Rainbow Cycle Number and EFX Allocations: (Almost) Closing the Gap

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Abstract

Recently, some studies on the fair allocation of indivisible goods notice a connection between a purely combinatorial problem called the Rainbow Cycle problem and a fairness notion known as EFX: assuming that the rainbow cycle number for parameter d (i.e. $\mathsf{R}(d)$) is $O(d^{\beta}\log^{\gamma} d)$, we can find a $(1-\epsilon)$-EFX allocation with $O_{\epsilon}(n^{\frac{\beta}{\beta+1}}\log^{\frac{\gamma}{\beta+1}} n)$ number of discarded goods. The best upper bound on $\mathsf{R}(d)$ is improved in a series of works to $O(d^4)$, $O(d^{2+o(1)})$, and finally to $O(d^2)$. Also, via a simple observation, we have $\mathsf{R}(d) \in \Omega(d)$.

In this paper, we introduce another problem in extremal combinatorics. For a parameter ℓ, we define the rainbow path degree and denote it by $\mathsf{H}(\ell)$. We show that any lower bound on $\mathsf{H}(\ell)$ yields an upper bound on $\mathsf{R}(d)$. Next, we prove that $\mathsf{H}(\ell) \in \Omega(\ell^2/\log \ell)$ which yields an almost tight upper bound of $\mathsf{R}(d) \in \Omega(d \log d)$. This in turn proves the existence of $(1-\epsilon)$-EFX allocation with $O_{\epsilon}(\sqrt{n\log n})$ number of discarded goods. In addition, for the special case of the Rainbow Cycle problem that the edges in each part form a permutation, we improve the upper bound to $\mathsf{R}(d) \leq 2d-4$. We leverage $\mathsf{H}(\ell)$ to achieve this bound.

Our conjecture is that the exact value of $\mathsf{H}(\ell)$ is $\lfloor \frac{\ell^2}{2} \rfloor - 1$. We provide some experiments that support this conjecture. Assuming this conjecture is correct, we have $\mathsf{R}(d) \in \Theta(d)$.

1 Introduction

Fair allocation of indivisible goods has been an important problem in economics and social choice theory [Aziz *et al.*, 2015; Brams and Taylor, 1996; Steinhaus, 1948; Dubins and Spanier, 1961; Brams and Taylor, 1995; Kurokawa *et al.*, 2018; Ghodsi *et al.*, 2018; Lipton *et al.*, 2004; Etkin *et al.*, 2007; Halpern *et al.*, 2020; Moulin, 2019; Procaccia, 2020; Pratt and Zeckhauser, 1990; Budish and Cantillon, 2012; Budish, 2011; Barman *et al.*, 2018] with many applications in the real world.[1] In a fair allocation problem, we have a set of n agents and a set of m indivisible goods, and each agent has a valuation function that represents her utility for receiving each subset of goods. The goal is to allocate the goods to the agents fairly [Baklanov *et al.*, 2021; Brams *et al.*, 2017; Amanatidis *et al.*, 2017; Barman and Krishnamurthy, 2020; Garg *et al.*, 2019; Garg and Taki, 2020; Bouveret and Lemaître, 2016].

A critical challenge in a fair allocation problem is to specify a reasonable notion of fairness that is simultaneously robust and practical. For the classic version of the problem that the resource is a single divisible good, a notion such as envy-freeness[2] perfectly satisfies these conditions: it is commonly accepted as a notion that represents fairness, and there are strong guarantees for the existence of envy-free divisions [Edward Su, 1999]. However, the applicability of this notion decreases significantly when dealing with indivisible goods: even for two agents and one good, envy-freeness can not be guaranteed. In recent years, several relaxations of envy-freeness have been introduced to adopt this notion to the indivisible setting. Among these notions, EFX is widely believed to be the most prominent [Caragiannis *et al.*, 2019b; Chaudhury *et al.*, 2021a; Chaudhury *et al.*, 2021b; Amanatidis *et al.*, 2020; Berger *et al.*, 2021; Chaudhury *et al.*, 2020; Plaut and Roughgarden, 2020].

Definition 1. *An allocation is EFX (α-EFX), if for every agents i and j, agent i does not envy (α-envy)*[3] *agent j after removal of any good from the bundle of agent j.*

See Figure 1 for examples of envy-free, EFX, and α-EFX allocations. Recent studies suggest that one can obtain strong guarantees on EFX by discarding a subset of goods [Chaudhury *et al.*, 2021b; Caragiannis *et al.*, 2019a]. In a pioneering work, Chaudhury, Kavitha, Mehlhorn, and Sgouritsa [2021b] show that it is possible to find an EFX allocation by discarding at most $n-1$ goods. Further investigations in this direction reveal an intriguing connection between EFX and a purely combinatorial problem called *Rainbow Cycle* problem[4] [Chaudhury *et al.*, 2021a]. For a multi-partite bidirected graph, a rainbow cycle is a cycle that passes each part at most

[1]See spliddit.org and www.fairoutcomes.com for example.

[2]An allocation is envy-free if each agent prefers her share over the other agents' share.

[3]For $\alpha < 1$, agent i α-envies agent j, if the value of i for his bundle is less than α times his value for bundle of agent j.

[4]The problem is also known as the Fixed Point Cycle.

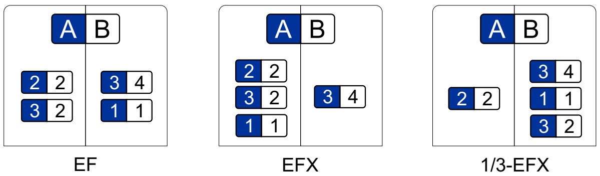

Figure 1: In this figure, three different allocations of four goods to two agents are shown. The valuation of A and B for a good are shown respectively on the left and right sides of the good, and the valuations are additive. The left allocation is envy-free since $v_A(X_A) = 2+3 = 5 > v_A(X_B) = 3+1 = 4$ and $v_B(X_B) = 4+1 = 5 > v_B(X_A) = 2+2 = 4$. The allocation in the middle is EFX since $v_A(X_A) = 6 \geq \max_{x \in X_B} v_A(X_B \setminus \{x\}) = 0$ and $v_B(X_B) = 4 \geq \max_{x \in X_A} v_B(X_A \setminus \{x\}) = 4$. Finally, the right allocation is 1/3-EFX, because $v_A(X_A) = 2 \geq (1/3)\max_{x \in X_B} v_A(X_B \setminus \{x\}) = (3+3)/3 = 2$ and $v_B(X_B) = 7 \geq (1/3)\max_{x \in X_A} v_B(X_A \setminus \{x\}) = 0/3 = 0$.

once. The Rainbow Cycle problem is then defined as follows.

Problem 1 (Rainbow Cycle)**.** *For a constant d, what is the maximum value ℓ such that there exists an ℓ-partite bidirected graph with no rainbow cycle and the following properties: (i) each part contains at least d vertices, and, (ii) each vertex receives an incoming edge from all other parts other than the one containing it. We call such a value ℓ the rainbow cycle number of d and denote it by $\mathsf{R}(d)$.*

We refer to Section 2 for a more formal definition of this problem. The connection between the Rainbow Cycle problem and EFX notion was first observed by Chaudhury *et al.* [2021a]: any upper bound on $\mathsf{R}(d)$ yields a corresponding upper bound on the number of discarded goods.

Theorem 1 (Proved in [Chaudhury *et al.*, 2021a])**.** *For any constant $\varepsilon \in (0, 1/2]$, if there exists constants β, γ such that $\mathsf{R}(d) \in O(d^\beta \log^\gamma d)$, then we can find a $(1-\varepsilon)$-EFX allocation with $O_\epsilon(n^{\frac{\beta}{\beta+1}} \log^{\frac{\gamma}{\beta+1}} n)$ number of discarded goods.*

The first upper bound on $\mathsf{R}(d)$ was also proposed by Chaudhury *et al.* [2021a]. They proved that $\mathsf{R}(d) \in O(d^4)$ which bounds the number of unallocated goods by $O_\epsilon(n^{\frac{4}{5}})$. Recently, in two parallel studies [Berendsohn *et al.*, 2022; Akrami *et al.*, 2022], the bound on $\mathsf{R}(d)$ is improved to $O(d^{2+o(1)})$ and $O(d^2)$, yielding an upper bound of $O_\epsilon(n^{\frac{2}{3}})$ on the number of unallocated goods. Note that a trivial lower bound on $\mathsf{R}(d)$ is $\Omega(d)$. [5] Therefore, previous results leave a gap of $[\Omega(d), O(d^2)]$ between the best upper bound and the best lower bound. There is a plausible conjecture that $\mathsf{R}(d) \in O(d)$.

In this paper, we almost close this gap by showing that $\mathsf{R}(d) \in O(d \log d)$. To obtain this bound, we introduce another invariant called rainbow path degree which might be of independent interest. We show that any lower bound on this invariant implies an upper bound on $\mathsf{R}(d)$. Next, we improve the lower bound on $\mathsf{R}(d)$ by providing an upper bound on the rainbow path degree.

[5] See [Chaudhury *et al.*, 2021a] for a matching example.

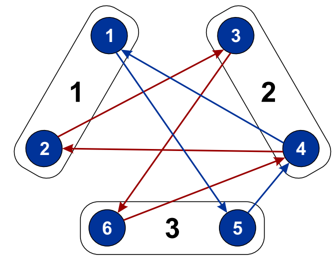

Figure 2: In this figure, path $1 \to 5 \to 4$ is a rainbow path, and if we add edge $4 \to 1$ to the end of the path, we have a rainbow cycle. On the other hand, path $2 \to 3 \to 6 \to 4$ and cycle $2 \to 3 \to 6 \to 4 \to 2$ are not rainbow path and rainbow cycle respectively as they go through part 2 twice.

Before ending this section, we mention that apart from EFX and fair allocation, bounding the rainbow cycles number itself is an interesting extremal problem. Recently, Berendsohn, Boyadzhiyska, and Kozma [2022] established a connection between two combinatorial problems: Permutation Rainbow Cycle problem which is a special case of Rainbow Cycle problem, and Zero-sum Cycle problem [Alon and Krivelevich, 2021; Mészáros and Steiner, 2021; Alon and Caro, 1993; Alon and Dubiner, 1993; Alon and Linial, 1989; Bialostocki, 1993; Caro, 1996; Schrijver and Seymour, 1991], which is a problem in zero-sum extremal combinatorics. Here we also give an improved upper bound on the permutation rainbow cycle number. We refer to Section 3 for more details on our results and techniques.

A short note on a parallel result. We note that parallel and concurrent to this work, Akrami *et al.* [2022] also updated their results on arXiv. In the updated version, their upper bound on $\mathsf{R}(d)$ is improved from $O(d^2)$ to $O(d \log d)$ via a probabilistic argument. We emphasis that these two studies are parallel and independent.

2 Preliminaries

In this paper, our focus is on multi-partite bidirected graphs. For an ℓ-partite bidirected graph G, we denote its parts by $V_1, V_2, \ldots, V_\ell$. Also, for a subset $W \subseteq \{V_1, V_2, \ldots, V_\ell\}$ we define $G[W]$ to be the induced subgraph of G that only includes vertices that belong to the parts in W. Thus, $G[W]$ has $|W|$ parts. A path in graph G is called rainbow if it passes through each part at most once. The same definition carries over to cycles. See Figure 2 for an example.

For integers $\ell, d \geq 0$, we define $\Phi_{\ell,d}$ to be the set of all multi-partite bidirected graphs G with the following properties:

- G has exactly ℓ parts,
- each part of G has at least 1 and at most d vertices,
- each vertex of G has exactly one incoming edge from every other part,
- G admits no rainbow cycle.

In Figure 3, we show an example of a graph in $\Phi_{\ell,d}$. We also define $\Phi_{*,d}$ and $\Phi_{\ell,*}$ as unions of $\Phi_{\ell,d}$ over all ℓ and d

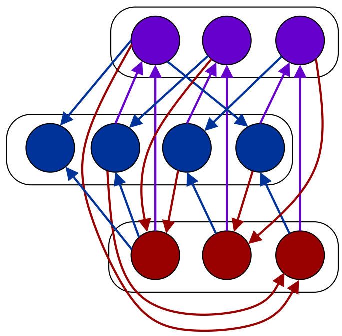

Figure 3: The graph shown in this figure is in $\Phi_{3,4}$: it contains exactly 3 parts, each part has at most 4 vertices, and one can check that two other conditions of $\Phi_{3,4}$ hold as well. Additionally, by the definition, this graph also belongs to $\Phi_{3,*}$ and $\Phi_{*,4}$.

respectively, that is,

$$\Phi_{*,d} = \bigcup_{\ell \geq 0} \Phi_{\ell,d} \qquad \text{and} \qquad \Phi_{\ell,*} = \bigcup_{d \geq 0} \Phi_{\ell,d}.$$

Also, we define $\mathsf{R}(d)$ as the largest ℓ such that an ℓ-partite graph exists in $\Phi_{*,d}$, i.e.,

$$\mathsf{R}(d) = \max_{\ell} \quad \text{s.t. } \Phi_{\ell,d} \neq \emptyset.$$

Our goal is to give an upper bound on $\mathsf{R}(d)$ for every d. To this aim, we introduce another property. Let G be a multipartite graph. For every vertex $v \in G$, we define $f_G(v)$ as the number of vertices in G that have a rainbow path to v except v itself. Given $f_G(v)$, for every constant ℓ, we define the rainbow path degree of ℓ, denoted by $\mathsf{H}(\ell)$ as follows:

$$\mathsf{H}(\ell) = \min_{G \in \Phi_{\ell,*}} \min_{v \in G} \quad f_G(v).$$

In other words, $\mathsf{H}(\ell)$ is the maximum possible value that we are guaranteed that for an ℓ-partite graph $G \in \Phi_{\ell,*}$, for every vertex $v \in G$ there are at least $\mathsf{H}(\ell)$ different vertices with a rainbow path to v. For brevity, we call $\mathsf{H}(\ell)$ the rainbow path degree of ℓ.

In order to prove an upper bound on $\mathsf{R}(d)$, we first prove a lower bound on $\mathsf{H}(\ell)$. Interestingly, though the definition of $\mathsf{H}(\ell)$ does not depend on d, our lower bound on $\mathsf{H}(\ell)$ results in an almost tight upper bound on $\mathsf{R}(d)$.

In the last part of this section, we mention Stirling's formula for approximating factorials. For every $n > 1$, we have:

$$\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}. \tag{1}$$

In the next section, we briefly review our results and techniques.

3 Our Results and Techniques

The main result of this paper is an almost tight upper bound on the rainbow cycle number by showing that $\mathsf{R}(d) \in \widetilde{O}(d)$. Our techniques are structurally different from previous methods. Indeed, a primary application of our techniques provides a simpler proof for $\mathsf{R}(d) \in O(d^2)$. Using a more in-depth analysis, we improve this bound to $O(d \log d)$. To show this, we prove a lower bound for the rainbow path degree and show that $\mathsf{H}(\ell) \in \Omega(\ell^2/\log \ell)$. This in turn implies that an EFX allocation exists that discards at most $O_\epsilon(\sqrt{n \log n})$ goods.

For a better understanding of our techniques, let us overview a simple proof for $\mathsf{R}(d) \in O(d^2)$.[6] We prove this bound by showing that $\mathsf{H}(\ell) \in \Omega(\ell\sqrt{\ell})$. Let $G \in \Phi_{\ell+1,*}$ be an $\ell+1$ partite graph with parts $\{V_1, V_2, \dots, V_{\ell+1}\}$ and let v be a vertex in $V_{\ell+1}$. By definition, we know that there are at least $\mathsf{H}(\ell+1)$ vertices that have a rainbow path to v. Denote the set of these vertices by C. Our goal is to provide a lower bound on $|C|$. Since the vertices in C belong to parts $V_1, V_2, \dots, V_\ell$, there is a part that contributes at most $\mathsf{H}(\ell+1)/\ell$ vertices to C. Without loss of generality, suppose that this part is V_ℓ. Therefore,

$$|V_\ell \cap C| \leq \mathsf{H}(\ell+1)/\ell.$$

In other words, at most $\mathsf{H}(\ell+1)/\ell$ of the vertices in V_ℓ have a rainbow path to v. Now, consider the vertices that have an outgoing edge to v. Since $G \in \Phi_{\ell+1,*}$, by definition, each part has a vertex with an outgoing edge to v. For each part V_i, we assume that v_i is the vertex with an outgoing edge to v. Also, note that for every $1 \leq i \leq \ell-1$, vertex v_i has an incoming edge from part V_ℓ. Since v_i has an outgoing edge to v, any vertex in V_ℓ that has an outgoing edge to v_i has a rainbow path of length 2 to v and thus belongs to C. Since $|V_\ell \cap C| \leq |C|/\ell$, there exists a vertex $u \in |V_\ell \cap C|$ that has outgoing edges to at least

$$(\ell-1)/(\mathsf{H}(\ell+1)/\ell) \simeq \ell^2/\mathsf{H}(\ell+1)$$

vertices in $\{v_1, v_2, \dots, v_{\ell-1}\}$. Denote these vertices by $\hat{C}$ and suppose without loss of generality that $v_{\ell-1} \in \hat{C}$. We know that in $G[V \setminus \{V_{\ell-1}, V_{\ell+1}\}]$, the number of vertices that have a rainbow path to u is at least $\mathsf{H}(\ell-1)$. These vertices also have a rainbow path to v: consider their rainbow path to u, then go to $v_{\ell-1}$ and then to v. Also, these vertices do not belong to $\hat{C}$; otherwise, since u has outgoing edges to the vertices in $\hat{C}$, we have a rainbow cycle. Therefore,

$$\mathsf{H}(\ell+1) \geq \ell^2/\mathsf{H}(\ell+1) + \mathsf{H}(\ell-1). \tag{2}$$

Using straightforward calculus one can show that Inequality (2) implies $\mathsf{H}(\ell+1) \in \Omega(\ell\sqrt{\ell})$.

A consequence of this lower bound is an upper bound on $\mathsf{R}(d)$. To see why, assume for simplicity that $\mathsf{H}(\ell+1)$ is exactly equal to $\ell\sqrt{\ell}$. We show $\Phi_{d^2+1,d}$ is empty. To see why, consider a vertex in V_{d^2+1} with a non-zero outgoing degree. By definition of $\mathsf{H}(d^2+1)$, the number of vertices with a rainbow path to this vertex is at least $d^2\sqrt{d^2} = d^3$, which is equal to the number of vertices in $\{V_1, V_2, \dots, V_{d^2}\}$. Thus, any outgoing edge from this vertex yields a rainbow cycle.

In Section 4, via a similar but more in-depth analysis, we show that $\mathsf{H}(\ell) \in \Omega(\ell^2/\log \ell)$. A consequence of this result is the upper bound of $O(d \log d)$ on the rainbow cycle number, which leaves a gap of $O(\log d)$ factor between the upper

[6]We emphasize that in the interest of simplicity, our discussion in this section is not completely accurate.

bound and the lower bound for the rainbow cycle number. Also, in Section 6, we show that $\mathsf{H}(\ell) \in O(\ell^2)$ that leaves a gap of $O(\log \ell)$ factor between the upper bound and lower bound for the rainbow path degree.

In Section 6, we represent our experimental results on finding the exact value of $\mathsf{H}(\ell)$. Our experiments suggest that for small values of ℓ, we have $\mathsf{H}(\ell) = \lfloor \frac{\ell^2}{2} \rfloor - 1$. Assuming that this conjecture is correct for every ℓ, we have $\mathsf{R}(d) \in O(d)$. As a future direction, one can think of improving the lower bound on $\mathsf{H}(\ell)$ to $\Omega(\ell^2)$.

Also, we consider a special case of the Rainbow Cycle problem called the Permutation Rainbow Cycle problem, where each vertex has exactly one outgoing edge to each part. As we mentioned earlier, this problem has some independent applications in extremal combinatorics. We improve the upper bound on the permutation rainbow cycle number to $2d - 3$. Next, we leverage the bounds we obtain on $\mathsf{H}(\ell)$ for small values of ℓ in Section 6 to improve the upper bound to $2d - 4$. Furthermore, In Section 6, we consider the relation between the rainbow cycle number and the rainbow path degree in the permutation case. We show that our conjecture of $\mathsf{H}(\ell) = \lfloor \frac{\ell^2}{2} \rfloor - 1$ implies the upper bound of $2d - 3$ on $\mathsf{R}_p(d)$ in the permutation case.

4 Upper Bound on the Rainbow Cycle Number

We now present our results for the Rainbow Cycle problem. This section is divided into three parts. In the first part, in Lemma 1, we show that any lower bound on rainbow path degree implies a corresponding upper bound on rainbow cycle number. Next, we prove two lower bounds on $\mathsf{H}(\ell)$. As a warm up, we start by showing that $\mathsf{H}(\ell) \in \Omega(\ell\sqrt{\ell})$. Next, we present the main result of this section, that is, $\mathsf{H}(\ell) \in \Omega(\ell^2/\log \ell)$. This, combined with Lemma 1, yields the upper bound of $\mathsf{R}(d) \in O(d \log d)$.

Lemma 1 shows a simple connection between $\mathsf{H}(\ell)$ and $\mathsf{R}(d)$. The idea behind the proof of Lemma 1 is simple: the rainbow path degree of a vertex cannot be more than the total number of the vertices. [7]

Lemma 1. *For every $\beta > 0$,γ if $\mathsf{H}(\ell) \in \Omega(\ell^{1+\beta} \log^{\gamma} \ell)$ then $\mathsf{R}(d) \in O(d^{\frac{1}{\beta}} \log^{-\frac{\gamma}{\beta}} d)$.*

We use Lemma 1 to prove two upper bounds on $\mathsf{R}(d)$. First, in Lemma 2, we show that $\mathsf{H}(\ell) \in \Omega(\ell\sqrt{\ell})$, which implies $\mathsf{R}(d) \in O(d^2)$.

Lemma 2. *For every $\ell \geq 1$, we have $\mathsf{H}(\ell + 1) \geq \ell\sqrt{\ell}/6$.*

Proof. In order to prove Lemma 2, we use induction on ℓ. For $\ell = 1, 2$ we have:

$$\frac{\ell\sqrt{\ell}}{6} \leq \frac{2\sqrt{2}}{6} < 1 \leq \mathsf{H}(\ell + 1).$$

[7] In the interest of space, some of the proofs are deffered to the appendix.

Now, suppose that the statement holds for every $\ell' < \ell$. Our goal is to prove the claim for ℓ. As a contradiction, suppose

$$\mathsf{H}(\ell + 1) < \ell\sqrt{\ell}/6. \tag{3}$$

This means that there exists a graph $G \in \Phi_{\ell+1,*}$ and a vertex $v \in G$, such that if we define C as the set of the vertices in G with a rainbow path to v, we have

$$|C| < \ell\sqrt{\ell}/6. \tag{4}$$

Suppose that $\{V_1, V_2, \ldots, V_{\ell+1}\}$ is the set of parts in G and suppose without loss of generality that $v \in V_{\ell+1}$.

Claim 4.1. *Fix a vertex u, and define P as the set of all rainbow paths with length at most 2 from u to v. Also, let $\hat{C}$ be the set of all different vertices that have an incoming edge from u and belong to a path in P. We have $|\hat{C}| \leq 2\sqrt{\ell}/3$.*

By Inequality (4), we know $|C| < \ell\sqrt{\ell}/6$. The vertices in C belong to parts $V_1, V_2, \ldots, V_\ell$. Therefore, at least one of these parts contributes less than $\sqrt{\ell}/6$ vertices to C. Suppose without loss of generality that V_ℓ is one of such parts, i.e., $|V_\ell \cap C| < \sqrt{\ell}/6$. Since $G \in \Phi_{\ell+1,*}$, each part other than $V_{\ell+1}$ has a vertex with an outgoing edge to v. For each part V_i ($i \leq \ell - 1$), we denote this vertex by v_i. Also, note that each vertex v_i has an incoming edge from V_ℓ. Since v_i has an outgoing edge to v, any vertex in V_ℓ that has an outgoing edge to v_i has a rainbow path of length 2 to v and thus belongs to C. Hence, at least one of the vertices in $V_\ell \cap C$ has outgoing edges to at least

$$\frac{\ell - 1}{\sqrt{\ell}/6} = \frac{6(\ell - 1)}{\sqrt{\ell}} \tag{5}$$

of the vertices in $\{v_1, v_2, \ldots, v_{\ell-1}\}$. On the other hand, by Claim 4.1, we know that each vertex in V_ℓ has at most $2\sqrt{\ell}/3$ outgoing edges to $\{v_1, v_2, \ldots, v_{\ell-1}\}$. Thus, we have

$$\frac{6(\ell - 1)}{\sqrt{\ell}} \leq \frac{2\sqrt{\ell}}{3},$$

which means

$$18(\ell - 1) \leq 2\ell,$$

that is, $\ell \leq 16/18$. But this contradicts the fact that $\ell > 2$. $\square$

Corollary 1 (of Lemma 2)**.** *By choosing $\beta = 0.5$ and $\gamma = 0$ in Lemma 1, we have $\mathsf{R}(d) \in O(d^2)$.*

Now, we are ready to prove our main result. In Theorem 1, we show that $\mathsf{H}(\ell + 1) \in \Omega(\ell^2/\log \ell)$. The structure of the proof of Theorem 1 is similar to the proof of Lemma 2. The difference is that here we generalize Claim 4.1 to consider paths with length more than 2.

Theorem 1. *For every $\ell \geq 3$, we have $\mathsf{H}(\ell+1) \geq \ell^2/20 \ln \ell$.*

Proof. We use induction on ℓ. For $\ell = 3, 4$ we have:

$$\frac{\ell^2}{20 \ln \ell} < 1 \leq \ell \leq \mathsf{H}(\ell + 1).$$

Now, suppose that for some $\ell \geq 5$ we know that the statement of Theorem 1 holds for every $3 \leq \ell' < \ell$ and our goal is to prove the claim for ℓ. As a contradiction, suppose

$$\mathsf{H}(\ell+1) < \frac{\ell^2}{20 \ln \ell}. \tag{6}$$

This means that there exists a graph $G \in \Phi_{\ell+1,*}$ and a vertex $v \in G$, such that exactly $\mathsf{H}(\ell+1)$ of the vertices in G have a rainbow path to v, which is less than $\ell^2/20 \ln \ell$. Suppose that $\{V_1, V_2, \ldots, V_{\ell+1}\}$ is the set of parts in G and suppose without loss of generality that $v \in V_{\ell+1}$. We start by proving Claim 4.2. Claim 4.2 plays a similar role as Claim 4.1. The main difference is that in Claim 4.2, we consider paths with length more than 2.

Claim 4.2. *Fix a vertex u and an integer $k \leq \ell - 3$, and define P_k as the set of all rainbow paths with length at most k from u to v. Also, let $\hat{C}$ be the set of all different vertices that have an incoming edge from u in the paths of P_k. Then, $|\hat{C}| \leq \ell k/4 \ln \ell$.*

Now, we use Claim 4.2 to prove Claim 4.3.

Claim 4.3. *Fix an integer $t \leq \ln \ell$. For every $k \leq t$ and subset W of $\{V_1, V_2, \ldots, V_\ell\}$ with $\ell - t + k$ parts, at least $t^{k-1}/(k-1)!$ vertices of each part in W have rainbow paths with lengths at most k to v in $G[W \cup \{V_{\ell+1}\}]$.*

By the pigeonhole principle, there exists a part V_i that contains at most $\mathsf{H}(\ell+1)/\ell$ vertices with a rainbow path to v. By choosing $k = t$ in Claim 4.3, we have:

$$\begin{aligned}
\frac{\mathsf{H}(\ell+1)}{\ell} &\geq \frac{t^{t-1}}{(t-1)!} \\
&\geq \frac{t^{t-1}}{(\frac{t-1}{e})^{t-1}\sqrt{2\pi(t-1)} \cdot e^{\frac{1}{12(t-1)}}} && \text{Inequality (1)} \\
&\geq \frac{e^{t-1}}{e^{\frac{1}{12t}}\sqrt{2\pi t}} = \frac{e^{t+1}}{e^{2+\frac{1}{12t}}\sqrt{2\pi t}}.
\end{aligned}$$

If we choose $t = \lfloor \ln \ell \rfloor$, we have[8]:

$$\begin{aligned}
\frac{e^{t+1}}{e^{2+\frac{1}{12t}}\sqrt{2\pi t}} &\geq \frac{\ell}{e^{2+\frac{1}{12t}}\sqrt{2\pi t}} && t+1 \geq \ln \ell \\
&\geq \frac{\ell}{e^{2+\frac{1}{12}}\sqrt{2\pi \ln \ell}} && 1 \leq t \leq \ln \ell \\
&\geq \frac{\ell}{21\sqrt{\ln \ell}} \\
&\geq \frac{\ell}{20 \ln \ell} && \ell \geq 5,
\end{aligned}$$

which contradicts Inequality (6). This Completes the proof of Theorem 1. □

Corollary 2 (of Theorem 1)**.** *By choosing $\beta = 1$ and $\gamma = 1$ in Lemma 1, we have $\mathsf{R}(d) \in O(d \log d)$.*

[8] Since $\lfloor \ln \ell \rfloor \leq \ln \ell$, the conditions of Claim 4.3 hold.

By Corollary 2, we have the upper bound of $O(d \log d)$ on $\mathsf{R}(d)$. Using this upper bound in Theorem 1 we obtain a new upper bound on the number of discarded goods in EFX allocations.

Corollary 3. *By choosing $\beta = 1$ and $\gamma = 1$ in Theorem 1, For every constant $\varepsilon \in (0, 1/2]$, we can find a $(1-\varepsilon)$-EFX allocation with $O_\epsilon(\sqrt{n \log n})$ number of discarded goods.*

5 Permutation Rainbow Cycle

In this section, we consider the Permutation Rainbow Cycle problem. For an integer $d > 0$, define $\Pi_{\ell,d}$, $\Pi_{*,d}$, and $\Pi_{\ell,*}$ respectively as subsets of $\Phi_{\ell,d}$, $\Phi_{*,d}$, and $\Phi_{\ell,*}$ consisting all graphs G with the additional property that each vertex in G has exactly one outgoing edge to every other part. Also, we define $\mathsf{R}_p(d)$ as the largest k such that a k-partite graph exists in $\Pi_{*,d}$, i.e.,

$$\mathsf{R}_p(d) = \max_{G \in \Pi_{*,d}} \#(G).$$

Our result in this section is an improved upper bound on $\mathsf{R}_p(d)$ for every $d \geq 3$. Our method slightly improves the method of Akrami *et al.* [2022], wherein the authors prove the upper bound of $2d-2$ on $\mathsf{R}_p(d)$. Throughout this section, we show that for $d \geq 4$ we have $\mathsf{R}_p(d) \leq 2d - 4$. In order to prove this bound, first in Theorem 2 we show that for $d \geq 3$, we have $\mathsf{R}_p(d) \leq 2d - 3$. In the proof of Theorem 2, we use the idea of constructing a sequence with certain properties. This idea has been previously used by Akrami *et al.* [2022] to prove the upper bound of $2d - 2$. Here, we strengthen the assumptions on the sequence. Lemma 3 plays a key role in route to proving our upper bound.

We next show how we can incorporate $\mathsf{H}(\ell)$ in the proof to improve the upper bound to $2d - 4$. Later in Section 6, we discuss the possibility of obtaining better upper bounds on $\mathsf{R}_p(d)$ via a more effective incorporation of $\mathsf{H}(\ell)$ in the proof. While it might be possible to obtain $2d - c$ upper bound for $c > 4$ with the same idea, we show that it is not possible to obtain an upper bound in the form of $d + c$ for a constant $c > 0$ using the same method.

Theorem 2. $\mathsf{R}_p(d) \leq 2d - 3$ *for* $d \geq 3$.

As a contradiction, suppose there is a graph G in $\Pi_{*,d}$ consisting of at least $2d - 2$ parts, i.e., $\#(G) \geq 2d - 2$. We denote by $v_{i,j}$ the j'th vertex in the i'th part of G.

The first important step in order to improve the previous result is stated in Lemma 3. In this lemma, we show that for every vertex v there is a vertex u with an outgoing edge to v, such that no other vertex has outgoing edge to both v and u.

Lemma 3. *For each vertex v, there exists some vertex u with an outgoing edge to v such that for any vertex w with an outgoing edge to v, w does not have an outgoing edge to u.*

Consider vertex $v_{1,1}$. We know that in every other part, there exist a vertex with an outgoing edge to $v_{1,1}$. Without loss of generality, we assume that for every j, vertex $v_{j,1}$ is the vertex with an outgoing edge to $v_{1,1}$.

Also, by Lemma 3, we know that there exists an index k such that $v_{k,1}$ has no incoming edge from any $v_{k',1}$ for $k' \notin \{1, k\}$. Again, without loss of generality, we suppose that $k = 2$. Therefore, we have that for every $i > 1$, vertex $v_{i,1}$

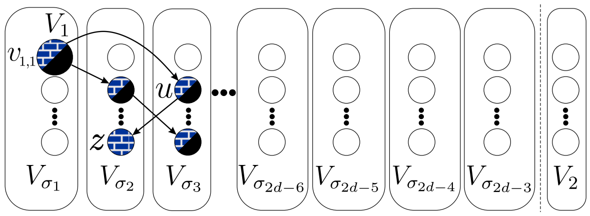

Figure 4: An illustration of the final setting of Lemma 4. Striped vertices are σ-reachable and black vertices are σ-rightward-reachable. Vertex z is σ-reachable but not σ-rightward-reachable because the path from $v_{1,1}$ uses the edge from u to z which is not rightward.

has an outgoing edge to $v_{1,1}$ and for every $i > 2$, $v_{i,1}$ does not have an outgoing edge to $v_{2,1}$. By definition, we know that for every $i \neq 2$, there exists a vertex in part V_i with an outgoing edge to $v_{2,1}$. Without loss of generality, we suppose that for every $i > 2$, this vertex in part V_i is $v_{i,2}$.

Definition 2. *Consider a sequence of indices $\sigma = \sigma_1, \sigma_2, ..., \sigma_k$, such that $\sigma_1 = 1$. Given σ, we say a vertex $v_{\sigma_i,j}$ is σ-reachable if there exists a rainbow path from $v_{1,1}$ to $v_{\sigma_i,j}$ in $G[\{V_{\sigma_1}, V_{\sigma_2}, \ldots, V_{\sigma_k}\}]$. Moreover, we say an edge in $G[\{V_{\sigma_1}, V_{\sigma_2}, \ldots, V_{\sigma_k}\}]$ is σ-rightward if it is of the form $(v_{\sigma_{j,k}}, v_{\sigma_{j',k'}})$ where $j < j'$. A vertex $v_{\sigma_i,j}$ is σ-rightward-reachable if there exists a rainbow path from $v_{1,1}$ to $v_{\sigma_i,j}$ via σ-rightward edges.*

As we mentioned before, it is sufficient to show if G contains at least $2d-2$ parts (and $d \geq 3$), then we have a rainbow cycle in G. We use induction to prove this claim. For the base case $d = 3$, it has already shown in [Chaudhury *et al.*, 2021a] that $\mathsf{R}_p(3) = 3$, which means $\mathsf{R}_p(3) \leq 2 \times 3 - 3$. Now, suppose that the claim holds for every $d' < d$ and our goal is to prove the claim for d. As a contradiction, we suppose that G does not admit any rainbow cycle. We start by proving Lemma 4.

Lemma 4. *There exists a sequence of form $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_{2d-3}$ such that for every $1 \leq i \leq 2d-3$, we have $\sigma_i \in [1, 2d-2]$ and the following properties hold:*

- $\sigma_1 = 1$.
- *For every $1 \leq i \leq 2d-3$, we have $\sigma_i \neq 2$.*
- *For every $2 \leq i \leq 2d-4$, there are $\lceil \frac{i}{2} \rceil$ σ-rightward-reachable vertices in V_{σ_i}.*
- *There are $d-2$ σ-rightward-reachable vertices in $V_{\sigma_{2d-3}}$.*

Let σ be the sequence that satisfies the properties of Lemma 4. In Lemma 5, we prove another property for such a sequence.

Lemma 5. *For every sequence σ with properties mentioned in Lemma 4 and every $2 \leq i \leq 2d-3$, vertices $v_{\sigma_i,1}$ and $v_{\sigma_i,2}$ are not σ-reachable.*

Definition 3. *If we consider S as a subset of $\{1, 2, ..., d\}$, part V_i is one-way S-corresponding to part V_j, if and only if for each vertex $v_{i,k}$ such that $k \in S$, it has an outgoing edge to $v_{j,l} \in V_j$, such that $l \in S$.*

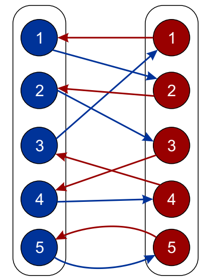

Figure 5: In this example, the left part is one-way $\{1, 2, 3\}$-corresponding to the right part. Moreover, the left part and the right part are $\{1, 2, 3, 4\}$-corresponding, since the left part is one-way $\{1, 2, 3, 4\}$-corresponding to the right part and vice versa.

Definition 4. *Parts V_i and V_j are S-corresponding, if and only if:*

- *V_i is one-way S-corresponding to V_j*
- *V_j is one-way S-corresponding to V_i*

See Figure 5 for an illustrative example.

Lemma 6. *There are three pairwise $\{1, 2\}$-corresponding parts.*

Finally, note that by Lemma 6, there exists three pairwise $\{1, 2\}$-corresponding parts in G. Therefore, if we consider the induced subgraph G' of G containing vertices with indices $1, 2$ in these three parts, since $G \in \Pi_{*,d}$, G' must belong to $\Pi_{*,2}$. However, we know that $\mathsf{R}_p(2) = 2$, which means that G' cannot have more than two parts. This contradiction shows that, if graph contains at least $2d-2$ parts, then it has a rainbow cycle. Hence, $\mathsf{R}_p(d) \leq 2d-3$.

Improving the upper-bound to $2d-4$. We end this section by a discussion on how we can improve the upper bound to $2d-4$. Recall the definition of $\mathsf{H}(\ell)$. As we show in Section 6, we have $\mathsf{H}(4) = 7$. This means that for every set W of parts with $|W| = 4$, for any vertex $v \in G[W]$, there are at least 7 other vertices that have a rainbow path to v in $G[W]$. Since the graph is a permutation graph, the inverse direction is also true: for any vertex $v \in G[W]$, v has rainbow paths to at least 7 different vertices in $G[W]$. We use this fact to decrease the upper bound on $\mathsf{R}_p(d)$ by one.

Consider part V_1 and three arbitrary parts other than V_2 (i.e., V_3, V_4, and V_5). It is guaranteed that vertex $v_{1,1}$ has rainbow paths to at least 7 different vertices in these three parts. Therefore, by the pigeonhole principle, vertex $v_{1,1}$ has rainbow paths to 3 vertices in one of these parts. Assume without loss of generality that this part is V_5. Now, we create a shortcut in the sequence by replacing σ_2, σ_3, σ_4, σ_5 with $3, 4, 5$. Note that though parts V_3 and V_4 might violate the properties of the sequence (e.g., rainbow paths to V_5 are not necessarily σ-rightward-reachable), but V_5 can be treated the same way as V_{σ_5} in the previous sequence, which was the first part with 3 σ-rightward-reachable vertices. Therefore, we can continue constructing the sequence from V_5 in the same way as we construct the sequence (first, add σ_6 and σ_7, next σ_8 and σ_9, and so on). This way, we save one part in the sequence and therefore, the length of the sequence is reduced to $2d-4$. Hence, we can conclude that if we have $\max(4, 2d-4)$ parts,

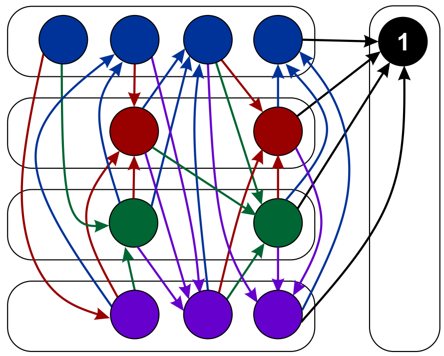

Figure 6: In this figure, you can find a compact form of a graph that shows $\mathsf{H}(5) = 11$. Due to lack of space and for convenience, here we only show the induced subgraph of the vertices that have a rainbow path to vertex 1. Let G be the graph in this figure. In order to construct the entire Graph, one can proceed as follows. Merge G and the graph constructed in the proof of Lemma 7 for $\ell = 5$ (G'). The vertices of each part in the union graph are the union of the vertices in the corresponding parts in G and G'. Similarly, the edges in the union graph are the union of the edges in G and G'. In addition, some of the vertices in G do not have incoming edges from some other parts. For such pairs of vertices and parts, we choose an arbitrary vertex from the corresponding part of G' and add a directed edge to that vertex.

then we have a rainbow cycle. As a result, $\mathsf{R}_p(d) \leq 2d - 4$ for $d \geq 4$.

6 Experiments

In order to evaluate $\mathsf{H}(\ell)$, we performed a set of experiments to calculate $\mathsf{H}(\ell)$ for small values of ℓ. Our algorithm inputs ℓ, x and performs an exhaustive search to find a counter-example for $\mathsf{H}(\ell) > x$. By the definition of $\mathsf{H}(\ell)$, this counter-example must have at most x vertices with a rainbow path to a specific vertex v. If such an example is found, we have $\mathsf{H}(\ell) \leq x$. Otherwise, when there is no such example, we can imply that $\mathsf{H}(\ell) > x$. The overall result of running this experiment is shown in Table 1.

ℓ	Lower bound	Upper bound
2	1	1
3	3	3
4	7	7
5	11	11
6	15	17
7	-	25

Table 1: Lower bounds and upper bounds on $\mathsf{H}(\ell)$ obtained by the experiments.

As you can see in Table 1, for $2 \leq \ell \leq 5$, the exact value of $\mathsf{H}(\ell)$ is determined by the experiments. Also, for $\ell = 6, 7$, our experiments provide an upper bound on $\mathsf{H}(\ell)$. Recall that by Theorem 1, we have $\mathsf{H}(\ell) \in \Omega(\ell^2/\ln \ell)$. In Lemma 7, we prove an upper bound of $O(n^2)$ on $\mathsf{H}(\ell)$.

Lemma 7. *we have* $\mathsf{H}(\ell) \leq (\ell - 1)(\ell - 2) + 1$.

Note that the upper bound provided by Lemma 7 exactly matches the upper bounds for $\mathsf{H}(2), \mathsf{H}(3)$, and $\mathsf{H}(4)$. However, for $\mathsf{H}(5)$ this upper bound is not tight. In Figure 6, a tight example for $\mathsf{H}(5)$ is shown. Based on the results extracted from the experiments, our conjecture is as follows.

Conjecture 1. *We conjecture that* $\mathsf{H}(\ell) = \lfloor \frac{\ell^2}{2} \rfloor - 1$.

Note that if Conjecture 1 holds, then using Lemma 1, we have $\mathsf{R}(d) \in O(d)$.

We also performed similar experiments to evaluate $\mathsf{H}_p(\ell)$ which is an analogous of $\mathsf{H}(\ell)$ for permutation graphs. Formally,

$$\mathsf{H}_p(\ell) = \min_{G \in \Pi_{\ell,*}} \min_{v \in G} \quad f_G(v),$$

where $f_G(v)$ is the number of the vertices in G that have a rainbow path to v. Interestingly, the results were exactly the same as the previous case stated.

Conjecture 2. *We conjecture that* $\mathsf{H}_p(\ell) = \mathsf{H}(\ell)$.

Similar to Lemma 1, we can prove a simple relation between $\mathsf{H}_p(\ell)$ and $\mathsf{R}_p(d)$.

Lemma 8. *Given that for some* ℓ, $\mathsf{H}_p(\ell) > (d-1)(\ell-1)$, *we have* $\mathsf{R}_p(d) < \ell$.

Lemma 9. *For* $d \geq 3$, $\mathsf{H}_p(\ell) \geq \frac{\ell^2}{2} - 1$ *implies* $\mathsf{R}_p(d) \leq 2d - 3$.

Proof. We have

$$\begin{aligned} \mathsf{H}_p(2d-2) &\geq \frac{(2d-2)^2}{2} - 1 \\ &= 2d^2 - 4d + 1 \\ &= (d-1)((2d-2)-1) + (d-2) \\ &> (d-1)((2d-2)-1). \qquad d > 2 \end{aligned}$$

Therefore, by Lemma 8, $\mathsf{R}_p(d) < 2d - 2$. Since $\mathsf{R}_p(d)$ is an integer, $\mathsf{R}_p(d) \leq 2d - 3$. □

Lemma 9 shows that even if we prove Conjecture 2 is correct, we cannot get a better upper bound for $\mathsf{R}_p(d)$ with a simple connection between $\mathsf{R}_p(d)$ and $\mathsf{H}_p(\ell)$. However, we believe proving Conjecture 2 would be a good warm-up in the way of proving Conjecture 1.

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