Couples Can Be Tractable: New Algorithms and Hardness Results for the Hospitals/Residents Problem with Couples

Gergely Csáji[1,2] , **David Manlove**[3] , **Iain McBride**[3] and **James Trimble**[3]
[1]ELTE Eötvös Loránd University, Budapest, Hungary
[2]HUN-REN KRTK KTI, Budapest, Hungary
[3]School of Computing Science, University of Glasgow, Glasgow G12 8QQ, UK
csaji.gergely@krtk.hun-ren.hu, david.manlove@glasgow.ac.uk

Abstract

In this paper, we study the HOSPITALS / RESIDENTS PROBLEM WITH COUPLES (HRC). We present a novel polynomial-time algorithm that can find a near-feasible stable matching (adjusting the hospitals' capacities by at most 1) in an HRC instance where the couples' preferences are sub-responsive (i.e., if one member switches to a better hospital, then the couple also improves) and sub-complete (i.e., each pair of hospitals that are individually acceptable to both members are jointly acceptable for the couple) by reducing it to an instance of the STABLE FIXTURES problem. We also present a polynomial-time algorithm for HRC in a sub-responsive, sub-complete instance that is a Dual Market, or where all couples are one of several possible types. Our polynomial-time solvability results greatly expand the class of known tractable instances of HRC.

We complement our algorithms with several hardness results. We show that HRC with sub-responsive and sub-complete couples is NP-hard, even with other strong restrictions. We also show that HRC with a Dual Market is NP-hard under several simultaneous restrictions.

1 Introduction

1.1 Background

The STABLE MARRIAGE PROBLEM, and its many-to-one extension, the HOSPITALS / RESIDENTS PROBLEM (HR), are well-studied and central problems in Computational Social Choice, Computer Science, Game Theory and Economics, with numerous applications including in entry-level labour markets, school choice and higher education allocation [Manlove, 2013]. In the medical sphere, centralised matching schemes that assign aspiring junior doctors to hospitals operate in many countries. One of the largest and best-known examples is the National Resident Matching Program (NRMP) in the US [NRMP website, 2024], which had just under 43,000 applicants for just over 40,000 residency positions in 2023 [NRMP data, 2024]. Canada [CARMS, 2024] and Japan [JRMP, 2024] have analogous matching schemes for junior doctor allocation.

The HR problem model represents a bipartite matching market with two-sided preferences, involving the preferences of doctors over acceptable hospitals, and those of hospitals over their applicants. Each hospital has a capacity, indicating the maximum number of doctors that it can admit. Roth [1984] argued that a key property to be satisfied by a matching M in an instance I of HR is *stability*, which ensures that there is no *blocking pair*, comprising a doctor and a hospital, both of whom have an incentive to deviate from their assignments in M and become matched to one another, undermining the integrity of M. It is known that every instance of HR admits a stable matching, which may be found in time linear in the size of I [Gale and Shapley, 1962].

In many of the above applications, there may be couples amongst the applying doctors, who wish to be allocated to hospitals that are jointly acceptable for the couple. Indeed, the NRMP matching algorithm was redesigned in 1983 specifically to allow couples to provide preferences over pairs of hospitals, with each pair representing a simultaneous assignment that is suitable for both members of the couple. We thus obtain a generalisation of HR called the HOSPITALS / RESIDENTS PROBLEM WITH COUPLES (HRC). By modifying the definition of stability, taking into account how a couple could improve relative to a matching, Roth [1984] showed that a stable matching need not exist. In HRC, the problem therefore is to find a stable matching or report that none exists. In contrast to the case for HR, Ronn [1990] showed that HRC is NP-hard.

Since then, further NP-hardness results for HRC have been established, even in very restrictive cases [Ng and Hirschberg, 1988; Ronn, 1990; Biró *et al.*, 2014], suggesting that HRC is a very challenging computational problem in theory. Until now, there have been very few special cases of HRC that have been shown to admit polynomial-time algorithms. Our work in this paper extends the techniques that can be used in HRC instances by providing novel polynomial-time algorithms for variants of the problem in a surprisingly wide range of settings, and thus contributes new ways to efficiently handle specific instances in several applications.

1.2 Related Work

Roth [1984] considered stability in the HRC context although did not define the concept explicitly. Whilst Gusfield and Irving [1989] defined stability formally in HRC, their definition neglected to deal with the case that both members of a couple wish to be assigned to the same hospital simultaneously. There are discussions of this point in [Biró and Klijn, 2013] and [Manlove, 2013, Section 5.3.2]. McDermid and Manlove [2010] provided a stability definition for HRC that does deal with this possibility. A detailed study of the effects of different stability definitions for HRC was carried out by Delorme et al. [2021].

In this paper, we will adopt the stability definition of McDermid and Manlove [2010]. With respect to this definition, several algorithmic results for HRC hold. Firstly, Ronn's NP-hardness result for HRC holds even in the case that each hospital has capacity 1 and there are no single doctors [Ronn, 1990] (note that all HRC stability definitions are equivalent in the unit hospital capacity case). Biró et al. [2014] showed that NP-hardness holds for HRC even when there are no single doctors, each couple ranks at most two pairs of hospitals, each hospital ranks at most two doctors, and all hospitals have capacity 1.

Ng and Hirschberg [1988] also proved NP-hardness for a "dual market" restriction of HRC, which we refer to as HRC-DUAL, where the two sets H_1 and H_2, comprising the hospitals appearing in the first (respectively second) positions of the acceptable pairs of the first (respectively second) members of each couple are disjoint. This reflects the "two body" problem that often exists for couples, where, for example, one member of a couple may be applying for a job at a university, whilst the other couple member may be applying for a job in industry. Another motivating application arises in a student allocation scheme, where students apply to both a university degree programme and to an internship at a company. This can also be viewed as a specific HRC-DUAL instance, where H_1 comprises the university degree programmes, and H_2 the companies, and each student corresponds to a couple with one member applying to universities and the other to internships.

Manlove et al. [2017] proved that HRC is solvable in polynomial time when each single doctor and hospital has a preference list of length at most two, and each couple finds only one hospital pair acceptable. Another simple tractable case of HRC is given by Klaus and Klijn [2005] (see also [Klaus *et al.*, 2009]), but also in a very restricted framework. They required that each couple's preference list must be *weakly responsive* (informally, a couple's preferences are weakly responsive if there are underlying preferences over individual hospitals for each member, such that if one member of the couple goes to a better hospital then the couple also improves), each couple must find acceptable all possible outcomes where at least one member is matched, and each hospital has capacity 1.

Given the prevalence of NP-hardness results for HRC, and the scarcity of polynomial-time algorithms, heuristics have been applied to the problem (see [Manlove, 2013, Section 5.3.3] for a survey) as well as approaches based on parameterised complexity and local search [Marx and Schlotter, 2011; Biró *et al.*, 2011], integer programming [Biró *et al.*, 2014; Delorme *et al.*, 2021] and more [Manlove *et al.*, 2017; Drummond *et al.*, 2015].

Nguyen and Vohra [2018] studied so-called *near-feasible* stable matchings in HRC. They showed that if hospital capacities can be modified by at most 2, then a stable matching with respect to the new capacities always exists. However, their algorithm computes a stable fractional matching first, which is known to be PPAD-hard to compute [Csáji, 2022], so it is not efficient.

Another direction, to cope with the possible non-existence of a stable matching, is to consider matchings that are "as stable as possible", i.e., admit the minimum number of blocking pairs. We refer to this problem as MIN BP HRC. Biró et al. [Biró *et al.*, 2014] showed that this problem is NP-hard and not approximable within $n_D^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP, where n_D is the number of doctors in a given instance.

1.3 Our Contributions

In this paper, we provide new polynomial-time algorithms for HRC in the case that the couples' preference lists are *sub-complete* and *sub-responsive*. Informally, a couple's preferences are sub-complete if there are underlying preferences for the couple members, such that if both members go to a hospital that is acceptable for them individually, then the pair of hospitals is also acceptable for the couple, and if one member is willing to be unmatched, then any assignment of the other member to an acceptable hospital is acceptable for the couple. The concept of sub-responsiveness is closely related to, but a lot less restrictive than weak responsiveness as described above, even together with sub-completeness.

Our main result is a novel and surprising polynomial-time algorithm to find a *near-feasible* stable matching in an HRC instance if the couples' preferences are sub-responsive and sub-complete. Here, our notion of *near-feasibility* is based on modifying the capacities of each hospital by at most 1. This strengthens Nguyen and Vohra's [2018] result, albeit for a special case of HRC, in two ways: (i) capacities are varied by at most 1 rather than 2, and (ii) we provide a polynomial-time algorithm to produce the desired outcome. This result is established via a reduction to the STABLE FIXTURES PROBLEM (SF), a non-bipartite many-to-many generalisation of HR [Irving and Scott, 2007]. Our algorithm has other nice properties: for example, it can indeed be very beneficial for a couple to apply together, as it can happen that by applying alone, only one of them would have been matched, but by applying together, both members are matched. We illustrate this property with an example.

Next, we provide another polynomial-time algorithm for HRC in the presence of sub-responsive and sub-complete preferences that can find a stable matching, or report that none exists, if all couples are one of several possible types. One of these types corresponds to the very practical and natural restriction of HRC-DUAL in the case of sub-responsive and sub-complete preference lists, and gives a contrast to the NP-hardness result of Ng and Hirschberg for general HRC-DUAL instances as mentioned earlier. Using our approach, we argue that this algorithm can potentially be extended to other types of couples, depending on the specific application.

On the structural side, we prove that a version of the classical Rural Hospitals Theorem for HR [Roth, 1984; Gale and Sotomayor, 1985; Roth, 1986] remains true even in our HRC setting with sub-responsive and sub-complete preferences, and couples belonging to one of several possible types. These are the first non-trivial classes of HRC instances that we are aware of where these structural properties hold.

We complement our positive results with several hardness results. We show that HRC is NP-hard, even with unit hospital capacities, short preference lists and other strong restrictions, including (i) sub-responsive and sub-complete couples, and (ii) dual markets and master preference lists [Irving *et al.*, 2008]. Hence, even in these settings we may not hope to find a stable matching in polynomial time, so our algorithm to find a near-feasible stable solution becomes even more appealing.

Finally, we show that MIN BP HRC is not approximable within $m^{1-\varepsilon}$, for any $\varepsilon > 0$, where m is the total length of the hospitals' preference lists, unless P=NP, even if each couple applies to only one pair of hospitals. This strengthens a claim made in [Manlove *et al.*, 2017] that MIN BP HRC is not approximable within $n_D^{1-\varepsilon}$, for any $\varepsilon > 0$, where n_D is the total number of doctors, unless P=NP, even if each couple applies to only one pair of hospitals.[1]

Our algorithm for the case that the couples' preferences are sub-responsive and sub-complete, and the couples are one of several possible types, also helps to identify the frontier between polynomial-time solvable and NP-hard cases, as our hardness results show that if we have weaker restrictions on the couples' preferences, then it becomes NP-hard to find a stable matching. The results in this paper are summarised in Table 1.

The remainder of this paper is structured as follows. Section 2 provides definitions of key notation and terminology, and motivation for sub-responsiveness, sub-completeness and other restrictions on couples. Our polynomial-time algorithms and structural results are given in Section 3, whilst our NP-hardness and inapproximability results follow in Section 4. Finally, Section 5 concludes. All omitted proofs can be found in the full version of the paper [Csáji *et al.*, 2023].

2 Preliminaries

An instance I of the HOSPITALS / RESIDENTS PROBLEM WITH COUPLES (HRC) involves a set $D = \{d_1, d_2, \dots, d_{n_D}\}$ of *single doctors*, a set $C = \{C_i = (c_{2i-1}, c_{2i}) : 1 \leq i \leq n_C\}$ of *couples*, and a set $H = \{h_1, h_2, \dots, h_{n_H}\}$ of *hospitals*. Let C' denote the members of the couples, i.e., $C' = \{c_{2i-1}, c_{2i} : 1 \leq i \leq n_C\}$. The notation $r_i \in D \cup C'$ denotes either a single doctor $r_i \in D$ or a member of a couple $r_i \in C'$, and collectively the members of $D \cup C'$ are referred to as *doctors*.

In I there is a vector $\mathbf{q}$ that gives each hospital $h_j \in H$ a *capacity* $q_j \in \mathbb{Z}^+$. We define a capacity vector $\mathbf{q}'$ to be *near-feasible* in an instance I of HRC if it holds that $|q'_j - q_j| \leq 1$ for all $h_j \in H$, that is, each capacity is changed by at most 1.

Each single doctor $d_i \in D$ has a strict ranking $\succ_{d_i}$ over a subset $A(d_i) \subseteq H$ of *acceptable* hospitals. Each couple $C_i = (c_{2i-1}, c_{2i})$ $(1 \leq i \leq n_C)$ has a set $A(C_i) \subseteq ((H \cup \{\emptyset\}) \times (H \cup \{\emptyset\})) \setminus \{(\emptyset, \emptyset)\}$ of *acceptable* pairs of hospitals, and has a strict ranking $\succ_{C_i}$ over $A(C_i)$, where $\emptyset$ represents the case that a doctor is unmatched. Each entry in this list is an ordered pair of the form $(h_j, \emptyset)$ $(\emptyset, h_k)$, (h_j, h_k), representing either one member of the couple being matched to a hospital and the other being unmatched, or both members of the couple being matched to a hospital, respectively, where h_j and h_k need not be distinct hospitals. We let $A(c_{2i-1}) = \{h_j \in H : (h_j, \emptyset) \in A(C_i) \vee (h_j, h_k) \in A(C_i)$ for some $h_k \in H\} \setminus \{\emptyset\}\}$ and $A(c_{2i}) = \{h_k \in H : (\emptyset, h_k) \in A(C_i) \vee (h_j, h_k) \in A(C_i)$ for some $h_j \in H\} \setminus \{\emptyset\}\}$ denote the acceptable hospitals in H for c_{2i-1} and c_{2i}, respectively. Finally, each $h_j \in H$ has a strict ranking $\succ_{h_j}$ over those doctors $r_i \in D \cup C'$ such that $h_j \in A(r_i)$.

Let M be a set of doctor–hospital pairs, i.e., a subset of $(D \cup C') \times H$. Given $r_i \in D \cup C'$, we denote $\{h_j \in H : (r_i, h_j) \in M\}$ by $M(r_i)$, and given $h_j \in H$ we denote $\{r_i \in D \cup C' : (r_i, h_j) \in M\}$ by $M(h_j)$. We say that $r_i \in D \cup C'$ is *unmatched* in M if $M(r_i) = \emptyset$. Similarly $C_i \in C$ is said to be *unmatched* in M if both c_{2i-1} and c_{2i} are unmatched in M. Hospital h_j is *undersubscribed*, *full*, or *oversubscribed*, if $|M(h_j)| < q_j$, $|M(h_j)| = q_j$ or $|M(h_j)| > q_j$, respectively. Given $C_i = (c_{2i-1}, c_{2i}) \in C$, we define $M(C_i) = \{(h_j, h_k) : (c_{2i-1}, h_j) \in M \wedge (c_{2i}, h_k) \in M\} \cup \{((h_j, \emptyset) : (c_{2i-1}, h_j) \in M \wedge c_{2i}$ is unmatched in $M\} \cup \{(\emptyset, h_k) : (c_{2i}, h_k) \in M \wedge c_{2i-1}$ is unmatched in $M \cup \{(\emptyset, \emptyset) : c_{2i-1}$ and c_{2i} are unmatched in $M\}$.

A *matching* is a subset M of $(D \cup C') \times H$ such that (i) $M(d_i) \subseteq A(d_i)$ for each $d_i \in D$; (ii) $M(C_i) \subseteq A(C_i) \cup \{(\emptyset, \emptyset)\}$ for each $C_i \in C$; (iii) $|M(r_i)| \leq 1$ for each $r_i \in D \cup C'$; and (iv) $|M(h_j)| \leq q_j$ for each $h_j \in H$. If $(r_i, h_j) \in M$ for some $r_i \in D \cup C'$, as before with a slight abuse of notation we let $M(r_i)$ denote h_j, and if $M(C_i) = \{p\}$ for some $C_i \in C$ and some $p \in A(C_i) \cup \{(\emptyset, \emptyset)\}$ then we also let $M(C_i)$ denote p.

We now define the concept of a *blocking pair* in HRC.

Definition 1 ([McDermid and Manlove, 2010])**.** Let I be an HRC instance and let M be a matching in I. A single doctor $d_i \in D$ forms a *blocking pair* of M with a hospital $h_j \in H$ if:

1. (i) $h_j \in A(d_i)$, (ii) d_i is unmatched, or $h_j \succ_{d_i} M(d_i)$ and (iii) h_j is undersubscribed, or $d_i \succ_{h_j} d_k$ for some $d_k \in M(h_j)$.

A couple $C_i = (c_{2i-1}, c_{2i}) \in C$ forms a *blocking pair* of M with a hospital pair $(h_j, h_k) \in (H \cup \{\emptyset\}) \times (H \cup \{\emptyset\})$ if $(h_j, h_k) \in A(C_i)$, $(h_j, h_k) \succ_{C_i} M(C_i)$, and *either*:

2. either $h_j = M(c_{2i-1})$ or $h_k = M(c_{2i})$, and *either*
   (a) $h_j = M(c_{2i-1})$, and either h_k is undersubscribed or $c_{2i} \succ_{h_k} r_s$ for some $r_s \in M(h_k) \setminus \{c_{2i-1}\}$, *or*
   (b) $h_k = M(c_{2i})$, and either h_j is undersubscribed or $c_{2i-1} \succ_{h_j} r_s$ for some $r_s \in M(h_j) \setminus \{c_{2i}\}$.
3. $h_j \neq M(c_{2i-1})$, $h_k \neq M(c_{2i})$, and *either*
   (a) $h_j \neq h_k$, and (i) h_j undersubscribed or $c_{2i-1} \succ_{h_j} r_s$ for some $r_s \in M(h_j)$, and (ii) h_k undersubscribed or $c_{2i} \succ_{h_k} r_t$ for some $r_t \in M(h_k)$; *or*
   (b) $h_j = h_k$, and h_j has at least two free posts, i.e., $q_j - |M(h_j)| \geq 2$; *or*
   (c) $h_j = h_k$, and h_j has one free post, i.e., $q_j - |M(h_j)| = 1$, and $c_s \succ_{h_j} r_t$ for some $r_t \in M(h_j)$, where $s \in \{2i-1, 2i\}$, *or*

[1]The proof of this result is based on a claim appearing in the supplementary material of [Manlove *et al.*, 2017] (Theorem 8) that HRC is NP-complete even if each couple applies to only one pair of hospitals. The latter claim is erroneous and indeed HRC is solvable in polynomial time in this case as we show in Theorem 10.

Problem	General	Sub-resp. & sub-comp	Type-a, -b or -c
HRC	NP-h (pref lists ≤ 2) [Biró *et al.*, 2014]	NP-h [Thm. 14]	P [Thm. 10]
HRC-DUAL	NP-h (pref lists ≤ 3 & master lists) [Thm. 15]	P [Cor. 12]	P [Cor. 12]
HRC-NEAR-FEAS	EXPTIME (± 2) [Nguyen and Vohra, 2018]	P (± 1) [Thm. 4]	P (± 1) [Thm. 4]
MIN BP HRC	NP-h, inapprox. (couple pref lists ≤ 1) [Thm 16]	NP-h [Thm. 14]	a: [Thm 16] ; b / c: ?

Table 1: A summary of results in this paper. HRC-NEAR-FEAS denotes the problem of finding a near-feasible stable matching in a HRC instance, which means that the capacities of the hospitals can be changed by at most the value indicated.

(d) $h_j = h_k$, h_j is full, $c_{2i-1} \succ_{h_j} r_s$ for some $r_s \in M(h_j)$, and $c_{2i} \succ_{h_j} r_t$ for some $r_t \in M(h_j) \setminus \{r_s\}$.

M is *stable* if it admits no blocking pair.

In HRC, the problem is to find a stable matching or report that none exists. An important special case of HRC, called HRC-DUAL (i.e., a *Dual Market*), arises when the set $H' = \bigcup_{c_i \in C'} A(c_i)$ of acceptable hospitals for the couple members can be partitioned into two sets H_1, H_2, such that for each couple $C_i \in C$, each acceptable pair $(h_j, h_k) \in A(C_i)$ satisfies $h_j \in H_1$ and $h_k \in H_2$.

Given that an instance I of HRC may not admit a stable matching, we define MIN BP HRC to be the problem of finding a matching in I with the minimum number of blocking pairs.

Next we define some properties regarding the preference lists of couples.

Definition 2. Let $C_i = (c_{2i-1}, c_{2i})$ be a couple with joint preference list $\succ_{C_i}$.

1. We say that $\succ_{C_i}$ is *sub-responsive* if there are preference orders $\succ_{c_{2i-1}}, \succ_{c_{2i}}$ over $A(c_{2i-1}) \cup \{\emptyset\}$ and $A(c_{2i}) \cup \{\emptyset\}$ for the individual members of the couple such that, for any three hospitals h_j, h_k, h_q in $H \cup \{\emptyset\}$,
   (i) if $((h_j, h_k) \in A(C_i)$ and $(h_j, h_q) \in A(C_i))$ then $((h_j, h_k) \succ_{C_i} (h_j, h_q)$ if and only if $h_k \succ_{c_{2i}} h_q)$, and
   (ii) if $((h_j, h_k) \in A(C_i)$ and $(h_q, h_k) \in A(C_i))$ then $((h_j, h_k) \succ_{C_i} (h_q, h_k)$ if and only if $h_j \succ_{c_{2i-1}} h_q)$.

   That is, there are underlying preferences for each member of the couple, such that if we send one of them to a better hospital, then the couple also improves, assuming that the new pair is acceptable.
2. We say that $\succ_{C_i}$ is *sub-complete* if $A(C_i) = (A'(c_{2i-1}) \times A'(c_{2i})) \setminus \{(\emptyset, \emptyset)\}$, where, for $j \in \{0, 1\}$, either $A'(c_{2i-j}) = A(c_{2i-j})$ or $A'(c_{2i-j}) = A(c_{2i-j}) \cup \{\emptyset\}$. That is, if both members go to an acceptable hospital for them, then the pair of hospitals is also acceptable for the couple, and if one member is willing to be unmatched, then any assignment of the other member to an acceptable hospital is acceptable for the couple.

We say that C_i is sub-responsive or sub-complete if $\succ_{C_i}$ is sub-responsive or sub-complete, respectively.

As we show in Theorem 14, HRC remains NP-hard even with sub-responsive and sub-complete preference lists. However, in Theorem 10, we provide a novel reduction to SF that allows us to deal with many different kinds of couples via a polynomial-time algorithm. To this end, we define SF, and then some special types of couples.

In an instance of SF, we are given a graph $G = (V, E)$, where each vertex $v \in V$ has a strict preference list $\succ_v$ over the edges incident to v. We are also given a capacity $b(v)$ for each $v \in V$. A *b-matching* is a set $M \subseteq E$ such that $|M(v)| \leq b(v)$ for each $v \in V$, where $M(v)$ is the set of edges in M that are incident to v. M is *stable* if M admits no *blocking pair*, which is an edge $e = \{u, v\} \in E$ such that, for each $w \in e$, either $|M(w)| < b(w)$ or $e \prec_w e'$ for some $e' \in M(w)$.

Definition 3. We say a sub-responsive and sub-complete couple $C_i = (c_{2i-1}, c_{2i})$ is

- *separable*, if $A(C_i) = (A(c_{2i-1}) \cup \{\emptyset\}) \times (A(c_{2i}) \cup \{\emptyset\}) \setminus \{(\emptyset, \emptyset)\}$ (that is, it is allowed to match only one member of the couple in a feasible matching),
- *half-separable* if $A(C_i) = A(c_{2i-1}) \times (A(c_{2i}) \cup \{\emptyset\})$ or $(A(c_{2i-1}) \cup \{\emptyset\}) \times A(c_{2i})$ (that is, it is allowed to leave at most one member of the couple unmatched, but this can only ever be one designated member),
- *connected*, if $A(C_i) = A(c_{2i-1}) \times A(c_{2i})$ (that is, both couple members must be matched),
- of *type-a*, if $A(c_{2i-1}) \cap A(c_{2i}) = \emptyset$ (that is the two members apply to distinct hospitals),
- of *type-b*, if it is connected, $A(c_{2i-1}) \cap A(c_{2i}) = \{h\}$ and if $c_{2i-1} \succ_h c_{2i}$, then $A(c_{2i}) = \{h\}$, otherwise $A(c_{2i-1}) = \{h\}$ (that is, the worse member for h only applies to h)
- of *type-c*, if it is connected, $A(c_{2i-1}) \cap A(c_{2i}) = \{h\}$ and for any $h \neq h' \in A(c_{2i-1})$, $h' \succ_{c_{2i-1}} h$ and for any $h \neq h'' \in A(c_{2i})$, $h'' \succ_{c_{2i}} h$, (that is, h is the worst hospital for both members of the couple).

To avoid collision, we consider a couple C_i that is both of type-b and -c, simply as a type-b couple.

We will show that HRC is solvable in polynomial time for couples of type-a, -b or -c. Moreover for Dual Markets, every couple is necessarily of type-a. However, every couple being of type-a is more general than HRC-DUAL, as it only means that for each couple C_i, the acceptable hospitals for each member of C_i are two disjoint sets. To illustrate this with a simple example, suppose there are three hospitals h_1, h_2, h_3 and three couples C_1, C_2, C_3. C_1 applies only to (h_1, h_2), C_2 applies only to (h_2, h_3), and C_3 applies only to (h_3, h_1). Then, every couple is type-a, but the market is not dual.

Motivation. We finally provide some motivation for the couple types given in Definitions 2 and 3. Firstly, in the Dual Market example given in Section 1, where each "couple" is a student who applies to both a university and a company, the underlying mathematical problem is HRC-DUAL. It is natural to assume that preferences are sub-responsive and sub-complete in this case, and all couples are thus of type-a as observed above.

Types-b and -c may seem a little more artificial. On the one hand, they will demonstrate the flexibility of our technique such that it can handle couples even if they apply to the same hospital. On the other hand, there may be applications related to HRC where these types of (sub-responsive and sub-complete) structures arise, especially in scenarios where it is better for a couple (or an agent) to obtain a joint allocation where the two assigned positions are more diverse. Imagine for example that two members of a couple apply to part-time positions, which also have associated shifts (indicating which days they have to attend). Then, a couple may prefer their shifts to be on different days, to minimise the amount of day-care needed for their children. For example, if one of them applies to shifts on days 1, 2 and 3, and the other on days 3, 4 and 5, then it is natural to assume that they express a sub-responsive and sub-complete preference list, such that (3,3) is ranked last, hence they give a type-c couple. Our preliminary results show that we can handle even more types of couples using our techniques, including those that may arise in other applications, but in order not to make the proofs extremely technical, we only consider the couple types defined in Definitions 2 and 3 in this paper.

We also note the important special case of HRC in which each couple applies only to one pair of hospitals. Then as per the proof of Corollary 11, each couple is type-a or -b. On the other hand, HRC is NP-hard even if each couple finds acceptable only two hospital pairs [Biró *et al.*, 2014]. This helps to push forward our understanding of the frontier between polynomial-time solvability and NP-hardness for restrictions of HRC. Indeed, couple structures that do not make the problem NP-hard may need to be very specific, as Theorem 14 shows that HRC remains NP-hard even with sub-responsive and sub-complete couples, such that each member of a couple applies to only two hospitals.

As a more theoretical application, types-a, -b and -c also arise when we model a new multigraph extension of SF where loop edges are allowed. Due to space constraints, this problem is not discussed in the current version of this paper. Finally, we note that even in cases where there may be different types of couples (beyond those defined in Definitions 2 and 3), it may speed up heuristics and IP techniques to first compute (in polynomial time) a stable solution restricted to the single doctors and the couples of types-a, -b and -c.

3 Polynomial Time Algorithms

In this section we describe our main positive results. We start by describing our most practical polynomial-time algorithm to find a near-feasible stable matching in any HRC instance I, where the participating couples have sub-responsive and sub-complete preference lists.

Theorem 4. *Given an* HRC *instance where each couple's preferences are sub-responsive and sub-complete, there is always a near-feasible capacity vector $\boldsymbol{q}'$ where $|\boldsymbol{q}' - \boldsymbol{q}|_\infty \leq 1$, such that there is a stable matching with respect to the modified capacities. Furthermore, these modified capacities and the stable matching can be found in $\mathcal{O}(m)$ time, where m denotes the total length of the preference lists of the hospitals.*

Proof. We prove our statement by providing a polynomial-time reduction of the problem to SF. Then, we proceed by finding a fractional stable solution and round it off in a specific way to obtain a near-feasible stable matching.

Let $I = (D, C, H, \mathbf{q}, \succ)$ be an instance of HRC where the couples' preferences are sub-responsive and sub-complete. We transform I to a SF instance I', and start by describing the vertices and the capacities in I'.

- For each single doctor $d_i \in D$, we just create a vertex d_i with capacity 1.
- For each hospital h_j with capacity q_j, we just create a vertex h_j with capacity q_j.
- For each couple $C_i = (c_{2i-1}, c_{2i})$ we create two vertices c_{2i-1} and c_{2i} with capacity 1; and if C_i is connected, we create connector vertices $a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}$ with capacity 1.

We proceed by describing the edges of I'.

- For each single doctor d_i and an acceptable hospital h_j, we add edge (d_i, h_j).
- For each couple $C_i = (c_{2i-1}, c_{2i})$ with acceptable hospitals $A(c_{2i-1}), A(c_{2i})$:
  - we add the edges $(c_{2i-1}, h_j), \forall h_j \in A(c_{2i-1})$, and edges $(c_{2i}, h_j), \forall h_j \in A(c_{2i})$,
  - if C_i is half-separable, then we add (c_{2i-1}, c_{2i}),
  - if C_i is connected, we add the edges (c_{2i-1}, a_{2i-1}), (a_{2i-1}, b_{2i-1}), (b_{2i-1}, c_{2i}), (c_{2i}, a_{2i}), (a_{2i}, b_{2i}), (b_{2i}, c_{2i-1}).

Finally, we give the preference lists of the created vertices over their neighbours.

- For each single doctor d_i, we just keep his preference list over the hospitals.
- For each hospital h_j, it just keeps its preference list over the doctors.
- For each member c_k of a half-separable couple, if c_k is the one that can be assigned alone, then he ranks the hospitals in $A(c_k)$ according to $\succ_{c_k}$, followed by his partner $c_{k'}$ last. If c_k is the member who cannot be assigned alone, then c_k ranks $c_{k'}$ first, and then the hospitals in $A(c_k)$.
- Let c_k be a member of a non-half-separable couple. The preferences of c_k in I' are then created such that he/she ranks a_k first (if it exists), followed by the hospitals in $A(c_k)$ according to $\succ_{c_k}$, and b_{k-1} or b_{k+1} last (depending on the parity of k, again, if it exists).
- For the additional a_k, b_k vertices we have that $b_{2i-1} \succ_{a_{2i-1}} c_{2i-1}$, $c_{2i} \succ_{b_{2i-1}} a_{2i-1}$, $b_{2i} \succ_{a_{2i}} c_{2i}$ and $c_{2i-1} \succ_{b_{2i}} a_{2i}$.

With a modification of the algorithm of Tan [1991], described by Fleiner [2010], we can find a *stable half-integral matching* M_f in I' in time $\mathcal{O}(|E|) = \mathcal{O}(m)$. That is, for each edge $(u, v) \in E$, $M_f(u, v) \in \{0, \frac{1}{2}, 1\}$, for each vertex $u \in V$, $\sum_{(u,v)\in E} M_f(u, v) \le q_v$ (q_v is the created capacity for v), and for each edge (u, v) with $M_f(u, v) < 1$, it holds that either u or v is *saturated* in M_f (i.e., $\sum_{(w,x)\in E} M_f(w, x) = 1$ for some $w \in \{u, v\}$) with partners that are at least as good as (u, v). Then, we say that (u, v) is *dominated* at that vertex. We start with an important observation.

Observation 5. *In M_f, each vertex is incident to either 0 or two incident fractional edges.*

Proof. First note that each fractional edge must be dominated at one endpoint, as M_f is stable. Orient each of the edges towards a vertex that dominates it. Then, there can be at most one incoming arcs into each vertex, because if there would be two, then the better one of them would not be dominated there, contradiction. Also, if there is one incoming arc to a vertex, then there is an outgoing arc too, as for the vertex to be saturated, there must be another fractional adjacent edge. □

We create a set of pairs M in I as follows. For each doctor $r_i \in D \cup C'$, let $S_i = \{h_j \in A(r_i) : M_f(r_i, h_j) > 0\}$. If $S_i \ne \emptyset$, add to M the edge (r_i, h_j) such that h_j is the most-preferred (according to $\succ_{r_i}$) member of S_i. (If $S_i = \emptyset$, no edge incident to r_i is added to M.) For each hospital $h_j \in H$, we modify h_j's capacity in I' as follows: if h_j was saturated in M_f then we set $q'_j = |M(h_j)|$, otherwise we set $q'_j = q_j$.

Claim 6. *The obtained matching M is a near-feasible stable matching in I with respect to the new capacities.*

Proof. **Feasibility.** First of all we have to show that M defines a feasible matching in I. Clearly, every doctor r_i is assigned to at most one hospital in M. If r_i is a single doctor, then it is clear that $M(r_i)$ is acceptable to him.

If r_i is a member of a connected couple, say (by symmetry) c_{2i-1}, who is matched in M, then we first show that his partner c_{2i} is also matched in M. Suppose for the contrary that c_{2i} is not matched in M. Then, c_{2i} is not matched to any hospital vertex h_j with positive weight in M_f. As the edge (b_{2i-1}, c_{2i}) does not block M_f, we obtain that c_{2i} must be saturated in M_f. Hence $M_f(c_{2i}, a_{2i}) + M_f(b_{2i-1}, c_{2i}) = 1$. As each of $\{a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}, c_{2i-1}, c_{2i}\}$ are a strict first choice of some other agent, all of them must be saturated in M_f. Therefore, if $x = M_f(c_{2i}, a_{2i})$, then $1 - x = M_f(b_{2i-1}, c_{2i}) = M_f(a_{2i}, b_{2i})$, hence $x = M_f(b_{2i}, c_{2i-1}) = M_f(a_{2i-1}, b_{2i-1})$ and $1 - x = M_f(c_{2i-1}, a_{2i-1})$. But this implies that $M_f(c_{2i-1}, a_{2i-1}) + M_f(b_{2i}, c_{2i-1}) = 1$, which contradicts the fact that c_{2i-1} is matched to a hospital in M. Hence, for each couple C_i, neither or both of them are matched.

If c_k is a member of a half-separable couple who cannot be assigned alone, then we show that c_k is assigned only if his partner $c_{k'}$ is too. Indeed, if $c_{k'}$ is not matched to a hospital, but c_k is, then $c_{k'}$ is not matched with weight 1 in M_f, and as the edge $(c_k, c_{k'})$ is the best for c_k, we get that it blocks M_f, contradiction.

Also, because every couple's preferences are sub-complete, we obtain that no matter which hospitals in $A(c_{2i-1})$ and $A(c_{2i})$ are assigned c_{2i-1} and c_{2i} in M, it must be the case that $(M(c_{2i-1}), M(c_{2i})) \in A(C_i)$. Hence each couple finds the created allocation feasible too.

Finally, we show that no hospital is oversubscribed in M. Obviously, only the hospitals whose capacity we do not change to $|M(h_j)|$ can become oversubscribed. But these hospitals were undersubscribed, and it follows by Observation 5 that the number of doctors it obtains in M is at most $|M_f(h_j)| + 1 \le q_j$.

The new capacities are near-feasible. Next we show that the new capacities are near-feasible, that is each capacity is changed by at most 1. Suppose we changed a hospital's capacity to $|M(h_j)|$. Then, h_j was saturated in M_f and it had at most two incident fractional edges and all of these edges were to vertices corresponding to doctors. Hence, after the rounding, $q_j - 1 = |M_f(h_j)| - 1 \le |M(h_j)| \le |M_f(h_j)| + 1 = q_j + 1$.

Stability. Finally we show that the created matching M is stable with respect to the new capacities.

Suppose that a single doctor d_i blocks with a hospital h_j. Then, by the creation of M, we know that $M_f(d_i, h_j) = 0$. But, the edge (d_i, h_j) did not block M_f, so h_j was filled with better, possibly fractional partners in M_f, each of them being vertices corresponding to doctors. By the definition of the new capacities, h_j remains saturated, and any doctor at h_j in M must have been at h_j with positive weight in M_f. Therefore, h_j is filled with better doctors than d_i, a contradiction.

Suppose that a couple $C_i \in C$ blocks M in I with a pair of hospitals $(h_j, h_k) \in A(C_i)$. The following argument holds regardless of whether $h_j = h_k$ or not. By the assumption that the couple's preferences are sub-responsive, it follows that either C_i is unmatched in M (in which case one of the members cannot be matched to vertices better than his hospital partners with weight 1 in M_f), or one of its members c_{2i-1} or c_{2i} is at a worse hospital according to his ranking. Suppose by symmetry that this member is c_{2i}. By the construction of M, it follows that $M_f(c_{2i}, h_k) = 0$ must hold. But the edge (c_{2i}, h_k) did not block M_f in I', so h_k was saturated in M_f with better partners than c_{2i}, all of them being doctors. Hence, we obtain again by our construction that h_k is full in M with better doctors than c_{2i}, contradiction.

Suppose a couple C_i blocks with one hospital and an application $(h_i, \emptyset)$ or $(\emptyset, h_j)$. In this case, C_i must be separable or half-separable. Suppose by symmetry that c_{2i} is the one who gets matched to a hospital in the blocking coalition. Then, if C_i is half-separable, then c_{2i} must be the member who can be assigned alone, so he ranks c_{2i-1} last. Hence, in any case, c_{2i} cannot be matched to better partners in M_f than h_j, and $M_f(c_{2i}, h_j) = 0$. But (c_{2i}, h_j) does not block M_f, so h_j was saturated with better doctors in M_f, so it still is in M, contradiction.

Therefore, we obtain that the matching M is stable too, proving the theorem. □ □

Remark 7. *We first remark that this algorithm extends in a straightforward way to the case where ties are allowed in the preference lists, as a stable half-integral matching can be found even if ties are allowed.*

Remark 8. *By the result of Nguyen and Vohra [2018], we know that there is always a near-feasible stable matching in* *any* HRC *instance, if we can change the capacities by 2 instead of 1. However, there is no known polynomial-time algorithm to find such capacities and a stable matching.*

In the HRC problem, especially with sub-responsive and sub-complete preferences, it might be unclear as to why it is beneficial for a couple to apply together to pairs of hospitals. Indeed, any stable matching would be stable even if they would have applied separately, however, by applying together it is possible that neither of them gets matched in the end, whereas one of them could have been matched if they had applied separately. However, a very nice feature of our algorithm as given in the proof of Theorem 4 is that now it may be very beneficial for a couple to apply together, as demonstrated below.

Example 9. Let h be a hospital with capacity 2. Suppose there is one single doctor d and a couple (c_1, c_2). Every doctor applies only to h, which has preference list $c_1 \succ_h d \succ_h c_2$. Then, if the two members of the couple would apply as separate doctors, the output matching would be to assign c_1 and d to h, and leave c_2 unmatched. However, if c_1 and c_2 apply together as a connected couple, then in the fractional stable matching M_f, d is assigned to h with weight 1, while c_1 and c_2 are both assigned to h with weight 0.5. Hence, in the output of the algorithm, the capacity of h is increased by 1, and both c_1 and c_2 are accepted. $\square$

We continue to our other main theorem of this section.

Theorem 10. HRC *is solvable in $\mathcal{O}(m)$ time, whenever each couple is sub-responsive, sub-complete and is of type-a, -b or -c, where m denotes the total length of the preference lists of the hospitals.*

The proof of Theorem 10 is complex and technical; here we describe the main underlying ideas and techniques only. As in Theorem 4, we reduce the problem to SF, which can be solved in $\mathcal{O}(m)$ time [Irving and Scott, 2007]. For each couple type, we carefully design different types of gadgets, building on the couple gadgets used in Theorem 4, that enable us to preserve the property that the HRC instance has a stable matching if and only if the SF instance does. We also split the vertices in the SF instance corresponding to the hospitals, such that we have a vertex of capacity 1 that corresponds to the worst doctor at h, and delete some edges incident to this new vertex (since, e.g., if a couple C_i has only one application (h, h), then the better member of C_i for h cannot be the worst doctor at h) which is a crucial idea to handle couples that apply to the same hospital.

We next make a simple observation about separable couples. If every couple were separable, then by treating each member as just a single doctor and running the doctor-oriented Gale-Shapley algorithm [Gusfield and Irving, 1989] we could always obtain a stable matching. This demonstrates that separable couples are easier to handle and indeed it is one of the main reasons that we consider mostly connected couple types (type-b and type-c). We could extend our technique to other separable and half-separable couple types, but it would make both the statement and the proof of Theorem 10 significantly more technical.

Theorem 10 has some interesting corollaries.

Corollary 11. HRC *is solvable in polynomial time if each couple only applies to one pair of hospitals.*

Corollary 12. HRC-DUAL *is solvable in polynomial time if each couple is sub-responsive and sub-complete. Furthermore, there always exists a stable matching.*

We can also show that a variant of the Rural Hospital theorem also extends to this framework.

Corollary 13. *Given an* HRC *instance where each couple is of type-a, -b or -c, in every stable matching, the same set of single doctors are matched and every hospital has the number of doctors. Furthermore, if a hospital is undersubscribed in one stable matching, then it is assigned the same set of doctors in all stable matchings.*

4 Hardness Results

In this section we present hardness results for three variants of HRC. We firstly consider sub-responsive and sub-complete preferences. It might be tempting to believe that sub-responsive and sub-complete preferences are sufficient to guarantee the polynomial-time solvability of HRC. However, it turns out that these assumptions are unfortunately not enough, as the following result demonstrates.

Theorem 14. HRC *is NP-hard even if each couple has sub-responsive and sub-complete preferences. This holds even with unit hospital capacities, and even if each preference list is of length at most 4. This also holds if either (i) the couples are half-separable, or (ii) every couple is separable or of type-a, in which (in both cases) each preference list is of length at most 8.*

We next consider dual markets, and we show that the hardness holds in restricted cases of HRC-DUAL too.

Theorem 15. HRC-DUAL *is NP-hard, even if all preference lists have length 3, all hospitals have capacity 1, and the preference list of each single doctor, couple and hospital is derived from a strictly ordered master list of hospitals, pairs of hospitals and doctors, respectively.*

We finally state our inapproximability result for MIN BP HRC.

Theorem 16. MIN BP HRC *is not approximable within $m^{1-\varepsilon}$, for any $\varepsilon > 0$, where m is the total length of the hospitals' preference lists, unless* P=NP. *The result holds even if there are no single doctors, every couple finds acceptable only one pair of hospitals, and each hospital has capacity 1.*

5 Conclusions

In this paper, we have presented a range of novel polynomial-time solvability and NP-hardness results for variants of HRC. An important future direction is to continue to expand our knowledge of the frontier between tractable and intractable variants of HRC. A key message from this paper, however, is that HRC *can* be tractable in a range of different scenarios, and perhaps much more so than might have been predicted previously, given the inherent intractability of the problem in general and in so many restricted settings.

Ethical Statement

There are no ethical issues.

Acknowledgements

Gergely Csáji was supported by the Hungarian Academy of Sciences, Momentum grant number LP2021-2/2021, by the Hungarian Scientific Research Fund, OTKA, grant number K143858 and by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation fund, financed under the KDP-2023 funding scheme (grant number C2258525). David Manlove was supported by the Engineering and Physical Sciences Research Council, grant numbers EP/X013618/1 and EP/P028306/1. Iain McBride was supported by a Scottish Informatics and Computer Science Alliance Prize Studentship. James Trimble was supported by the Engineering and Physical Sciences Research Council, Doctoral Training Partnership number EP/N509668/1. For the purpose of open access, the authors have applied a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising from this submission.

References

[Biró and Klijn, 2013] P. Biró and F. Klijn. Matching with couples: a multidisciplinary survey. *International Game Theory Review*, 15(2), 2013. article number 1340008.

[Biró *et al.*, 2011] P. Biró, R.W. Irving, and I. Schlotter. Stable matching with couples: an empirical study. *ACM Journal of Experimental Algorithmics*, 16, 2011. Section 1, article 2, 27 pages.

[Biró *et al.*, 2014] P. Biró, D.F. Manlove, and I. McBride. The Hospitals/Residents problem with Couples: Complexity and integer programming models. In *Proceedings of SEA 2014: the 13th International Symposium on Experimental Algorithms*, volume 8504 of *Lecture Notes in Computer Science*, pages 10–21. Springer, 2014.

[CARMS, 2024] `https://www.carms.ca` (Canadian Resident Matching Service website), 2024.

[Csáji, 2022] G. Csáji. On the complexity of stable hypergraph matching, stable multicommodity flow and related problems. *Theoretical Computer Science*, 931:1–16, 2022.

[Csáji *et al.*, 2023] G. Csáji, D. Manlove, I. McBride, and J. Trimble. Couples can be tractable: New algorithms and hardness results for the Hospitals / Residents problem with Couples. Technical Report 2311.00405, Computing Research Repository, Cornell University Library, 2023. Available from http://arxiv.org/abs/2311.00405.

[Delorme *et al.*, 2021] M. Delorme, S. García, J. Gondzio, J. Kalcsics, D. Manlove, and W. Pettersson. Stability in the hospitals/residents problem with couples and ties: Mathematical models and computational studies. *Omega*, 103:102386, 2021.

[Drummond *et al.*, 2015] J. Drummond, A. Perrault, and F. Bacchus. SAT is an effective and complete method for solving stable matching problems with couples. In *Proceedings of IJCAI '15: the 24th International Joint Conference on Artificial Intelligence*, pages 518–525. AAAI Press, 2015.

[Fleiner, 2010] T. Fleiner. The stable roommates problem with choice functions. *Algorithmica*, 58(1):82–101, 2010.

[Gale and Shapley, 1962] D. Gale and L.S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.

[Gale and Sotomayor, 1985] D. Gale and M. Sotomayor. Some remarks on the stable matching problem. *Discrete Applied Mathematics*, 11:223–232, 1985.

[Gusfield and Irving, 1989] D. Gusfield and R.W. Irving. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.

[Irving and Scott, 2007] R.W. Irving and S. Scott. The stable fixtures problem. *Discrete Applied Mathematics*, 155:2118–2129, 2007.

[Irving *et al.*, 2008] R.W. Irving, D.F. Manlove, and S. Scott. The stable marriage problem with master preference lists. *Discrete Applied Mathematics*, 156(15):2959–2977, 2008.

[JRMP, 2024] `https://jrmp.jp` (Japan Resident Matching Program website), 2024.

[Klaus and Klijn, 2005] B. Klaus and F. Klijn. Stable matchings and preferences of couples. *Journal of Economic Theory*, 121(1):75–106, 2005.

[Klaus *et al.*, 2009] B. Klaus, F. Klijn, and T. Nakamura. Corrigendum: Stable matchings and preferences of couples. *Journal of Economic Theory*, 144(5):2227–2233, 2009.

[Manlove *et al.*, 2017] D.F. Manlove, I. McBride, and J. Trimble. "Almost-stable" matchings in the Hospitals / Residents problem with Couples. *Constraints*, 22:50–72, 2017.

[Manlove, 2013] D.F. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific, 2013.

[Marx and Schlotter, 2011] D. Marx and I. Schlotter. Stable assignment with couples: parameterized complexity and local search. *Discrete Optimization*, 8:25–40, 2011.

[McDermid and Manlove, 2010] E.J. McDermid and D.F. Manlove. Keeping partners together: Algorithmic results for the hospitals / residents problem with couples. *Journal of Combinatorial Optimization*, 19(3):279–303, 2010.

[Ng and Hirschberg, 1988] C. Ng and D.S. Hirschberg. Complexity of the stable marriage and stable roommate problems in three dimensions. Technical Report UCI-ICS 88-28, Department of Information and Computer Science, University of California, Irvine, 1988.

[Nguyen and Vohra, 2018] T. Nguyen and R. Vohra. Near-feasible stable matchings with couples. *American Economic Review*, 108(11):3154–3169, 2018.

[NRMP data, 2024] `https://www.nrmp.org/match-data-analytics/`

residency-data-reports (NRMP Main Residency Match Data and Reports), 2024.

[NRMP website, 2024] https://www.nrmp.org (National Resident Matching Program website), 2024.

[Ronn, 1990] E. Ronn. NP-complete stable matching problems. *Journal of Algorithms*, 11:285–304, 1990.

[Roth, 1984] A.E. Roth. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of Political Economy*, 92(6):991–1016, 1984.

[Roth, 1986] A.E. Roth. On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica*, 54:425–427, 1986.

[Tan, 1991] J.J.M. Tan. A necessary and sufficient condition for the existence of a complete stable matching. *Journal of Algorithms*, 12(1):154–178, 1991.