Bidirectional Search while Ensuring Meet-In-The-Middle via Effective and Efficient-to-Compute Termination Conditions

Yi Wang[1] , **Eyal Weiss**[2] , **Bingxian Mu**[3] and **Oren Salzman**[2]
[1]University of New Hampshire, New Hampshire, USA
[2]Technion-Israel Institute of Technology, Haifa, Israel
[3]University of Prince Edward Island, Prince Edward Island, Canada
yw1055@usnh.edu, {eweiss@campus,osalzman@cs}.technion.ac.il, bingxianmu@gmail.com

Abstract

In bidirectional heuristic search, the *meeting-in-the-middle* property (MMP) and the theory of *must-expand pairs* (MEP) have driven significant recent developments in search efficiency. However, these methodologies typically terminate the search based on minimal priority metrics in the forward and backward OPEN lists, requiring exploration of all potentially better solutions and potentially incurring substantial computational burden. In this paper, we investigate the reasons that contribute to the potential inefficiency in MM, and introduce a tighter termination condition that enables earlier termination without exhaustive exploration while still ensuring both MMP and optimality. This results in a highly efficient bidirectional search algorithm. Experimental comparisons demonstrate that our algorithm outperforms MM in terms of running time by at least two orders of magnitude and is on par or better compared to A*, highlighting its potential in a wide range of applications.

1 Introduction and Related Work

Search is a fundamental task in AI and robotics in which, in its simplest form, we wish to compute a minimal-cost path on a given graph between given start and goal vertices [Russell and Norvig, 2020]. Heuristics, which estimate the cost between states, allow to improve planner efficiency by focusing search efforts to smaller portions of the search space [Pearl, 1984]. *Unidirectional heuristic search* (Uni-HS) algorithms, such as A* [Hart *et al.*, 1968], exemplify this principle by expanding the search from the start state to the goal.

Bidirectional search algorithms compute a solution by conducting two separate searches simultaneously: a forward search from the start, and a backward search from the goal [Nicholson, 1966]. In theory, a bidirectional search is capable of reducing the number of expanded states exponentially due to the search proceeding only to half the depth of the solution path [Nicholson, 1966; Pearl and Korf, 1987]. This theoretical efficiency renders *bidirectional heuristic search* (Bi-HS) a promising algorithmic approach. However, orchestrating the meeting of the forward and backward searches presents a significant challenge using heuristics (e.g., [Sint and de Champeaux, 1977; Kaindl and Kainz, 1997; Auer and Kaindl, 2004; Felner *et al.*, 2010; Arefin and Saha, 2010; Barker and Korf, 2015; Siag *et al.*, 2023]).

Arguably, there have been two conceptual breakthroughs in the field in the last decade [Sturtevant and Felner, 2018]: The first was the introduction of the meeting-in-the-middle propriety (MMP), which guarantees finding an optimal solution while also ensuring that each expanded state has a cost that is less than or equal to half of the cost of an optimal solution [Holte *et al.*, 2016]. The second, which is an extension and generalization of the first, was a new theory of *must expand pairs* (MEP) for Bi-HS [Eckerle *et al.*, 2017] which characterizes the set of forward and backward node pairs that must be expanded to guarantee solution optimality. Indeed, the majority of work in Bi-HS has been devoted to developing approaches that use these two concepts within Bi-HS algorithms in a computationally efficient manner.

Thus far, MM and its variants [Holte *et al.*, 2017] did not achieve competitive running times when compared to the state-of-the-art in Uni-Hs and Bi-HS. In contrast, using the theory of MEP, many highly efficient, though arguably more complex, Bi-HS algorithms were suggested (e.g., [Shaham *et al.*, 2017; Chen *et al.*, 2017; Shaham *et al.*, 2018; Shperberg *et al.*, 2019a; Shperberg *et al.*, 2019b; Sturtevant *et al.*, 2020; Alcázar *et al.*, 2020]). When the heuristic used is known to be consistent[1], the efficiency of these algorithms may be improved. Siag et al. (2023) presented an algorithmic framework unifying many of these algorithms. This also resulted in slight improvements to existing Bi-HS algorithms such as BAE* [Sadhukhan, 2012; Sewell and Jacobson, 2021]. As a general guideline, Siag et al. (2023) suggest that BAE* should be the default algorithm to use for the consistency case, as it strikes a good balance between node expansions and runtime across different domains and heuristics. Consequently, we use BAE* as a baseline.

In this paper, we revisit the MM-family of algorithms and study the reason to their practical inefficiency. We claim that this can be attributed to the *termination condition* (TC) used to decide when to stop MM's search. We suggest a tighter TC that preserves the theoretical properties of MM while being

[1]A heuristic function is said to be consistent if its value is always less than or equal to the estimated distance from any neighboring vertex to the goal, plus the cost of reaching that neighbor.

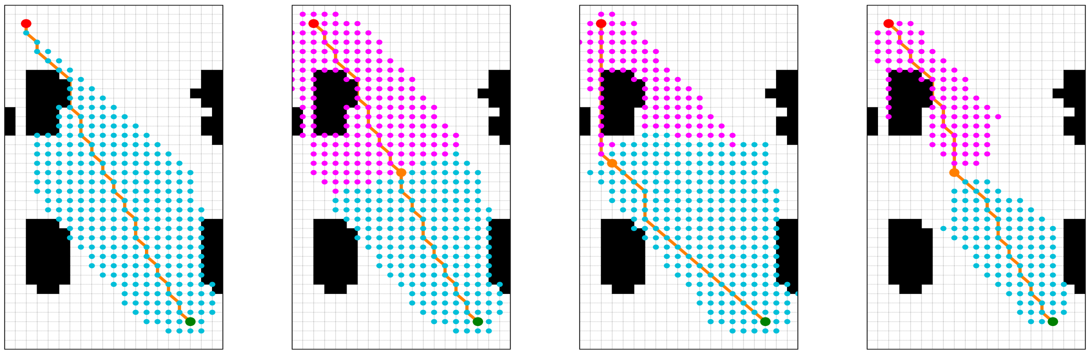

(a) A* (0.106 ms). (b) MM (4.118 ms). (c) BAE* (0.232 ms). (d) MEET (0.075 ms).

Figure 1: Comparative illustration of A* (a), MM (b), BAE* (c), and MEET (d) when finding an optimal path (—) from a start state (●) to the goal state (●), using the Euclidean distance as a heuristic and breaking ties according to larger g-values. When relevant, (●) denotes the *meeting state* (see Sec. 2), and (●) and (●) denote states expanded in the forward and backward search, respectively. All algorithms return a path with the same optimal cost but the number of expanded states differ: A* expands 292 states, MM expands a total of 381 with 196 and 185 in the forward and backward searches, respectively, BAE* expands a total of 352 with 262 and 97 in the forward and backward searches, respectively, and MEET expands a total of 202 with 111 and 91 in the forward and backward searches, respectively.

highly efficient to compute: In contrast to the TC employed by MM, our new TC does not require computing the minimal cost-to-come among all states in the forward and backward search. This allows (i) to perform state culling (i.e., to remove states that cannot be part of an optimal solution); and (ii) to efficiently test if the TC holds without the need to iterate over all states in the forward and backward search queues.

Meet-in-the-middle with Early and Efficient Termination (MEET) incorporates this new TC into an efficient Bi-HS algorithm that satisfies the MMP and is guaranteed to compute optimal solutions (i.e., it retains the same theoretical guarantees as MM). In contrast to conventional heuristic search algorithms, such as A*, which use a predefined admissible heuristic, MEET improves heuristic values *on-the-fly*, in a similiar fashion to MM, which increases its efficiency. Unlike Bi-HS algorithms that build on MEP theory, MEET halts the search by relying on cost metrics of a *meeting state*. Compared to BAE*, MEET achieves optimality without relying on the expansion of a *meeting state* that intersects on both search trees and lies on an optimal solution. Additionally, MEET is the first algorithm to address the open question of whether Bi-HS can terminate early while still ensuring optimality and MMP.

In this paper, we focus on the arguably simple MM-family of algorithms that do not require much of the advanced algorithmic building blocks developed using the general MEP theory. However, as we will see in the experimental results (Sec. 5), MEET is competitive and often outperforms state-of-the-art Bi-HS algorithms such as BAE*. To illustrate this, we visualize in Fig. 1 the states expanded by A*, MM, BAE*, and MEET in an instance on the grid map brc203 [Sturtevant, 2012], where MEET outperforms the other algorithms both in terms of the number of nodes expanded and the running time.

2 Problem Definitions and Notations

Let $G = (S, E, c)$ be a weighted graph, where S is a set of states, $E \subseteq S \times S$ is a set of edges connecting states, and $c : E \to \mathbb{R}^{\geq \varepsilon}$ is a cost function over the edges (i.e., we assume that the minimal edge cost is ε for some $\varepsilon > 0$). For a state s, we define the *neighbors* of s, denoted as $\text{nbr}(s)$ to be all states that share an edge with s. Namely, $\text{nbr}(s) := \{s' | (s, s') \in E\}$. A path $\pi = \langle s_0, \ldots, s_k \rangle$ is a sequence of states such that $\forall i,\ s_i \in S$ and $(s_i, s_{i+1}) \in E$, where s_i is called the parent of s_{i+1}, and s_{i+1} is called the child of s_i. We denote the parent of a state s in a path as $\text{prt}(s)$. The cost of a path $c(\pi)$ is defined as the sum of edge costs, i.e., $c(\pi) := \sum_i c(s_i, s_{i+1})$. Given start and goal states $s_{\text{start}}, s_{\text{goal}} \in S$, a path connecting s_{start} to s_{goal} is also called a *solution*. A solution whose cost is minimal is called an *optimal solution*.

A Bi-HS algorithm simultaneously performs a search from s_{start} to s_{goal} (forward) and from s_{goal} to s_{start} (backward) until the searches meet at a state that lies on an optimal solution. We use subscripts '$\mathcal{F}$' and '$\mathcal{B}$' to denote the forward and backward search directions, respectively. For brevity, we primarily concentrate on the forward search direction, as the backward search (notations, definitions, analyses, etc.) is symmetrical and will be referenced only when specifically stated. When referring to an arbitrary direction $\mathcal{D} \in \{\mathcal{F}, \mathcal{B}\}$, we denote its opposite by $\bar{\mathcal{D}}$.

We denote by Π^* and C^* the set of all optimal solutions and their cost. Given a state s we denote $g_{\mathcal{F}}(s)$ the estimated cost to reach s from s_{start} as currently computed by the forward search. Furthermore, we define $h^*_{\mathcal{F}}(s)$ to be the minimal cost among all paths connecting s to s_{goal} and assume that we are provided a heuristic function $h_{\mathcal{F}}(s)$ that estimates $h^*_{\mathcal{F}}(s)$. We say that $h_{\mathcal{F}}$ is *admissible* if $\forall\ s \in S,\ h_{\mathcal{F}}(s) \leq h^*_{\mathcal{F}}(s)$ and call $h_{\mathcal{F}}$ *static* if it never changes its values during the search. Throughout this work, we assume that all *static* heuristics are admissible. Finally, we set $f_{\mathcal{F}}(s) := g_{\mathcal{F}}(s) + h_{\mathcal{F}}(s)$.

As common in best-first heuristic search, all algorithms utilize $\text{OPEN}_{\mathcal{F}}$, a priority queue that stores states generated in the forward search but not expanded (i.e., states with $g_{\mathcal{F}}(s) < \infty$), ordered by their $f_{\mathcal{F}}$-value. Similarly, $\text{CLOSED}_{\mathcal{F}}$ is a set of all states that have been expanded in the forward search.

We use π_{curr} and C_{curr} to denote the current-best path and its cost as produced by a search algorithm at any given point in time. When an algorithm starts, π_{curr} is undefined and $C_{\text{curr}} = \infty$. If it is guaranteed that $\pi_{\text{curr}} \in \Pi^*$ and $C_{\text{curr}} = C^*$ upon termination, then the algorithm is said to be *optimal*.

We use $f_{\min\text{-}\mathcal{F}}$ and $g_{\min\text{-}\mathcal{F}}$ to denote the minimal f-value and g-value among all states in $\text{OPEN}_{\mathcal{F}}$, respectively. For $\text{OPEN}_{\mathcal{B}}$ we define $f_{\min\text{-}\mathcal{B}}$ and $g_{\min\text{-}\mathcal{B}}$, analogously. Additionally, we set $f_{\min} := \min(f_{\min\text{-}\mathcal{F}}, f_{\min\text{-}\mathcal{B}})$ and $g_{\min} := \min(g_{\min\text{-}\mathcal{F}}, g_{\min\text{-}\mathcal{B}})$. Finally, with a slight abuse of notation, we set $g_{\min}(s) := \min(g_{\mathcal{F}}(s), g_{\mathcal{B}}(s))$.

We say that a state $s \in S$ is an *intersecting state* if $s \in \text{OPEN}_{\mathcal{F}} \cap \text{OPEN}_{\mathcal{B}}$. We denote by $\mathcal{I}_{\text{curr}}$ the set of all intersecting states that lie on paths with a cost equal to C_{curr}. For the intersecting states, we set

$$I_{\text{b}} := \arg\min_{s \in \mathcal{I}_{\text{curr}}} (g_{\min}(s)) .$$

Namely, I_{b} is an intersecting state that lies on a solution with C_{curr} and has the minimal g-value among such states. We call I_{b} the *current-best intersecting state*. When the search ends with $C_{\text{curr}} = C^*$, I_{b} serves as a *meeting state*, s_{meet}.

Definition 1 (Meet-in-the-middle property [Holte *et al.*, 2017])**.** *A Bi-HS algorithm satisfies* meet-in-the-middle property *(MMP) if it only expands states s with $g_{\mathcal{D}}(s) \leq C^*/2$.*

We say that a Bi-HS algorithm is ***MM-optimal*** if it is ensured to (i) return an optimal solution and to (ii) satisfy MMP.

3 Algorithmic Background

3.1 Algorithmic Framework

We start by outlining an algorithmic framework for an MM-optimal Bi-HS algorithm to standardize the core operations (Alg. 1). These key operations include state priority, state selection for expansion, state culling (or pruning), and TCs.

- **State priority:** A function that dictates how states are stored in the priority queues based on the estimated path cost in each search direction (Lines 1, 3 and 14).
- **State selection for expansion:** An operation that selects a state for expansion from the queues of both search directions. In a best-first approach, the selection should follow a priority function, which could be identical to the state priority, but not necessarily (Line 3).
- **State pruning:** A function that tests if discarding a state does not compromise MM-optimality (Line 12).
- **Termination conditions:** A termination criterion that allows the search to stop once no further expansion can improve C_{curr}, ensuring *MM-optimality*, (Line 4).

3.2 Challenges in Bidirectional Heuristic Search

Unlike in Uni-HS, where the search termination is tied to a singular, clearly defined goal state, in Bi-HS the state in which the search should terminate, i.e., a meeting state, is *implicitly defined*. This raises a challenge: There is no guarantee of optimality when the forward and backward search fronts meet (i.e., when a solution is found). Thus, verifying that a solution is optimal might incur unnecessary expansions if the TC is not tight. Conversely, tighter TCs can be computationally expensive to evaluate, and earlier termination poses a high risk of compromising optimality.

Algorithm 1: Generic Framework for *MM-Optimal* Bi-HS

```
Input: $G = (S, E, c)$, $s_{\text{start}}$, $s_{\text{goal}}$
Output: Solution $\pi$
1  Insert $s_{\text{start}}$ into $\text{OPEN}_{\mathcal{F}}$ and $s_{\text{goal}}$ into $\text{OPEN}_{\mathcal{B}}$ using
     priority function
2  while both $\text{OPEN}_{\mathcal{F}}$ and $\text{OPEN}_{\mathcal{B}}$ are not empty do
3      Select $s \in \text{OPEN}_{\mathcal{F}} \cup \text{OPEN}_{\mathcal{B}}$ with min priority
       function
4      if ToTerminate then
5          return ReconstructPath
6      if $s \in \text{OPEN}_{\mathcal{F}}$ then
7          Remove $s$ from $\text{OPEN}_{\mathcal{F}}$
8          Add $s$ to $\text{CLOSED}_{\mathcal{F}}$
9          foreach $s' \in \text{nbr}(s)$ do
10             if $s$ can't improve $g_{\mathcal{F}}(s')$ then
11                 continue
12             if ToPrune($s'$) then
13                 continue
14             Insert $s'$ into $\text{OPEN}_{\mathcal{F}}$ using priority function
15     else
16         Analogous operation for the backward search
17 return $\emptyset$
```

Indeed, significant research efforts were taken to derive effective TCs that ensure solution optimality in Bi-HS. This culminated in the work by Siag et al. (2023) who introduced a set of 17 search bounds (which included bounds from previous works) that can be used for determining optimality. Notably, utilizing combinations of them for obtaining a tighter TC introduces non-trivial computational overhead.

3.3 Meet-In-The-Middle (MM)

To obtain MMP, MM uses the following priority function when inserting a state s into $\text{OPEN}_{\mathcal{F}}$ where states are prioritized from low to high:

$$\text{pr}_{\mathcal{F}}(s) := \max(f_{\mathcal{F}}(s), 2g_{\mathcal{F}}(s)) .$$

The ordering in $\text{OPEN}_{\mathcal{B}}$ is defined analogously. To understand the intuition behind this priority function, recall that to maintain MMP, an algorithm should only expand a state s if $g_{\mathcal{F}}(s) \leq C^*/2$, identical to the condition $2g_{\mathcal{F}}(s) \leq C^*$. In addition, to ensure optimality, a state s should be expanded if $f_{\mathcal{F}}(s) \leq C^*$, indicating it may lie on an optimal path.

Unfortunately, when a heuristic fails to accurately estimate the cost to reach the goal, then a state s with $g_{\mathcal{F}}(s) > C^*/2$ may have $f_{\mathcal{F}}(s) < C^*$, thereby violating MMP. This happens when $g_{\mathcal{F}}(s) > h_{\mathcal{F}}(s)$. In this case, we say that h is a *weak heuristic* [Barker and Korf, 2015], and this may cause MM to expand states that can not be part of an optimal solution.

As a termination condition, MM halts the search once

$$C_{\text{curr}} \leq \max(C_{\text{pr}}, f_{\min\text{-}\mathcal{F}}, f_{\min\text{-}\mathcal{B}}, g_{\min\text{-}\mathcal{F}} + g_{\min\text{-}\mathcal{B}} + \varepsilon).$$

Here, C_{pr} represents the minimum priority on both search fronts. Holte et al. [2017] show that each of the last three terms is a lower bound on the cost of any solution that might be found by continuing to search.

Importantly, this TC requires access to $g_{\min}$ and $f_{\min}$ from both OPEN lists at each iteration. Since MM does not sort its lists by g- or f-values (rather by $\text{pr}_{\mathcal{F}}$ defined above), each iteration in MM needs to iterate over all members of each OPEN list to obtain $g_{\min}$ and $f_{\min}$. This comes with a significant computational overhead due to the exponential growth in the number of generated states as the search progresses. Consequently, a TC that can be computed efficiently, rather than with complexity $O(|\text{OPEN}_{\mathcal{F}}|+|\text{OPEN}_{\mathcal{B}}|)$, could dramatically speed up the search process.

Finally, MM does not prune any states.

4 Meet-in-the-middle with Early and Efficient Termination (MEET)

In this section, we introduce MEET, our Bi-HS algorithm, which follows the structure as detailed in Alg. 1 and Sec. 3.1.

4.1 Algorithm Description

State Priority

Similar to MM, MEET defines the priority of a state s in $\text{OPEN}_{\mathcal{D}}$ according to its $f_{\mathcal{D}}$-value. However, the heuristic used is *not* the static heuristic $h_{\mathcal{D}}$ but an updated one which we denote by $\tilde{h}_{\mathcal{D}}$. Specifically, and with slight abuse of notation, $f_{\mathcal{D}}(s) := g_{\mathcal{D}}(s) + \tilde{h}_{\mathcal{D}}(s)$ where $\tilde{h}_{\mathcal{D}}(s)$ is initialized to be $h_{\mathcal{D}}(s)$. In case $s \notin \text{CLOSED}_{\bar{\mathcal{D}}} \cup \{\text{OPEN}_{\bar{\mathcal{D}}}\}$ and $g_{\mathcal{D}}(s) > h_{\mathcal{D}}(s)$, $\tilde{h}_{\mathcal{D}}(s)$ is updated to be $g_{\mathcal{D}}(s)$. Otherwise, if $s \in \text{OPEN}_{\bar{\mathcal{D}}}$, $\tilde{h}_{\mathcal{D}}(s)$ is set to be $g_{\bar{\mathcal{D}}}(s)$. Note that for the simplicity of the presentation we make a slight abuse of notation and use the notation f rather than define a new notation for f with $\tilde{h}$. Hence, whenever we describe MEET or results related to it, consider that f is used with $\tilde{h}$.

State Selection for Expansions

Similar to MM, MEET selects the state whose priority equals $f_{\min}$.

State Pruning

Before generating a new state s in direction $\mathcal{D} \in \{\mathcal{F}, \mathcal{B}\}$, MEET tests whether

$$\min\{f_{\mathcal{D}}(s), g_{\mathcal{D}}(s) + g_{\bar{\mathcal{D}}}(s) \text{ if } g_{\bar{\mathcal{D}}}(s) < \infty\} > C_{\text{curr}}.$$

If so, s is discarded.

Termination Condition

MEET makes use of four termination conditions and terminates if one of them holds. When describing the termination conditions, we assume that a solution has been found (namely, $C^* \leq C_{\text{curr}}$) and that $\mathcal{H} \in \{\mathcal{F}, \mathcal{B}\}$ is the direction for which I_{b} has a larger g-value (i.e., $g_{\mathcal{D}}(I_{\text{b}}) > g_{\bar{\mathcal{D}}}(I_{\text{b}})$). $\text{prt}_{\mathcal{H}}(I_{\text{b}})$ denote the parent of I_{b} in $\mathcal{H}$ direction. Furthermore, we denote s to be the state chosen for expansion (i.e., $f_{\mathcal{D}}(s) = f_{\min}$) and $s' \in \text{nbr}(s)$ to be a neighbor that is generated in the expansion of s with minimal $f_{\mathcal{D}}$-value. Finally, we denote t as the state with $f_{\bar{\mathcal{D}}}(t) = f_{\text{min-}\bar{\mathcal{D}}}$ (i.e., the state with the minimal f-value in the opposite search direction).

MEET terminates if one of the following conditions holds:

TC 1. *$f_{\mathcal{D}}(s) \geq C_{curr}$.*

Intuitively, **TC1**, also used by MM, indicates that no further expansion can improve the current best solution.

TC 2. *(i) $g_{\mathcal{D}}(I_b) \leq g_{\bar{\mathcal{D}}}(I_b)$, (ii) $g_{\mathcal{D}}(s) \leq h_{\mathcal{D}}(s)$, (iii) $g_{\mathcal{D}}(s)+g_{\bar{\mathcal{D}}}(t) + \varepsilon > C_{curr}$, (iv) $I_b \notin \{s, t\}$, and (v) $g_{\bar{\mathcal{D}}}(t) \leq h_{\bar{\mathcal{D}}}(t)$.*

TC2 triggers if the lowest-priority states in both OPEN lists are too far apart, namely if the sum of their g-values plus the minimal edge cost exceeds C_{curr}, so no further expansions can improve it, and thus the search can terminate early.

MEET has two more termination conditions (**TC3** and **TC4**), for which we require further notation: We denote $S(I_{\text{b}}, s)$ to be the set of states generated after I_{b} has been found (i.e., generated by both search directions) but before s was expanded. Note that $S(I_{\text{b}}, s)$ may be empty. We now introduce an auxiliary precondition that needs to hold before using **TC3** and **TC4**.

PC 1. *$\forall \tilde{s} \in \{S(I_b, s) \neq \emptyset\}$, the inequalities $f_{\mathcal{D}}(\tilde{s}) \geq C_{curr}$ and $g_{\mathcal{H}}(\tilde{s}) > g_{\mathcal{H}}(I_b)$ hold.*

For the specific case where $S(I_{\text{b}}, s)$ is empty, we define that **PC1** does not hold. If **PC1** holds, MEET terminates if one of the following conditions hold:

TC 3. *(i) $g_{\mathcal{D}}(s) \geq g_{\mathcal{H}}(prt_{\mathcal{H}}(I_b))$, (ii) $g_{\mathcal{D}}(s) > h_{\mathcal{D}}(s)$, (iii) $c(s, s') = \varepsilon$, and (iv) $g_{\mathcal{H}}(s') > g_{\mathcal{H}}(I_b)$,*

TC3 captures the intuition that if C_{curr} is not yet optimal, then before expanding any child s' of s with the minimal f-value, there must already exist a newly generated $\tilde{s}$ such that $f_{\{\mathcal{D},\bar{\mathcal{D}}\}}(\tilde{s}) \leq 2g_{\mathcal{H}}(I_{\text{b}})$. In other words, if no such $\tilde{s}$ exists, and TC3 holds, then no further expansions can improve C_{curr} and the search can stop early.

TC 4. *(i) $g_{\mathcal{D}}(s) \geq g_{\bar{\mathcal{H}}}(prt_{\bar{\mathcal{H}}}(I_b))$, (ii) $g_{\mathcal{D}}(s) > h_{\mathcal{D}}(s)$, (iii) $c(s, s') = \varepsilon$, and (iv) $f_{\mathcal{D}}(s') \geq C_{curr}$.*

The logic and intuition of **TC4** are similar to those of **TC3**. The two key differences in **TC4** are that (1) $f_{\{\mathcal{D},\bar{\mathcal{D}}\}}(\tilde{s}) < C_{\text{curr}}$, and (2) it is applied when $g_{\mathcal{D}}(I_{\text{b}}) \leq g_{\bar{\mathcal{D}}}(I_{\text{b}})$.

It is noteworthy that in all termination conditions, we do not need to compute the minimal g-value in $\text{OPEN}_{\mathcal{F}}$ and $\text{OPEN}_{\mathcal{B}}$ as in MM. In contrast, we rely on the readily available g and $\tilde{h}$ values that can be computed in constant time (e.g., of the best intersecting state and its parent). This is demonstrated in **TC2** which is similar to the condition $g_{\min\text{-}\mathcal{F}} + g_{\min\text{-}\mathcal{B}} + \varepsilon$ used by MM but is much cheaper to compute from a computational point of view.

4.2 Theoretic Properties and Guarantees

We now prove the correctness of MEET. We start by assuming the existence of an optimal solution with the length C^* going through a meeting state s_{meet}, following $C^* = g_{\mathcal{F}}(s_{\text{meet}})+g_{\mathcal{B}}(s_{\text{meet}})$. For all our proofs in this section, we set $S^{g>h}{}_{\mathcal{D}} := \{s \mid g_{\mathcal{D}}(s) > h_{\mathcal{D}}(s)\}$. Furthermore, we prove all lemmas for the forward direction, where the backward direction is analogous. We first prove that MEET satisfies MMP during expansions.

Lemma 1. *MEET never expands a state s with $g_{\mathcal{D}}(s) > \frac{C^*}{2}$.*

Proof. Assume, for contradiction, that MEET expands s with $g_{\mathcal{D}}(s) > \frac{C^*}{2}$. When s is generated, MEET sets $\tilde{h}_{\mathcal{D}}(s)$

to either $h_{\mathcal{D}}(s)$, $g_{\mathcal{D}}(s)$, or $g_{\bar{\mathcal{D}}}(s)$. As a result, $f_{\mathcal{D}}(s) = g_{\mathcal{D}}(s) + \tilde{h}_{\mathcal{D}}(s) \geq g_{\mathcal{D}}(s) + g_{\mathcal{D}}(s) > C^*$. Since MEET expands the states in a monotonically non-decreasing order of priority and $f_{\mathcal{D}}(s) = f_{min}$, it implies that all nodes with priority $\leq C^*$ have been expanded. As a result, MEET must already have found the optimal solution and such solution has been recorded, which means $C_{\text{curr}} = C^*$. This contradicts the fact that s is selected for expansion only when $f_{\mathcal{D}}(s) < C_{\text{curr}}$. Thus, MEET satisfies MMP. ☐

We now prove that $\tilde{h}$ is an admissible heuristic for every state that is expanded.

Lemma 2. *When MEET expands a state s, then $\tilde{h}_{\mathcal{D}}(s) \leq h^*_{\mathcal{D}}(s)$ holds.*

Proof. By Lemma 1, when s is selected for expansion, we directly have $f_{\mathcal{D}}(s) = g_{\mathcal{D}}(s) + \tilde{h}_{\mathcal{D}}(s) \leq C^*$. Thus, it provides

$$\tilde{h}_{\mathcal{D}}(s) = f_{\mathcal{D}}(s) - g_{\mathcal{D}}(s) \leq C^* - g_{\mathcal{D}}(s).$$

According to [Pearl, 1984], $g^*_{\mathcal{D}}(s) \leq g_{\mathcal{D}}(s)$ holds once $f_{\mathcal{D}}(s) \leq C^*$. Moreover, for any s, we also have $f^*_{\mathcal{D}}(s) = g^*_{\mathcal{D}}(s) + h^*_{\mathcal{D}}(s) \geq C^*$. Substituting $g^*_{\mathcal{D}}(s) \leq g_{\mathcal{D}}(s)$ gives

$$h^*_{\mathcal{D}}(s) = f^*_{\mathcal{D}}(s) - g^*_{\mathcal{D}}(s) \geq C^* - g_{\mathcal{D}}(s).$$

Consequently, $\tilde{h}_{\mathcal{D}}(s) \leq h^*_{\mathcal{D}}(s)$ holds. ☐

We continue to examine each stopping condition individually to ensure optimality.

Lemma 3. *If **TC1** holds, MEET returns with $C^* = C_{curr}$.*

Proof. Since states are ordered according to their f-value from low to high, and due to $\tilde{h}_{\mathcal{F}}(s)$ being admissible when s is expanded (Lemma 2), once **TC1** is met, we are guaranteed that no solution with an f-value lower than C_{curr} exists. ☐

Lemma 4. *If **TC2** holds, MEET returns with $C^* = C_{curr}$.*

Proof. We first prove that $g_{\mathcal{D}}(s) + g_{\overline{\mathcal{D}}}(t) + \varepsilon > C_{\text{curr}}$ implies both $2(g_{\mathcal{D}}(s) + \varepsilon) > C_{\text{curr}}$ and $2(g_{\overline{\mathcal{D}}}(t) + \varepsilon) > C_{\text{curr}}$. We proceed by contradiction. Suppose $2(g_{\mathcal{D}}(s) + \varepsilon) \leq C_{\text{curr}}$, equivalent to $g_{\mathcal{D}}(s) + \varepsilon \leq \frac{C_{\text{curr}}}{2}$. Then, it follows $C_{\text{curr}} - g_{\mathcal{D}}(s) \geq \frac{C_{\text{curr}}}{2} + \varepsilon$. Since $g_{\mathcal{D}}(s) + g_{\overline{\mathcal{D}}}(t) + \varepsilon > C_{\text{curr}}$, we have $g_{\overline{\mathcal{D}}}(t) + \varepsilon > C_{\text{curr}} - g_{\mathcal{D}}(s)$. Substituting the previous inequality, we obtain $g_{\overline{\mathcal{D}}}(t) + \varepsilon > \frac{C_{\text{curr}}}{2} + \varepsilon$, indicating $g_{\overline{\mathcal{D}}}(t) > \frac{C_{\text{curr}}}{2}$, which contradicts the fact that $g_{\overline{\mathcal{D}}}(t) < \frac{C_{\text{curr}}}{2}$. By symmetry, assuming $2(g_{\overline{\mathcal{D}}}(t) + \varepsilon) \leq C_{\text{curr}}$ similarly leads to a contradiction. Since $C_{\text{curr}} > C^*$ and $g_{\mathcal{D}}(I_{\text{b}}) \leq g_{\bar{\mathcal{D}}}(I_{\text{b}})$, it implies that $f_{\mathcal{D}}(s_i) < C_{\text{curr}}$ must hold in at least one direction before finding C^* , contradicting the fact that $2(g_{\mathcal{D}}(s) + \varepsilon) > C_{\text{curr}}$ and $2(g_{\overline{\mathcal{D}}}(t) + \varepsilon) > C_{\text{curr}}$. Thus, $C_{\text{curr}} = C^*$. ☐

Lemma 5. *If **TC3** holds, MEET returns with $C^* = C_{curr}$. If $C_{curr} > C^*$, then the expansion of s, which occurs after I_b is found, will not have a child s' such that $g_{\mathcal{D}}(s') = g_{\mathcal{D}}(s) + \varepsilon$ and $g_{\mathcal{D}}(s') > g_{\mathcal{H}}(I_b)$.*

Proof. **TC3** requires that when I_{b} appears in $\mathcal{D}$, $2g_{\mathcal{D}}(I_{\text{b}}) > C_{\text{curr}}$ and implies that any s' in $\{\mathcal{D}, \bar{\mathcal{D}}\}$ satisfies $f_{\{\mathcal{D},\bar{\mathcal{D}}\}}(s') > 2g_{\mathcal{D}}(I_{\text{b}})$ after expanding $\text{prt}_{\mathcal{D}}(I_{\text{b}})$. By contradiction, since $C_{\text{curr}} > C^*$, the search will eventually generate a state $\tilde{s}$ with $f_{\{\mathcal{D},\bar{\mathcal{D}}\}}(\tilde{s}) < C_{\text{curr}} < 2g_{\mathcal{D}}(I_{\text{b}})$ before reaching $C_{\text{curr}} = C^*$. Consequently, $\text{prt}_{\{\mathcal{D},\bar{\mathcal{D}}\}}(\tilde{s})$ must have been expanded before s, contradicting **TC3**. ☐

Lemma 6. *If **TC4** holds, MEET returns with $C^* = C_{curr}$. If $C_{curr} > C^*$, then the expansion of s, which occurs immediately after I_b is found, will not have a child s' such that $g_{\mathcal{D}}(s') = g_{\mathcal{D}}(s) + \varepsilon$ and $f_{\mathcal{D}}(s') \geq C_{curr}$.*

Proof. The proof of Lemma 6 is similar to the proof of Lemma 5. ☐

Theorem 1. *MEET returns an optimal solution, if it exists.*

Proof. By Lemmas 3-6, we establish that when either **TC1**, **TC2**, **TC3** or **TC4** hold it necessarily implies that $C^* = C_{\text{curr}}$. Thus, MEET only terminates the search when an optimal solution is found. Furthermore, if a solution exists, then **TC1** must hold at some point in the search, unless the search is terminated beforehand by one of the other termination conditions. Namely, MEET necessarily returns the optimal solution if a solution exists. ☐

5 Experiments

In this section we compare MEET with A*, MM and BAE*, where the latter was chosen as a representative of the non-MM family of Bi-HS algorithms that provides state-of-the-art performance [Alcázar *et al.*, 2020].[2] Additionally, we conducted an ablation study for MEET to separately assess the impact of using the $\tilde{h}$ heuristic instead of the static h heuristic and the impact of the pruning mechanism.

We ran experiments on two distinct domains: grid maps and 10-Pancake puzzles with all algorithms implemented in C++ on a desktop with 16GB RAM and an Intel i7 CPU running Ubuntu 20.04. The implementations of A*, MM, and BAE* were adapted from the source code of [Uras and Koenig, 2015], [Holte *et al.*, 2017], and [Hu and Speck, 2022], respectively. Table 1 displays summarized experimental results for runtime statistics, while Fig. 2, 3, and 4 illustrate, respectively, the distributions of each triggered TC, the average percentages of runtime reduction achieved by each triggered TC vs. the baseline **TC1**, and detailed time distributions. The code implementation and additional results, including details on state expansions and reduction percentages, can be found in the supplementary material [Wang *et al.*, 2025].

[2]We also ran A* in the backward direction and MMe (a variant of MM with a different state priority function [Holte *et al.*, 2017]), but these yielded very similar results to forward A* and MM, respectively, and were thus omitted.

Domain	Heuristic	A*	MM	BAE*	MEET
brc203	Euclidean Distance	**1.4** (0.42, 2.4)	216 (52, 366)	3.9 (1.3, 6.1)	2.2 (0.65, 3.3)
	Octile Distance	**1.6** (0.36, 3.1)	406 (82, 879)	4.3 (1.3, 6.5)	2.4 (0.65, 3.8)
orz100d	Euclidean Distance	**8.5** (1.8, 14.5)	3.467×10^3 (588, 6.411×10^3)	23.8 (5.2, 34.3)	13.4 (2.7, 22.1)
	Octile Distance	**9.2** (1.4, 16.5)	5.214×10^3 (864, 11.339×10^3)	25 (4.6, 37.2)	14.7 (2.4, 24.5)
10-Pancake puzzle	GAP-1	**0.31** (0.08, 0.75)	7.1 (1.5, 22.3)	0.46 (0.2, 1.5)	0.48 (0.14, 0.74)
	GAP-2	5.6 (3.6, 17.9)	447 (110, 2.275×10^3)	1.3 (0.74, 3.1)	**1.0** (0.45, 2.1)
	GAP-3	81 (45.8, 291)	12.189×10^3 (403, 27.420×10^3)	8.2 (3.0, 24.8)	**3.9** (2.2, 11.1)
	GAP-4	510 (224, 1.605×10^3)	36.1326×10^3 (529, 40.419×10^3)	26.1 (13.7, 78.8)	**6.5** (3.8, 32.5)

Table 1: Experimental results for grid maps and 10-Pancake puzzle domains. For each algorithm, we report the median runtime (rounded to integer values if greater than 50 milliseconds) and the interquartile range (IQR), indicated in parentheses as (25th percentile, 75th percentile).

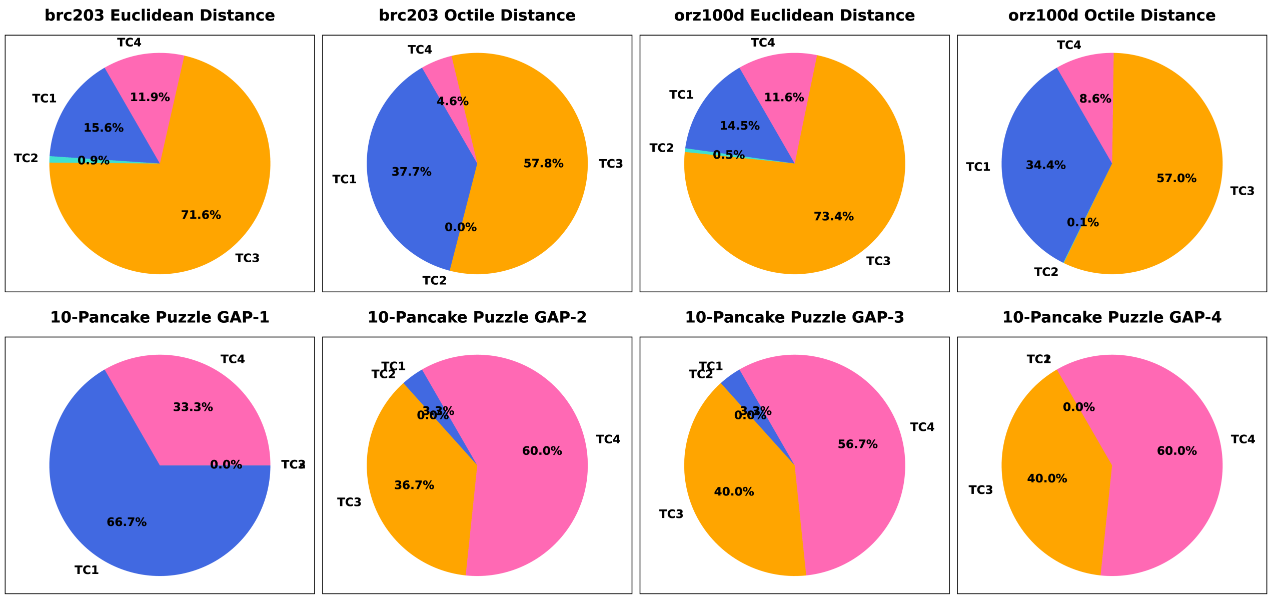


Figure 2: Triggered TC distributions. The colors (●), (●), (●), and (●) denote **TC1, TC2, TC3,** and **TC4**, respectively.

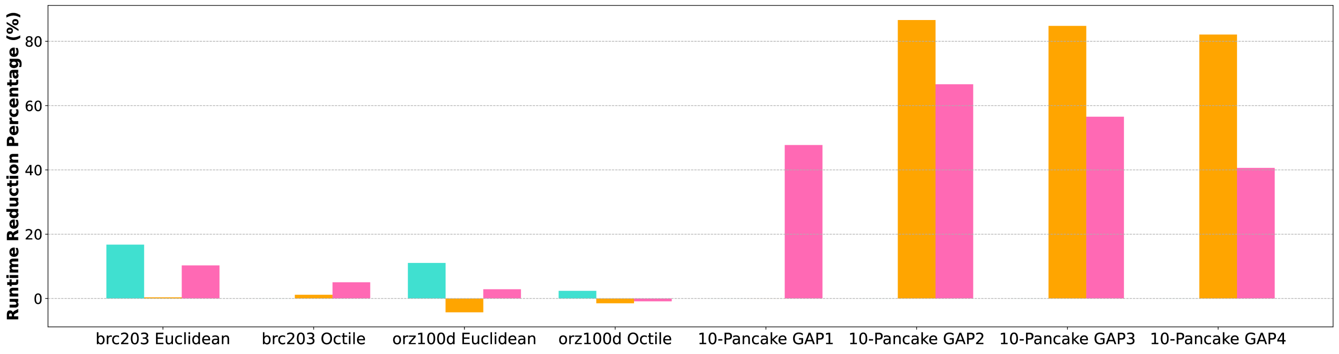
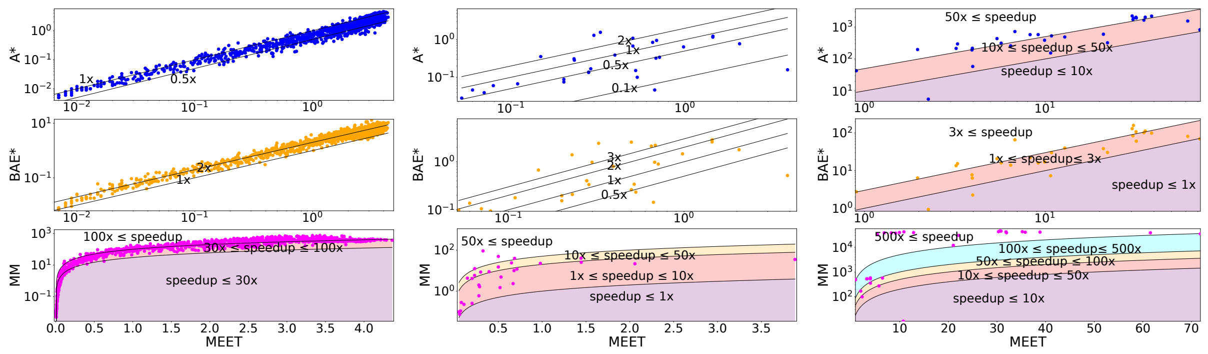


Figure 3: Average runtime reduction (%) vs. stopping on **TC1**. The bars (—), (—), and (—) denote **TC2, TC3,** and **TC4**, respectively.

5.1 Grid Maps

We evaluate each algorithm on two grid maps from the MovingAI benchmark repository [Sturtevant, 2012], both originating from Dragon Age: Origins (DAO). These maps feature traversable regions that vary significantly across instances, providing substantial variability and ensuring a thorough evaluation of each algorithm's performance. An agent can move in an 8-way movement, and all evaluations use the Octile and Euclidean distance heuristics. The first map, *brc203d* (274 × 391), has 1290 instances, following [Holte *et al.*, 2017]. The second map, *orz100d* (412 × 395), contains 2420 instances, and is considered relatively challenging, as the heuristics are often less informative on this map.

The experimental results demonstrate that MEET signifi-



(a) brc203 (Euclidean distance) (b) 10-Pancake puzzle (GAP-1) (c) 10-Pancake puzzle (GAP-4)

Figure 4: Time distributions, in seconds. The symbols (●), (●), and (●) denote MEET vs. A*, BAE*, and MM, respectively.

cantly improves the search speed as an optimal Bi-HS algorithm, outperforming MM by at least two orders of magnitude (Fig. 4a and Table 1). The runtime of MEET is still on par with A* in *brc203d* and roughly twice as slow in *orz100d*. MEET expands slightly more states than BAE* in *orz100d* but achieves better runtime.

Considering the ablation study, running MEET with the static heuristic h results in similar runtimes compared to running it with $\tilde{h}$, but the solutions found are often sub-optimal, which aligns with the theory, as the termination conditions **TC1**–**TC4** rely on f-values defined by $f = g + \tilde{h}$. Finally, the results show that the pruning process reduces the runtime of MEET by an average of roughly 2% over all the instances in grid maps. Results omitted for breivity.

5.2 10-Pancake Puzzle

Our second domain is the well-known Pancake puzzle: A state in the 10-Pancake puzzle is represented by a vector of 10 different numbers, where the goal is to sort them in ascending order through a sequence of flips from a given start state. In our experiments, we apply both GAP, as described in [Helmert, 2010] and GAP-X heuristics [Holte *et al.*, 2017]. The GAP-X heuristics are variants of the original GAP heuristic, devised to be less accurate. GAP-X heuristics exclude gaps that include pancakes among the smallest X in size, where the smallest X in our test dataset is 1. As X increases, GAP-X becomes less accurate. We conduct experiments on 30 random instances, with optimal solution costs (C^*) ranging between 7 and 10.

As indicated by the median search time in Table 1, it is at least $100\times$ faster than MM and an order of magnitude faster than A* as the heuristic accuracy decreases, which suggests the improved efficiency of our algorithm in scenarios with weaker heuristics. Compared to BAE*, MEET consistently achieves better performance, where the performance gap expands with weaker heuristics.

Fig 4b and 4c illustrate the advantages of MEET in runtime over A* and MM. We note that the performance of A* is highly dependent on the accuracy of the heuristic, and when compared to Bi-HS algorithms this dependence is exacerbated, since Bi-HS algorithms also exploit information on g-values from both sides, thus reducing the importance of heuristic accuracy. This explains the dominance of MEET over A* with GAP-X for $X > 1$.

As for the ablation study, the findings in this domain are consistent with those described for the grid maps domain: optimality can only be guaranteed if $\tilde{h}$ is utilized, and pruning reduces the runtime of MEET by an average of roughly 3%.

Lastly, we analyze differences between the TCs using Fig. 2 and 3. Fig. 2 indicates that **TC1**, **TC3** and **TC4** are all frequently triggered in different cases, providing meaningful contributions. Fig. 3 shows that **TC3** and **TC4** save more runtime as the heuristic weakens, whereas **TC1** leads when the heuristic is nearly perfect (e.g., GAP-1). Fig. 3 also shows that it is almost always preferable to use **TC2**, **TC3** and **TC4** in addition to **TC1** in terms of runtime. This is because the computation time required to evaluate them is insignificant, while the potential runtime reduction is significant.

6 Discussion and Future Work

While significant progress and achievements have been made in Bi-HS, no algorithm thus far has been able to satisfy MM-optimality while providing competitive runtime. Additionally, the challenge of terminating the search early while still ensuring both optimality and MMP was unsolved. This paper closes these gaps by introducing a new bidirectional best-first search algorithm, MEET, that guarantees MM-optimality with significantly improved runtime compared to MM. Key to the efficiency of our approach is a set of new termination conditions that test states when they are generated and *not* when they are expanded. A predicate that is computationally cheap to test. This is in contrast to MM whose termination condition can only be evaluated when a meeting state is expanded.

Interestingly two of the termination conditions (**TC3** and **TC4**) make use of the minimal edge cost ε. This is natural in settings such as grid worlds where many, if not all, edges have the same minimal cost. However, in settings where edge costs can take arbitrary values, these termination conditions can prove to be ineffective. However, by inflating ε by some scalar w, we can use these termination conditions to terminate while ensuring a *bounded suboptimal* solution whose approximation factor is defined by w. Further exploration of the implication of ε in **TC3** and **TC4** is left for future work.

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