200$. Since x_2 is not a neighbor of x_0, x_2 updates CX_2 from $\{(x_0, 0), (x_1, 0)\}$ to $\{(x_0, 1), (x_1, 0)\}$. Then the bounds of x_2 are reinitialized and updated by the bounds in the message: $LB_2(0) = UB_2(0) = 200, LB_2(1) = 0$, and $UB_2(1) = \infty$. x_2 also changes its value d_2 to 1 because $LB_2(0) > TH_2 = 1$, and sends a VALUE message to x_3. After x_3 receives only the VALUE message from x_2 but not the messages from x_1, x_3 updates the current context and the bounds as $CX_3 = \{(x_0, 1), (x_2, 1)\}$ and $LB_3 = UB_3 = 100$. Next, x_3 sends a COST message to x_2 again, and x_2 receives it. Then x_2 computes the bounds as $LB_2(1) = UB_2(1) = 100$ and updates the threshold as $TH_2 = 100$ because of ThresholdInvariant.

Step 5. Here, x_1 receives the COST message with $CX_2 = \{(x_0, 0), (x_1, 0)\}$ and $LB_2 = UB_2 = 1$, sent from x_2 in Step 3. Since this context is compatible with $CX_1 = \{(x_0, 0)\}$, x_1 updates the bounds as $LB_1 = UB_1 = 1$. At this moment, the termination condition is satisfied because $TH_1 = UB_1 = 1$. Thus, x_1 sends two messages to x_2: a THRESHOLD message with $th_1(0, x_2) = 1$ and $CX_1 = \{(x_0, 0)\}$ and a TERMINATE message with $CX_1 \cup \{(x_1, 0)\} = \{(x_0, 0), (x_1, 0)\}$. Then x_1 terminates.

Step 6. When x_2 receives the THRESHOLD message from x_1, x_2 does not update TH_2 since context $\{(x_0, 0)\}$ in the message is incompatible with $CX_2 = \{(x_0, 1), (x_1, 0)\}$, and therefore retains its threshold as $TH_2 = 100$. Next, x_2 receives the TERMINATE message from x_1. Although CX_2 is changed to $\{(x_0, 0), (x_1, 0)\}$, the bounds of x_2 are not changed since reinitialization is not performed when an agent receives a TERMINATE message. Therefore, x_2 terminates with the suboptimal value $d_2 = 1$ because x_2 has already satisfied the termination condition with $UB_2 = UB_2(1) = TH_2 = 100$ and received the TERMINATE message. This result contradicts Property 3.

Cause of the Counterexample

The counterexample presented above is attributed to the following two factors. The first cause is that an agent does not reinitialize its bounds when it receives a TERMINATE message whose context is incompatible with the current context.

However, as shown below, termination with a suboptimal solution can occur even if reinitialization was performed. To demonstrate this fact, let us consider a scenario that the bounds of x_2 are reinitialized when x_2 receives the TERMINATE message from x_1 in Step 6. In this case, the bounds of x_2 are obtained as $LB_2(0) = LB_2(1) = 0$ and $UB_2(0) = UB_2(1) = \infty$; and the threshold of x_2 is maintained as $TH_2 = 100$. After x_3 receives the VALUE message with the value $d_0 = 0$ from x_0, x_3 updates the current context as $CX_3 = \{(x_0, 0), (x_2, 1)\}$ and the bounds as $LB_3 = UB_3 = 11$. Then x_3 sends a COST message to x_2, and x_2 receives it. At this point, x_2 computes the bounds as $LB_2(1) = UB_2(1) = 11$ because context $\{(x_0, 0), (x_2, 1)\}$ in the COST message is compatible with $CX_2 = \{(x_0, 0), (x_1, 0)\}$. Then x_2 updates the threshold as $TH_2 = 11$ due to ThresholdInvariant. Since $UB_2 = UB_2(1) = TH_2$, x_2 does not change its variable value d_2 from 1. Therefore, x_2 satisfies the termination condition and terminates with the suboptimal value $d_2 = 1$.

The second cause of the counterexample is an incorrect threshold at termination. Even if the threshold of an agent differs from that in the THRESHOLD message sent just be-

fore a TERMINATE message, the termination condition of the agent can be satisfied. Therefore, the agent can terminate with a suboptimal value.

Occurrence in Variants

In ADOPT-N and IDB-ADOPT, this counterexample occurs for the same reasons as the flaws previously shown. Moreover, BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT all have similar flaws. The causes of them are that agents do not send their current contexts in VALUE messages, which include thresholds, or TERMINATE messages. Thus, they can terminate with incorrect contexts.

The incorrectness of contexts means that the statements of the proofs in the studies of ADOPT(k) and BD-ADOPT are not appropriate. They imply that agents obtain the optimal costs for each of the current contexts, which may be incorrect at termination. The study of BnB-ADOPT also presents an inappropriate statement. It implies that only the root agent obtains the optimal cost at termination, however, the other agents can terminate with incorrect contexts, resulting in a suboptimal solution.

By contrast, ADOPT$^+$ does not have this flaw because it is obtained by fixing the causes of the flaw in ADOPT. Specifically, in ADOPT$^+$, a VALUE message contains the current context and threshold of the agent, and furthermore, a time stamp for a variable value is introduced. These modifications guarantee the correct context and threshold at termination, and agents can reinitialize the bounds if the current context is changed.

4 Amendment to ADOPT

In this section, we propose an amended version of ADOPT to avoid the flaws. The amendment consists of three parts, each of which corresponds to one of the causes described in Section 3. The pseudocode of our version of ADOPT is presented in Appendix C.

First, we modify the update rules for the lower and upper bounds $lb_i(d, x_c)$ and $ub_i(d, x_c)$ for a child so as to change monotonically. In the amended version, when agent x_i receives a COST message from child x_c with bounds LB_c and UB_c and context $CX_c \ni (x_i, d)$, x_i updates the lower and upper bounds for d and x_c by the following:

$$lb_i(d, x_c) := \max\{lb_i(d, x_c), LB_c\}, \quad (1)$$

$$ub_i(d, x_c) := \min\{ub_i(d, x_c), UB_c\}. \quad (2)$$

The other calculations of the lower and upper bounds (i.e., $LB_i(d), UB_i(d), LB_i$, and UB_i) are not changed from the original definitions.

Second, we modify the initialization of a current context so as to store the assignments of all higher neighbors. Agent x_i initializes the current context as follows:

$$CX_i := \{(x_p, ValInit(x_p)) \mid x_p \in SCP(x_i)\},$$

where $ValInit(x_p) \in D_p$ is the initial value for x_p and $SCP(x_i)$ is a set of the ancestors of x_i that are parents or pseudo-parents of agents (including x_i) in the subtree rooted at x_i. Note that for x_i to avoid flaws, it is sufficient to initialize the current context with the assignments of the parent and pseudo-parents of x_i. However, our amended version uses the assignments of all agents in $SCP(x_i)$ since the current context CX_i eventually stores their assignments. The two aforementioned amendments have been introduced in BnB-ADOPT, BnB-ADOPT$^+$, ADOPT(k), and BD-ADOPT.

Finally, we modify TERMINATE messages and their receiving procedure. The amended procedure when agent x_i receives a TERMINATE message from the parent x_p is as follows: if the context $cx_i(d, x_c)$ for value d and child x_c is incompatible with the current context CX_i updated by the TERMINATE message, x_i reinitializes the lower and upper bounds $lb_i(d, x_c)$ and $ub_i(d, x_c)$, threshold $th_i(d, x_c)$, and context $cx_i(d, x_c)$. Additionally, we introduce the threshold $th_i(d_i, x_c)$ for the current value d_i and child x_c into the TERMINATE message sent to x_c. The amendment to TERMINATE messages guarantees the threshold of an agent after receiving a TERMINATE message to be correct, which is proven by a theorem in the next section. Therefore, the termination condition is satisfied with the correct threshold and optimal value.

5 Proof of Termination and Optimality

Our amended algorithm guarantees termination and optimality, which are stated as Theorems 1 and 2 below, and the full proofs and lemmata are shown in Appendix D. The arguments of the proofs are based on the study of BnB-ADOPT. However, the statements of the theorems are modified to derive our aimed properties.

For the theorem of optimality, we define the correctness of the current context of an agent.

Definition 1. *The current context CX_i of agent x_i is correct iff CX_i contains the assignments of all agents in $SCP(x_i)$, and the values of all agents in CX_i are equal to the values of the agents.*

The theorems for the amended version of ADOPT are as follows:

Theorem 1. *The amended version of ADOPT terminates after a finite amount of time.*

Theorem 2. *For any agent $x_i \in X$ when the amended version of ADOPT terminates, the current context CX_i is correct, and $TH_i = \gamma_i(d_i, CX_i) = \gamma_i(CX_i)$.*

6 Conclusion

We presented flaws in ADOPT and its variants with respect to termination and optimality, which have been believed to be correct. The flaws are classified into three types, one of which is related to termination, while the other two of which are related to optimality. The causes of the flaws are the update rules for the bounds, the initialization of a current context, and TERMINATE messages and their receiving procedure. Additionally, we proposed an amended version of ADOPT that avoids these flaws, whose amendment corresponds to the causes of the flaws. Furthermore, we proved termination and optimality in the amended version of ADOPT. For future work, we plan to verify the properties of ADOPT-ing, which was not analyzed in this work.

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