Facility Location Problems with Capacity Constraints: Two Facilities and Beyond

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Abstract

In this paper, we investigate the Mechanism Design aspects of the m-Capacitated Facility Location Problem (m-CFLP) on a line. We focus on two frameworks. In the first framework, the number of facilities is arbitrary, all facilities have the same capacity, and the number of agents is equal to the total capacity of all facilities. In the second framework, we aim to place two facilities, each with a capacity of at least half of the total agents. For both of these frameworks, we propose truthful mechanisms with bounded approximation ratios with respect to the Social Cost (SC) and the Maximum Cost (MC). When $m > 2$, the result sharply contrasts with the impossibility results known for the classic m-Facility Location Problem, where capacity constraints are not considered. Furthermore, all our mechanisms are optimal with respect to the MC and optimal or nearly optimal with respect to the SC among anonymous mechanisms. For both frameworks, we provide a lower bound on the approximation ratio that any truthful and deterministic mechanism can achieve with respect to the SC and MC.

1 Introduction

Mechanism Design aims to establish procedures for aggregating the private information of a group of agents to optimize a social objective. However, optimizing the social objective solely based on reported preferences frequently results in undesirable manipulation due to the self-interested behavior of the agents. Therefore, one of the most crucial requirements for a mechanism is the property of *truthfulness*, which ensures that no agent can gain an advantage by misreporting their private information. Unfortunately, this strict property often conflicts with optimizing the social objective, so the output of a truthful mechanism is oftentimes sub-optimal. To quantify the loss in efficiency, Nisan and Ronen introduced the concept of *approximation ratio*. This quantity represents the highest achievable ratio between the social objective obtained by a truthful mechanism and the optimal social objective among all possible agents' reports [Nisan and Ronen, 1999]. One of the classic examples of these problems is the m-Facility Location Problem (m-FLP). In its most basic form, the m-FLP consists in locating m facilities amongst n self-interested agents. Every agent needs to access a facility, so they would prefer to have one of the facilities placed as close as possible to their position. Moreover, each facility can serve any number of agents, thus every mechanism just needs to return the positions of the facilities. The agents are then free to decide which facility to use, without considering any possible overload.

In this paper, we study the m-Capacitated Facility Location Problem (m-CFLP) on the line [Pal *et al.*, 2001]. The m-CFLP is a natural extension of the m-FLP in which every facility has a capacity limit. Considering facilities with capacity constraints is a natural approach for modeling scenarios where facilities offer a limited resource, as it happens in distribution planning [Pochet and Wolsey, 1988] and telecommunication network design [Boffey, 1989; Chardaire, 1999]. For instance, the facilities represent servers while the agents represent tasks awaiting execution, or facilities could be grocery shops and agents the customers in need of service.

From a mechanism design perspective, the study of the m-FLP and m-CFLP differs significantly, allowing to elude the impossibility results known for the m-FLP [Fotakis and Tzamos, 2014; Walsh, 2020]. In particular, we show that this is the case when we have m facilities with equal capacity k and the number of agents is $n = km$. For this class of problems, we characterize both the upper and lower bounds of the approximation ratio of anonymous, truthful, and deterministic mechanisms for $m \geq 2$, showing that both are bounded, and that they coincide, making the bounds tight. It is also noteworthy that the study of the m-CFLP is contingent upon the specifics of the problem. Indeed, the properties of any mechanism depend on factors such as whether different facilities have different capacities [Aziz *et al.*, 2020a], whether the total capacity is larger than the number of agents [Walsh, 2022], or whether the capacity of each facility is lower than a critical threshold[Aziz *et al.*, 2020a]. For this reason, the few results providing tight lower bounds on the approximation ratio are limited to very specific settings [Aziz *et al.*, 2020a].

Our Contribution. In this paper, we study two relevant frameworks for the m-CFLP from a Mechanism Design perspective. First, we study the m-CFLP with equi-capacitated facilities and no spare capacity, i.e. the m-CFLP in which all the facilities have the same capacity, namely k, and the total capacity of the facilities equals the number of agents. We present two truthful and anonymous mechanisms, the Propagating Median Mechanism (PMM) and the Propagating InnerPoint Mechanism (PIPM). We show that both the PMM and the PIPM have a bounded approximation ratio with respect to the Social Cost (SC) and the Maximum Cost (MC), regardless of the value of m. This result stands in contrast with the classic results for the m-FLP, according to which no mechanism can be deterministic, anonymous, truthful, and achieve a finite approximation ratio when $m > 2$, even on the line [Fotakis and Tzamos, 2014; Walsh, 2020]. We then present three lower bounds for the approximation ratio for the m-CFLP with equi-capacitated facilities and no spare capacities. In particular: (i) no truthful and deterministic mechanism achieves an approximation ratio lower than 2 with respect to MC. Thus, PMM and PIPM are optimal with respect to this metric. (ii) When $k > 3$, no truthful and deterministic mechanism can achieve an approximation ratio lower than 3 with respect to the SC. (iii) No truthful, deterministic, and anonymous mechanism achieves an approximation ratio with respect to SC lower than $\left(\frac{k(m-1)}{2}+1\right)$, if m is odd, or lower than $\left(\frac{km}{2}-1\right)$ if m is even. In particular, the PMM and the PIPM are the best possible truthful, deterministic, and anonymous mechanisms for odd and even m, respectively.

We then study the 2-CFLP with abundant facilities, in which we have two facilities capable to accommodate at least half of the agents. This framework has been studied under further assumptions: (i) in [Aziz *et al.*, 2020a], the authors studied the case in which n is even, $c_1 = c_2 = \frac{n}{2}$, and proposed the InnerPoint (IM) Mechanism, (ii) in [Walsh, 2022], the author studied the case in which n is odd and $c_1 = \left\lceil \frac{n}{2} \right\rceil$, $c_2 = \left\lfloor \frac{n}{2} \right\rfloor$, and proposed the InnerChoice (IC) Mechanisms introduced, (iii) again, in [Walsh, 2022], the author also studied the case in which n is arbitrary but $c_1 = c_2$, and proposed the InnerGap (IG) Mechanism. However, this is the first time that a study of a framework encompassing all these different cases has been conducted. We propose the Extended InnerGap (EIG) Mechanism, which generalizes and includes IM, IC, and IG. The EIG is strong Group Strategyproof, thus truthful, and attains a bounded approximation ratio with respect to the SC and MC. We then provide a lower bound on the approximation ratio of any truthful mechanism with respect to the SC and MC and show that the EIG is optimal with respect to the MC. Moreover, the EIG is optimal with respect to the SC whenever $n \geq \bar{c} + \sqrt{\bar{c}}$, where $\bar{c} = \max\{c_1, c_2\}$.

In Table 1, we summarize our findings in terms of lower and upper bounds for the cases we study. Due to space limits, some proofs are deferred to the Appendix.

Related works. The m-Facility Location Problem (m-FLP) and its variants are relevant problems in several applied fields such as disaster relief [Balcik and Beamon, 2008], supply chain management [Melo *et al.*, 2009], healthcare [Ahmadi-Javid *et al.*, 2017], clustering [Hastie *et al.*, 2009], and public facilities accessibility [Barda *et al.*, 1990]. The Mechanism Design study of the m-FLP was first explored by Procaccia and Tennenholtz, who laid the foundation of this field in their pioneering work [Procaccia and Tennenholtz, 2013]. Subsequently, several mechanisms with small constant approximation ratios for locating one or two facilities on trees, circles, and general graphs have been introduced [Alon *et al.*, 2010; Dokow *et al.*, 2012; Feldman and Wilf, 2013; Filimonov and Meir, 2021; Lu *et al.*, 2010; Lu *et al.*, 2009; Filos-Ratsikas *et al.*, 2017; Meir, 2019; Tang *et al.*, 2020]. However, these positive results are limited to cases where the mechanism designer needs to place 2 facilities or the mechanism is not deterministic. Indeed, no deterministic, anonymous, and truthful mechanism can place more than two uncapacitated facilities while achieving a bounded approximation ratio even on the line [Fotakis and Tzamos, 2014; Walsh, 2020].

The m-Capacitated Facility Location Problem (m-CFLP) is a natural extension of the m-FLP, in which each facility has a maximum number of agents that it can serve [Brimberg *et al.*, 2001; Pal *et al.*, 2001; Aardal *et al.*, 2015]. The Mechanism Design aspects of the m-CFLP have received relatively little attention until recently. Indeed, the game theoretical framework for the m-CFLP that we consider was first introduced in [Aziz *et al.*, 2020a]. In this paper, the authors studied various truthful mechanisms (such as the InnerPoint Mechanism and the Extended Endpoint Mechanism) and studied their approximation ratios. Notably, only mechanisms capable of locating two facilities achieve a bounded approximation ratio. A more theoretical analysis of the problem has been then presented in [Walsh, 2022], where the author demonstrated that no mechanism can locate more than two capacitated facilities while being truthful, anonymous, and Pareto optimal. Lastly, papers that deal with different Mechanism Design aspects of the m-CFLP are [Auricchio *et al.*, 2023], where the m-CFLP is studied in a Bayesian setting, and [Aziz *et al.*, 2020b], where the authors investigate the case in which there is only one capacitated facility to place and it cannot accommodate all the agents.

2 Preliminaries

In this section, we introduce the two m-CFLP frameworks and fix the notation. Throughout the paper, we assume that the agents lay on a line and denote with $\vec{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$ the vector containing their positions. Moreover, we assume $m > 1$, as 1-CFLP is equivalent to the classic 1-FLP.

The m-CFLP with equi-capacitated facilities and no spare capacity. In the first framework, we have m facilities whose capacity is the same, *i.e.* $c_j = k > 1$ for every $j \in [m]$, and the total capacity of the facilities equals the total number of agents, hence $n = mk$. We call this framework the m-CFLP with *equi-capacitated facilities* and *no spare capacity*. Since in this setting all the facilities have the same capacity, a facility location is defined by two objects: (i) a m-dimensional vector $\vec{y} = (y_1, \dots, y_m)$ whose entries are the positions of the facilities on the line, and (ii) a matching $\mu \subset [n] \times [m]$ that determines how the agents are assigned to facilities, i.e. $(i, j) \in \mu$ if and only if the agent at x_i is assigned to y_j. Due to the capacity constraints, the degree of every vertex $j \in [m]$ according to μ is at most k. Since every agent is assigned to only one facility, the degree of $i \in [n]$ according to μ is 1.

The 2-CFLP with abundant facilities. In the second framework we consider, we have two facilities whose capaci-

	Social Cost			Maximum Cost	
	LB	LB*	UB	LB	UB
$c_j = k$ $n = km$	3	$\frac{k(m-1)}{2} + 1$ (m odd) $\frac{km}{2} - 1$ (m even)	$\frac{k(m-1)}{2} + 1$ (m odd) $\frac{km}{2} - 1$ (m even)	2	2
$c_1, c_2 \geq \lfloor \frac{n}{2} \rfloor$ $c_1 + c_2 \geq n$	3	$n - \bar{c} - 1$	$\max\{n - \bar{c}, \frac{\bar{c}}{n-\bar{c}}\} - 1$	2	2

Table 1: Each row contains the Lower and Upper Bounds (LB and UB, respectively) with respect to the Social and Maximum Cost for a different class of problems for the m-CFLP. The value $\bar{c}$ is the maximum capacity of the facilities. The LB column contains the lower bounds for the class of truthful and deterministic mechanisms. The LB^* column contains the lower bounds for the class of mechanisms that are truthful, deterministic, and anonymous.

ties, namely c_1 and c_2 are such that $\lfloor \frac{n}{2} \rfloor \leq c_2, c_1 \leq n-1$. We call this framework 2-CFLP with *abundant capacities*. Since the facilities may have different capacity, eliciting two positions, namely y_1 and y_2, and an agent-to-facility assignment μ is not sufficient, as we need to also specify the capacity of the two facilities. In particular, in this framework, a facility location is defined by three objects: (i) a bi-dimensional vector $\vec{y} = (y_1, y_2)$ whose entries are the positions of the facilities, (ii) a permutation $\pi : [2] \to [2]$ that specifies the capacity of each facility, so that if $\pi(1) = j$ the facility at y_1 has capacity c_j and the facility built at y_2 has capacity c_i with $i \neq j$ and $i, j \in [2]$, and (iii) a matching $\mu \subset [n] \times [2]$ that determines how the agents are assigned to facilities. The degree of every vertex $j \in [2]$ according to μ must be at most $c_{\pi(j)}$, while the degree of every $i \in [n]$ according to μ is 1.

Mechanism Design Framework for the m-CFLP. In both frameworks, given the positions of the facilities $\vec{y}$ and a matching μ, we define the cost of an agent positioned in x_i as $c_{i,\mu}(x_i, \vec{y}) = |x_i - y_j|$, where (i, j) is the unique edge in μ adjacent to i. Finally, a cost function is a map $C_\mu : \mathbb{R}^n \times \mathbb{R}^m \to [0, +\infty)$ that associates to $(\vec{x}, \vec{y})$ the overall cost of placing the facilities at $\vec{y}$ and assigning the agents positioned at $\vec{x}$ according to μ.[1] For both frameworks, given a vector $\vec{x} \in \mathbb{R}^n$ containing the agents' positions, the *m-Capacitated Facility Location Problem* with respect to the cost C, consists in finding the locations for m facilities and a matching μ that minimize the function $\vec{y} \to C(\vec{x}, \vec{y})$. Throughout the paper, we consider the *Social Cost* (SC), defined as the sum of all the agents' costs, i.e., $SC(\vec{x}, \vec{y}) = \sum_{i\in[n]} c_i(x_i, \vec{y}) = \sum_{i\in[n]} |x_i - y_j|$ and the *Maximum Cost* (MC), defined as the maximum cost among all agents' costs, i.e. $MC(\vec{x}, \vec{y}) := \max_{i\in[n]} c_i(x_i, \vec{y})$.

A mechanism for the m-CFLP is a function f that takes the private information of n self-interested agents as input and returns a facility location. Thus, for the m-CFLP with equi-capacitated facilities and no spare capacity, the mechanism returns a set of locations $\vec{y}$ and a matching μ between the agents and the facilities. A mechanism f is said to be *truthful* (or *strategy-proof*) if, for every agent, its cost is minimized when it reports its true position, i.e., for any $x_i' \in \mathbb{R}$, we have $c_i(x_i, f(\vec{x})) \leq c_i(x_i, f(\vec{x}_{-i}, x_i'))$ where x_i is the agent's real position and $\vec{x}_{-i}$ is the vector $\vec{x}$ without its i-th component.

[1] In what follows, we omit μ from the indexes of c and C if it is clear from the context which matching we are considering.

A mechanism f is strong Group Strategyproof (GSP) if no group of agents can misreport their positions in such a way that (i) the cost of every agent in the group after manipulating is less than or equal to the cost they would get by reporting truthfully, (ii) at least one of the agents in the group incurs a strictly lower cost after the group manipulation.

Albeit a truthful mechanism prevents agents from misreporting their positions, their output is usually suboptimal. To evaluate this efficiency loss, we consider the approximation ratio of the mechanism introduced in [Nisan and Ronen, 1999]. Given a truthful mechanism f, its approximation ratio with respect to the SC is defined as $ar_{SC}(f) := \sup_{\vec{x}\in\mathbb{R}^n} \frac{SC_f(\vec{x})}{SC_{opt}(\vec{x})}$, where $SC_f(\vec{x})$ is the SC of the solution returned by f and $SC_{opt}(\vec{x})$ is the optimal SC achievable on instance $\vec{x}$. Similarly, the approximation ratio of f with respect to the MC (namely, $ar_{MC}(f)$) is the highest ratio between the MC achieved by f and the optimal MC.

3 The m-CFLP with Equi-capacitated Facilities and No Spare Capacity

In this section, we focus on the Mechanism Design aspects of the m-CFLP with equi-capacitated facilities and no spare capacity, i.e. given m the number of facilities and their capacity k, it holds $n = mk$. Given $j \in [m]$ and a vector $\vec{x}$ containing the agents' positions ordered from left to right, i.e. $x_i \leq x_{i+1}$ for every $i \in [n-1]$, we define I_j as follows

$$I_j = \{x_{(j-1)k+i} \;\text{ where }\; i \in [k]\}. \tag{1}$$

Notice that, since every facility has the same capacity, the optimal solution to the m-CFLP with respect to the SC places the facilities at the positions $\vec{y} = (y_1, \ldots, y_m)$, where each y_j is a median of set I_j, and then it assigns every agent whose position is in I_j to y_j. Likewise, the optimal solution with respect to the MC places the facilities at $y_j = \frac{x_{(j-1)k+1} + x_{jk}}{2}$ and then it assigns the agents in I_j to y_j.

3.1 The Propagating Median Mechanism

We now introduce and study our first truthful mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity, the Propagating Median Mechanism (PMM).

Mechanism 1 (Propagating Median Mechanism (PMM))**.** *Let n be the total number of agents, m the number of facilities to place, and $k = \frac{n}{m} \in \mathbb{N}$ the capacity of each facility.*

Let us set $r = \left\lfloor \frac{m+1}{2} \right\rfloor$. The routine of the PMM is as follows: (i) First, we locate the facility y_r at one of the medians of I_r, i.e. $y_r = x_{k(r-1)+\lfloor \frac{k+1}{2} \rfloor}$. (ii) Second, we determine the positions of the other facilities via the following iterative routine. For any $l \geq r$, let us be given the position of the l-th facility, namely y_l. Then the position of the $(l+1)$-th facility is $y_{l+1} = \max\{x_{kl+1}, x_{kl} + d_l\}$, where d_l is the distance between y_l and $x_{kl} := \max_{x \in I_l} x$. Similarly, given $l \leq r$ and the position of the l-th facility, namely y_l, the $(l-1)$-th facility is placed at $y_{l-1} = \min\{x_{k(l-1)}, x_{k(l-1)+1} - d_l\}$, where d_l is the distance between y_l and $x_{k(l-1)+1} := \min_{x \in I_l} x$. (iii) Finally, all the agents in I_j are assigned to y_j.

Let $\vec{y} = (y_1, \ldots, y_m)$ be the position of the facilities returned by the PMM on a given instance $\vec{x}$. It is easy to see that the entries of $\vec{y}$ are non-decreasing, i.e. $y_j \leq y_{j+1}$ for every $j \in [m-1]$. Moreover, the PMM assigns every agent to its closest facility, so that $\min_{\ell \in [m]} |x_i - y_\ell| = |x_i - y_j|$, where $j \in [m]$ is the only index for which it holds $x_i \in I_j$.

Theorem 1. *The PMM is truthful.*

Proof. Toward a contradiction, let x_i be the real position of an agent able to manipulate by reporting x_i' instead of its real position x_i. We denote with $\vec{y}$ and I_j the positions of the facilities returned by the PMM and the sets in (1) on the truthful input, respectively. Similarly, we denote with $\vec{y}\,'$ and I_j' the positions of the facilities returned by the PMM and the sets in (1) when the agent at x_i reports x_i', respectively. Since the other case is symmetric, we assume that $y_r \leq x_i$, where $r = \left\lfloor \frac{m+1}{2} \right\rfloor$ and $y_r = x_{k(r-1)+\lfloor \frac{k+1}{2} \rfloor}$, is a median of I_r.

We show that no agent in I_r can manipulate, the case in which $x_i \notin I_r$ is similar and deferred to the Appendix. Toward a contradiction, let us assume that $x_i \in I_r$. Then, if $x_i = y_r$, the cost of the agent is null, thus it cannot benefit by misreporting. Therefore, it must be that $y_r < x_i$. If $x_i' < y_r$, we have that $x_i' \in I_\ell'$ with $\ell \leq r$. In this case, we have that $y_r' \leq y_r$, thus $y_l' \leq \cdots \leq y_r' \leq y_r < x_i$, which means that the manipulating agent is assigned to a facility that is not closer than y_r, hence its cost does not decrease after the manipulation. Finally, let us consider the case $y_r < x_i'$. In this case, we have that $y_r' = y_r$, since the median of I_r is the same regardless of whether the manipulating agent reports truthfully or not. Thus, if $x_i' \in I_r'$, it will still be assigned to $y_r' = y_r$, which brings no benefit to the manipulative agent. So it must be that $x_i' \notin I_r'$, then $x_i' \in I_\ell'$, where $\ell > r$, thus the manipulating agent is assigned to $y_\ell' \geq y_{r+1}'$, since $\ell > r$. Let us denote with x_{rk}' the position of the (rk)-th agent from the left in the manipulated instance (x_i', x_{-i}). Since $x_i' > x_i$, we have $x_{rk}' \geq x_{rk} \geq x_i$. We have that $y_{r+1}' = \max\{x_{rk+1}', x_{rk}' + |y_r' - x_{rk}'|\} \geq x_{rk}' + |y_r' - x_{rk}'| \geq x_i + |y_r - x_{rk}'| \geq x_i + |y_r - x_i|$, thus the cost of being assigned to y_ℓ' is no less than the cost of being assigned to y_r. □

Although the PMM is truthful, it is not strong GSP, as the following example shows.

Example 1. *Let us fix $n = 9$, $m = 3$, and $k = 3$. Let us consider the following instance: $x_1 = x_2 = x_3 = 0$, $x_4 = x_5 = 1$, $x_6 = 2$, $x_7 = 2.5$, and $x_8 = x_9 = 4$. The PMM places the facilities at $y_1 = 0$, $y_2 = 1$, and $y_3 = 3$. In this instance, the agents at x_6 and x_7 can collude: indeed, if x_6 reports $x_6' = 1$, the PMM places the facilities at $y_1 = 0$, $y_2 = 1$, and $y_3 = 2.5$, thus the cost of x_7 decreases.*

To conclude, we provide an analysis of the approximation ratio of the PMM with respect to the SC and MC. In particular, we prove that $ar_{SC}(\text{PMM})$ and $ar_{MC}(\text{PMM})$ are finite.

Theorem 2. *It holds $ar_{SC}(PMM) = k \left\lfloor \frac{m}{2} \right\rfloor + 1$.*

Proof. Let $r = \left\lfloor \frac{m+1}{2} \right\rfloor$ and $\vec{x}$ be a vector containing all the agents' reports ordered from left to right. Let $SC_{opt}(\vec{x})$ be the optimal SC for $\vec{x}$, then it holds that $SC_{opt}(\vec{x}) = \sum_{j \in [m]} SC_{opt}(I_j)$, where $SC_{opt}(I_j)$ is the SC of the agents whose report is in I_j according to the optimal solution. Similarly, let $SC_{\text{PMM}}(\vec{x})$ be the SC of instance $\vec{x}$ according to the output of PMM, then it holds that $SC_{\text{PMM}}(\vec{x}) = \sum_{j \in [m]} SC_{\text{PMM}}(I_j)$, where $SC_{\text{PMM}}(I_j)$ is the SC of the agents whose report is in I_j according to the output of PMM. We denote with $\vec{y}$ the vector containing the locations of the facilities returned by the PMM and define $J \subset [m]$ as the set of indexes $j \in [m]$ such that $y_j \in [x_{(j-1)k+1}, x_{jk}]$. We notice that J is non-empty since $r \in J$ by definition of the PMM. We notice that $SC_{opt}(I_r) = SC_{\text{PMM}}(I_r)$. Let us now consider $j \in J$ such that $j \neq r$. Since $y_j \in [x_{(j-1)k+1}, x_{jk}]$, we have that $SC_{\text{PMM}}(I_j) \leq (k-1)|x_{(j-1)k+1} - x_{jk}| \leq (k-1)SC_{opt}(I_j)$. Let us now consider $j \notin J$ and, without loss of generality, let us assume that $r < j$ since the other case is symmetric. Since $j \notin J$, there exists an index $\ell \in J$ such that $y_j = x_{k\ell} + |y_\ell - x_{k\ell}|$. If $\ell \neq r$, we have that $SC_{\text{PMM}}(I_j) \leq k|y_\ell - x_{k\ell}| \leq kSC_{opt}(I_\ell)$ since, for every $x_t \in I_j$, we have $x_{k\ell} \leq x_t \leq y_j$. Similarly, if $\ell = r$, we have that $SC_{\text{PMM}}(I_j) \leq k|y_r - x_{kr}|$. Therefore, it holds

$$SC_{\text{PMM}}(\vec{x}) \leq \sum_{J \ni j \neq r} (k\lambda_j + k - 1)SC_{opt}(I_j) + SC_{opt}(I_r) + k\gamma_r|y_r - x_{kr}| + k\gamma_l|y_r - x_{k(r-1)+1}|.$$

Hence $SC_{\text{PMM}} \leq \sum_{j \in J, j \neq r}(k\lambda_j + k - 1)SC_{opt}(I_j) + (k\Gamma + 1)SC_{opt}(I_r)$ where (i) λ_j, for $j \in J$ and $j \neq r$, is the number of $i \in [m]$ such that $i \neq j$ and $y_i = x_{jk} + |y_j - x_{jk}|$ if $j > r$ and the number of $i \in [m]$ such that $i \neq j$ and $y_i = x_{jk} - |y_j - x_{jk}|$ if $j < r$, (ii) γ_l is the number of $i \in [m]$ such that $i < r$ and $y_i = x_{k(r-1)+1} - |y_r - x_{k(r-1)+1}|$, (iii) γ_r is the number of $i \in [m]$ such that $i > r$ and $y_i = x_{kr} + |y_r - x_{kr}|$, and (iv) $\Gamma = \max\{\gamma_r, \gamma_l\}$.

If we set $K_r = (k\Gamma + 1)$ and $K_j = (k\lambda_j + k - 1)$ for every $j \in J$ such that $j \neq r$, we have $ar_{SC}(\text{PMM}) \leq \frac{\sum_{j \in J} K_j SC_{opt}(I_j)}{\sum_{j \in J} SC_{opt}(I_j)}$, thus $ar_{SC}(\text{PMM}) \leq \max_{j \in J}\{K_j\}$. Since, $\lambda_j \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$ and $\Gamma \leq \left\lfloor \frac{m}{2} \right\rfloor$, we have $K_j \leq k \left\lfloor \frac{m}{2} \right\rfloor + 1$ for every $j \in J$, therefore $ar_{SC}(\text{PMM}) \leq k \left\lfloor \frac{m}{2} \right\rfloor + 1$.

Finally, to prove that $ar(\text{PMM}) = \left(k \left\lfloor \frac{m}{2} \right\rfloor + 1\right)$, consider the following instance: $x_1 = \cdots = x_{kr-1} = 0$ and $x_{kr} = \cdots = x_n = 1$. The optimal cost of this instance is 1. The PMM places the facilities as it follows $y_1 = \cdots = y_r = 0$ and $y_{r+1} = \cdots = y_m = 2$, thus the Social Cost of the mechanism is $n - (k \left\lfloor \frac{m+1}{2} \right\rfloor - 1) = k \left\lfloor \frac{m}{2} \right\rfloor + 1$. □

Through a similar argument, we retrieve the approximation ratio of PMM with respect to the MC.

Theorem 3. *It holds ar_{MC}(PMM) $= 2$.*

3.2 The Propagating InnerPoint Mechanism

We now present our second truthful mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity, the Propagating InnerPoint Mechanism (PIPM). The routine of the PIPM is similar to the routine of the PMM, the main difference lies in how it determines the initial facilities. Indeed, the PMM places a facility at the median of $I_{\lfloor \frac{m+1}{2} \rfloor}$, while the PIPM places two facilities: one at the maximum value of $I_{\lfloor \frac{m}{2} \rfloor}$ and one at the minimum value of $I_{\lfloor \frac{m}{2} \rfloor+1}$.

Mechanism 2 (Propagating InnerPoint Mechanism)**.** *Let n be the total number of agents, m be the number of facilities to place, and $k = \frac{n}{m} \in \mathbb{N}$ be the capacity of each facility. Let us set $r = \lfloor \frac{m}{2} \rfloor$. The mechanism runs as follows: (i) First, we locate the facilities y_r and y_{r+1} at the positions x_{rk} and x_{rk+1}, respectively. (ii) To place the other facilities, we run an iterative routine. For $l \geq r+1$, given the position of the l-th facility, namely y_l, we place the $(l+1)$-th facility at the position $y_{l+1} = \max\{x_{kl+1}, x_{kl} + d_l\}$, where d_l is the distance between y_l and x_{kl}. For $l \leq r$, we run a similar iterative routine. Given the position of the l-th facility y_l, the $(l-1)$-th facility is placed at $\min\{x_{k(l-1)}, x_{k(l-1)+1} - d_l\}$, where d_l is the distance between y_l and $x_{k(l-1)+1}$. (iii) Finally, all the agents in I_j are assigned to y_j.*

Due to the similarities between the definition of the PMM and the PIPM, it is possible to adapt the arguments used in the proof of Theorem 1, 2, and 3 to this mechanism. In particular, the PIPM is truthful and achieves a bounded approximation ratio with respect to both the SC and MC.

Theorem 4. *The PIPM is truthful. Moreover, we have that ar_{SC}(PIPM) $= k \lceil \frac{m}{2} \rceil - 1$ and ar_{MC}(PIPM) $= 2$.*

Albeit ar_{MC}(PMM) $= ar_{MC}$(PIPM), the approximation ratios of the two mechanisms with respect to the SC are different. Indeed, we have that ar_{SC}(PMM) $< ar_{SC}$(PIPM) when m is odd and, vice-versa, ar_{SC}(PIPM) $< ar_{SC}$(PMM) when m is even. Finally, it is easy to adapt Example 1 to show that PIPM is not strong GSP.

3.3 Lower Bounds for the Approximation Ratio

To conclude the section, we study the lower bounds for the approximation ratio of truthful mechanisms for the m-CFLP with equi-capacitated facilities and no spare capacity. First, we show that 2 is the best approximation ratio for any truthful and deterministic mechanism with respect to the MC.

Theorem 5. *No deterministic truthful mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity can achieve an approximation ratio with respect to the Maximum Cost that is lower than 2.*

Proof. Let k be the capacity of m facilities, hence $n = mk$ is the total number of agents. Toward a contradiction, let M be a truthful and deterministic mechanism such that $ar_{MC}(M) = 2 - \delta$ where $\delta > 0$. Let us consider the following instance: $x_1 = 0$ and $x_2 = \cdots = x_n = 2$. It is easy to see that the optimal MC is 1. Let y_1 denote the position at which M places the facility to which agent at x_1 and $k-1$ of the agents at 2 are assigned. Since $ar_{MC}(M) < 2$, we have that $y_1 \in [0, 2]$. Given $t > 0$, let us now consider the instance $x_1' = -t$ and $x_2 = \cdots = x_n = 2$. For every $t > 0$, the optimal MC of these instances is $\frac{t+2}{2}$. Since M is truthful, we have that $y_1' \geq y_1$, thus $ar_{MC}(M) \geq \frac{t}{\frac{t+2}{2}} = \frac{2t}{t+2}$. Finally, we notice that the right hand-side of the inequality converges to 2 as $t \to \infty$, thus we have that $ar(M) > 2 - \delta$ for every $\delta > 0$, which is a contradiction. □

We now move to the lower bound for the Social Cost.

Theorem 6. *No deterministic truthful mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity can achieve an approximation ratio with respect to the Social Cost that is lower than 3 whenever $k > 3$.*

Proof. Given m facilities with capacity k, let $n = mk$ be the number of agents. Let us consider the following instance: $x_1 = \cdots = x_{k+1} = 0$ and $x_{k+2} = \cdots = x_n = 2$. It is easy to see that the optimal Social Cost is 2. We now show that the mechanism must place at least one facility at 0 and at least one facility at 2. If all the facilities are placed at 0, the approximation ratio of the mechanism would be higher than $k-1$, which would conclude the proof. Similarly, we conclude that not all the facilities are placed at 2. Finally, let us assume that the facility serving the agents placed at 2 and one of the agents placed at 0, namely y is such that $y \in (0, 2)$. Without loss of generality, it suffices to consider the case in which one agent at 0 shares the facility with agents at 2, since in all other cases the cost of the mechanism increases. In this case, if we move the agent placed at 0 to y, we have that the facility does not change its position (as otherwise an agent placed at y could manipulate by reporting 0), thus the approximation ratio of the mechanism would be at least equal to $k-1$. So the facility that serves both an agent at 0 and $k-1$ agents at 2 must be placed at 2.

Let us consider one of the agents placed at 0 that is not assigned to a facility placed at 2. For every $\epsilon > 0$, we have that if the agent was placed at $1-\epsilon$, it would be still assigned to a facility at 0, as otherwise, it could manipulate by reporting 0 rather than its real position. In this case, the cost of the mechanism is $3-\epsilon$, while the optimal cost is $1+\epsilon$. Since this holds for every $\epsilon > 0$, the approximation ratio with respect to the SC of the mechanism is greater or equal to 3. □

Finally, we present a lower bound for the approximation ratio with respect to the SC of deterministic, anonymous, and truthful mechanisms. We recall that a mechanism M is anonymous if every agent's outcome depends only on its reports, i.e. two agents swapping two different reports causes the mechanism to swap their outcomes.

Theorem 7. *No deterministic, anonymous, and truthful mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity can achieve an approximation ratio with respect to the Social Cost that is lower than $\left(\frac{k(m-1)}{2}+1\right)$ if m is odd or lower than $\left(\frac{km}{2}-1\right)$ if m is even.*

Proof. Let M be a deterministic, anonymous, and truthful mechanism. Let us consider the following instance $x_1 = \cdots = x_{kr+1} = 0$ and $x_{kr+2} = \cdots = x_n = 1$, where

$r = \lfloor \frac{m}{2} \rfloor$. The optimal cost of this instance is 1. First, we show that, according to M, the locations of the facilities serving the agents at 0 are all placed at the same distance from 0. Toward a contradiction, let us assume that M places two facilities at two positions, namely y and y', such that $|y - 0| \neq |y' - 0|$ and that both facilities serve an agent that reported 0. Without loss of generality, let us assume that $|y - 0| = |y| < |y'| = |y' - 0|$. Let us denote with x_i one of the agents who reported 0 that is assigned to y. Let us now consider the instance (y, x_{-i}). Since M is truthful, we must have that the mechanism places a facility at y and that the agent at y is assigned to it, as otherwise, it could misreport by reporting 0. Let us now denote with x_j one of the agents in 0 that is assigned to y'. Since M is anonymous, if x_j reports y, it is assigned to y, which is closer to 0 than y', which contradicts the truthfulness of M. In particular, we infer that all the agents placed at 0 incur the same cost. Similarly, all the agents placed at 1 incur the same cost. Since there is no spare capacity, there exists at least one facility that serves an agent placed at 0 and an agent placed at 1, let us denote with $\lambda \in \mathbb{R}$ its position on the line. Then, the total cost of the mechanism is $C = |\lambda|\big(k\lfloor \frac{m}{2} \rfloor + 1\big) + |1-\lambda|\big(n - k\lfloor \frac{m}{2} \rfloor - 1\big)$. Finally, we notice that $C \geq \big(\frac{k(m-1)}{2} + 1\big)$ if m is odd and $C \geq \big(\frac{km}{2} - 1\big)$ if m is even, which concludes the proof. □

Since the PMM and the PIPM are anonymous, the lower bound in Theorem 7 is tight. Indeed PMM achieves the lower bound for odd m, while PIPM does so for even m. Therefore, for the m-CFLP with equi-capacitated and no spare capacity, PMM and PIPM are the best anonymous, deterministic, and truthful mechanisms for odd and even m, respectively.

4 The 2-CFLP With Abundant Facilities

We now consider the case in which we have to place two facilities capable of accommodating half of the agents. We present the Extended InnerGap (EIG) mechanism, a truthful mechanism that generalizes and includes mechanisms that operate under further assumptions: the InnerPoint (IM) Mechanism [Aziz *et al.*, 2020a], the InnerGap (IG) Mechanism [Walsh, 2022], and the InnerChoice (IC) Mechanism [Walsh, 2022] (see Table 2). We show that EIG achieves a finite approximation ratio with respect to the SC and the MC and corroborate these results by providing lower bounds on the approximation ratio achievable by truthful and deterministic mechanisms with respect to SC and MC. As a consequence, we infer the approximation ratio of the IC and IG mechanisms, which, to the best of our knowledge, were previously unknown.

Mechanism 3 (Extended InnerGap Mechanism). *Let $\bar{c} := \max\{c_1, c_2\}$ and let $\vec{x} \in \mathbb{R}^n$ be the vector containing the agents' report ordered from left to right. Let us fix $y_1 = x_{n-\bar{c}}$, $y_2 = x_{\bar{c}+1}$, and $z = \frac{y_1+y_2}{2}$, let n_1 be the number of agents in $[y_1, z] \cap \{x_i\}_{i\in[n]}$ and n_2 be the number of agents in $(z, y_2] \cap \{x_i\}_{i\in[n]}$. Finally, the output of the EIG over $\vec{x}$ is (i) to place the facility with the largest capacity at y_1 and the other at y_2 if $n_1 \geq n_2$; or (ii) to place the facility with the lowest capacity at y_1 and the other at y_2 if $n_2 > n_1$. In both cases, every agent is assigned to the facility closer to its report.*

Theorem 8. *The EIG is strong GSP, hence truthful.*

	$\forall n \in \mathbb{N}$	$n < c_1 + c_2$	$c_1 \neq c_2$
EIG	Yes	Yes	Yes
EG	Yes	Yes	No
IC	No	No	Yes
IM	No	No	No

Table 2: Frameworks under which the mechanisms operate when $c_1, c_2 \geq \lfloor \frac{n}{2} \rfloor$. From right to left, the column tell us whether the mechanism is capable of working (1) for every number of agents n, (2) when the total capacity is larger than the number of agents, and (3) when the two facilities have different capacities. The EIG (Exended InnerGap) Mechanism is the only mechanism capable of working under no further restriction.

Proof. Let $\vec{x}$ be the true positions of the agents. We denote with $y_1 \leq y_2$ the positions of the facilities according to the EIG on the truthful input. Let $I := \{x_{i_1}, \dots, x_{i_s}\}$ be the real positions of the agents that form a coalition able to manipulate the output of the EIG. Without loss of generality, we assume that I is minimal, that is no subset of the agents in I can collude. We recall that the EIG places the two facilities: one at the $(n - \bar{c})$-th agents' report from the left, namely y_1, and one at the $(\bar{c} + 1)$-th agents' report from the left, namely y_2. Since I is minimal, none of the agents whose true position coincides with y_1 or y_2 takes part in the group manipulation. Hence, if we denote with y_1' and y_2' the positions of the facilities after the group manipulation, we cannot have that $y_1' < y_1$ and $y_2 < y_2'$ at the same time. Let us now consider a coalition of agents I that is able to lower the cost of an agent, whose real location is x_{i_1}, without increasing the cost of the other agents in I. Without loss of generality, let us assume that $x_{i_1} < y_1$, hence $y_1' < y_1$. If $y_1' < y_1$, it must be the case that at least one agent whose real position, namely x_t, was on the right of y_1 reports a position on the left of y_1, i.e. $x_t' \in I_1'$, where x_t' is the misreport of the agent whose real position was x_t. If that agent was assigned to y_2 in the truthful input, it must be that $|x_t - y_2| \geq |x_t - y_1| > |x_t - y_1'|$, since $y_1' < y_1 \leq x_t$. Thus, the agent at x_t is increasing its cost, which contradicts $x_t \in I$. Similarly, if x_t was assigned to y_1 according to the truthful input, its cost still increases after the manipulation, which concludes the proof. □

The EIG mechanism determines the facility position using the same routine used by a percentile mechanism, [Sui *et al.*, 2013]. However, the percentile mechanisms are not strong GSP in general, while the EIG mechanism is (see Example in the Appendix). This difference is due to the fact that the EIG forces the agents to use a specific facility, while the percentile mechanism does not.

Theorem 9. *It holds that $ar_{SC}(EIG) = \max\{(n - \bar{c} - 1), (\frac{\bar{c}}{n-\bar{c}} - 1)\}$. Moreover, it holds that $ar_{MC}(EIG) = 2$.*

Proof. We prove the statement only for the MC, the study of the SC is deferred to the Appendix. Let us denote with I_i the set of agents that are assigned to the facility with capacity c_i according to the optimal solution and, without loss of generality, we assume that all the agents in I_1 are placed to the left of the agents in I_2. The optimal MC is then

$\frac{1}{2}\max\{|\min\{I_1\}-\max\{I_1\}|,|\min\{I_2\}-\max\{I_2\}|\}$. Let $y_1 \leq y_2$ be the position at which the mechanism places the two facilities. Then the MC of the EIG is lower or equal to the MC of assigning all the agents in I_i to the facility at y_i. Finally, since $x_{n-\bar{c}} \in I_1$ and $x_{\bar{c}+1} \in I_2$, we infer that

$$\begin{aligned} MC_{EIG}(\vec{x}) &\leq \max\{\max_{x\in I_1}|x-x_{n-\bar{c}}|,\max_{x\in I_2}|x-x_{\bar{c}+1}|\} \\ &\leq \max\{|x_1-\max_{x\in I_1}\{x\}|,|\min_{x\in I_2}\{x\}-x_n|\} \\ &\leq 2MC_{opt}(\vec{x}), \end{aligned}$$

thus $ar_{MC}(EIG) \leq 2$. Lastly, let us define $\vec{x}$ as $x_1 = \cdots = x_{\bar{c}+1} = 0$, and $x_{\bar{c}+2} = \cdots = x_n = 1$. The optimal cost is 0.5, while the cost of the EIG mechanism is 1. □

We now provide lower bounds on the approximation ratio with respect to both the MC and SC of any truthful and deterministic mechanism for this framework. Our results show that the EIG is optimal or almost optimal for both costs.

Theorem 10. *Let M be a truthful and deterministic mechanism that places two facilities with capacity $c_1, c_2 \geq \lfloor \frac{n}{2} \rfloor$, then we have that $ar_{MC}(M) \geq 2$ and $ar_{SC}(M) \geq 3$. If M is also anonymous, then we have that $ar_{SC}(M) \geq (n-\bar{c}-1)$.*

Proof. We prove only the lower bound with respect to the SC for truthful, deterministic, and anonymous mechanisms. The proof for the other two cases, follow an argument similar to the ones used in the proof of Theorem 5 and 6 and are reported in the Appendix. Let us consider the following instance: $x_1 = \cdots = x_{\bar{c}+1} = 0$ and $x_{\bar{c}+2} = \cdots = x_n = 1$. By the same argument used in Theorem 7, any truthful, deterministic, and anonymous mechanism places the two facilities at the same distance from 0. Since we have that $\bar{c} \geq \lfloor \frac{n}{2} \rfloor$, we have that $\bar{c}+1 \geq n-\bar{c}-1$, hence we get that the approximation ratio of any truthful, anonymous, and deterministic mechanism is larger than $(n-\bar{c}-1)$. □

In particular, the EIG is the best truthful, anonymous, and deterministic mechanism whenever $n \geq \bar{c}+\sqrt{\bar{c}}$.

4.1 The EIG and Previous Mechanisms

To conclude, we show that the EIG mechanisms extends and includes three already-known mechanisms. In particular, (i) when n is an even number and $c_1 = c_2 = \frac{n}{2}$, the EIG mechanism coincides with the InnerPoint Mechanism, presented in [Aziz *et al.*, 2020a]. (ii) When $n = 2k+1$ is odd, $c_1 = k+1$, and $c_2 = k$, the EIG mechanism coincides with the InnerChoice Mechanism, presented in [Walsh, 2022]. (iii) When $c_1 = c_2$, the EIG mechanism coincides with the InnerGap Mechanism, presented in [Walsh, 2022].

For the sake of argument, we limit our discussion to the InnerChoice (IC) mechanism, and defer the other two cases to the Appendix. Given an odd number $n = 2k+1$ and two facilities whose capacities are $c_1 = k+1$ and $c_2 = k$, the routine of the IC mechanism is as follows: (i) Given $\vec{x} = (x_1, \dots, x_n)$ the vector containing the agents' reports ordered from left to right, i.e. $x_i \leq x_{i+1}$, we define $\delta_1 = |x_{k+1}-x_k|$ and $\delta_2 = |x_{k+2}-x_{k+1}|$. (ii) If $\delta_1 \leq \delta_2$, we locate the facility with capacity c_1 at x_k and the other one at x_{k+2}. Otherwise, we locate the facility with capacity c_1 at x_{k+2} and the other one at x_k. (iii) Lastly, every agent is assigned to its closest facility.

Since $\bar{c} = k+1$, we have that $x_{n-\bar{c}} = x_k$ and $x_{\bar{c}+1} = x_{k+2}$, hence, for every $\vec{x} \in \mathbb{R}^n$, the output of EIG and IC are the same, thus the two mechanisms do coincide. It was shown in [Walsh, 2022] that the IC is truthful, however, owing to Theorem 8, we have that IC is strong GSP.

Theorem 11. *The IC is strong Group Strategyproof.*

Similarly, we extend the results on the approximation ratio of the IC mechanism with respect to the SC and MC.

Theorem 12. *Let n be an odd number, then $ar_{MC}(IC) = 2$. Moreover, if $n > 5$, it holds $ar_{SC}(IC) = k-1 = \frac{n-3}{2}$, otherwise $ar_{SC}(IC) = 1$.*

Since $n \geq k+1+\sqrt{k+1}$, the IC is the optimal truthful, deterministic, and anonymous mechanisms to place two facilities of capacities $k+1$ and k amongst $n = 2k+1$ agents. Moreover, the IC is also optimal with respect to the MC.

Theorem 13. *Given $k \in \mathbb{N}$, let $n = 2k+1 > 1$. Then, every truthful deterministic mechanism M that places two facilities with capacity $k+1$ and k is such that $ar_{MC}(M) \geq 2$. Moreover, if $k > 2$, $ar_{SC}(M) \geq 3$. Lastly, if M is also anonymous, then $ar_{SC}(M) \geq k-1$.*

Lastly, we notice that the only other mechanism known that is not extended by the EIG is the Extended Endpoint Mechanism (EEM). However, the approximation ratio of EEM is equal to $\frac{3n}{2}$, which is larger than the one attained by the EIG, making it suboptimal [Aziz *et al.*, 2020a].

5 Conclusion and Future Works

In this paper, we investigated two frameworks for the m-CFLP from a Mechanism Design perspective. First, we considered the m-CFLP with equi-capacitated facilities and no spare capacity. We propose two truthful mechanisms: the Propagating Median Mechanism (PMM) and the Propagating InnerPoint Mechanism (PIPM). Both the mechanisms have bounded approximation ratios with respect to the Social and Maximum Costs. We then established lower bounds on the approximation ratio of any truthful and deterministic mechanism for the m-CFLP with equi-capacitated facilities and no spare capacity. Notably, both PMM and PIPM achieved optimal approximation ratios for the Maximum Cost. Additionally, we demonstrated that PMM and PIPM achieve the minimum possible approximation ratio for the Social Cost among truthful, deterministic, and anonymous mechanisms. In the second framework, we considered the case in which we have two facilities to place and both facilities can accommodate half of the agents. We proposed the Extended InnerGap mechanism, which is strong Group Strategyproof, achieves finite approximation ratio, is optimal with respect to the MC and almost optimal with respect to the SC.

In future research avenues, we aim to improve the lower bounds for non-anonymous mechanisms concerning the Social Cost, to explore higher-dimensional scenarios for agent placements, and to adapt existing randomized mechanisms to enhance approximation ratios results for this problem class [Procaccia and Tennenholtz, 2013].

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