Tight Asymptotic Bounds for Fair Division with Externalities

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Abstract

We study the problem of allocating a set of indivisible items among agents whose preferences include externalities. Unlike the standard fair division model, agents may derive positive or negative utility not only from items allocated directly to them, but also from items allocated to other agents. Since exact envy-freeness cannot be guaranteed, prior work has focused on its relaxations. However, two central questions remained open: does there always exist an allocation that is envy-free up to one item (EF1), and if not, what is the optimal relaxation EF-k that can always be attained?

We settle both questions by deriving tight asymptotic bounds on the number of items sufficient to eliminate envy. We show that for any instance with n agents, an allocation that is envy-free up to $\mathcal{O}(\sqrt{n})$ items always exists and can be found in polynomial time. Additionally, via a reduction from fair division with externalities to discrepancy theory combined with recent discrepancy lower bounds, we prove a matching $\Omega(\sqrt{n})$ lower bound showing that this result is tight, even when the valuations are binary and satisfy the no-chores condition, which refutes a conjecture from previous work and resolves the main open question in the area, ruling out the existence of EF1 allocations when agents have externalities.

1 Introduction

Fair division of indivisible items, including administrative tasks, laboratory equipment, or heritage between heirs, is an active research area that connects computer science and economics [Brams and Taylor, 1996; Thomson, 2016; Amanatidis *et al.*, 2023; Nguyen and Rothe, 2023]. Traditionally, a fair division instance consists of a set of indivisible items and a set of agents, where each agent has a valuation function assigning a numerical utility to each possible bundle. Crucially, in the standard model, the utility of an agent depends solely on the items allocated to it. The goal is to partition the items among agents so that a given fairness criterion is satisfied. A prominent example is *envy-freeness* (EF) [Foley, 1967], which requires that each agent weakly prefers its own bundle over the bundle of any other agent. Since EF allocations are not guaranteed to exist (consider two agents and one positively-valued item), several relaxations have been studied, most notably *envy-freeness up to one item* (EF1) [Lipton *et al.*, 2004; Budish, 2011; Caragiannis *et al.*, 2019] and *envy-freeness up to any item* (EFX) [Caragiannis *et al.*, 2019; Plaut and Roughgarden, 2020]. It is known that EF1 allocations always exist in the standard fair division model [Lipton *et al.*, 2004].

Nevertheless, there are many scenarios where the utility an agent experiences cannot depend solely on its own bundle. Consider a dormitory allocating newly purchased equipment among students: a student benefits not only from items given directly to them, but also from items assigned to their roommate, with whom they can share. Conversely, when dividing resources among competing firms, such as exclusive licenses, patents, or territory rights, a firm may be harmed when a rival acquires a valuable asset, even if the firm's own allocation remains unchanged. Externalities can also be mixed: when distributing research grants among academics, a researcher may benefit if a collaborator receives funding for a joint project but suffer if a competitor in the same field does. All these examples have one aspect in common: the inherent *externalities* underlying the agents' valuations.

Although externalities are well studied in the broader economics literature on resource allocation [Ayres and Kneese, 1969; Katz and Shapiro, 1985; Velez, 2016], most prior work in fair division focuses on *divisible* items [Brânzei *et al.*, 2013b; Li *et al.*, 2015]. The model of fair division of indivisible items with externalities was formulated only recently [Seddighin *et al.*, 2019; Mishra *et al.*, 2022; Aziz *et al.*, 2023; Deligkas *et al.*, 2024]. Most relevant to our work are Aziz *et al.* [2023], who adapted the standard envy-based fairness notions to the externalities setting, and Deligkas *et al.* [2024], who investigated the computational complexity of deciding whether a fair allocation exists.

Despite this progress, perhaps the most fundamental question in this area has remained open:

> Is an EF1 allocation guaranteed to exist for every instance of fair division with externalities?

Our current understanding of this question is limited. EF1 allocations are known to always exist for two agents, and for three agents with binary valuations and no chores[1] [Aziz *et al.*, 2023]. Beyond these special cases, even the existence of EF-k allocations for any constant k was unknown. Aziz *et al.* [2023] explicitly state that:

> An important open question that remains from our work is whether there always exists an EF1 allocation [...] If the answer is negative, it would be reasonable to ask for the optimal relaxation EF-k that can be attained.

Deligkas *et al.* [2024] further conjectured that EF1 allocations always exist for binary valuations, while acknowledging that a proof "seems highly non-trivial" given the extensive case analysis and computer verification required even for the case of three agents.

1.1 Our Contribution

We show that EF1 allocations do not always exist, even if agents' preferences are binary, hence resolving the open question posed by both Aziz *et al.* [2023] and Deligkas *et al.* [2024]. We supplement this by showing that the optimal relaxation EF-k that can be attained is for $k = \Theta(\sqrt{n})$, where n is the number of agents in the instance, answering the second open question of Aziz *et al.* [2023].

After a formal definition of the model of fair division with externalities in Section 2, we introduce the technical tools that we use to prove our results in Section 3. There are two main ingredients. First, we introduce an *asymmetric envy model* and show its equivalence with the externalities model. Valuations in the asymmetric envy model are closer to the standard additive valuations than the valuations in the original externalities model. That said, this model is mostly of theoretical interest: the valuations are not easy to interpret, and even a social-welfare maximizing allocation is not well-defined with these valuations. The second key component of our work is the framework of *multicolored discrepancy* [Doerr and Srivastav, 2003; Manurangsi and Suksompong, 2022; Caragiannis *et al.*, 2025; Manurangsi and Meka, 2026]. Importantly, we generalize some of the discrepancy bounds known from prior work to settings with a mixture of goods and chores, which may be of independent interest.

In Section 4, we use the technical toolbox developed in Section 3 to show that, for any fair division with externalities, there exists a deterministic polynomial-time mechanism $\mathcal{M}_1$ that finds an allocation that is envy-free up to $\mathcal{O}(\sqrt{n})$ items. Importantly, in Section 5, we show that the mechanism is asymptotically optimal. Specifically, we show that there is an instance in the externalities model such that the best relaxation EF-k we can achieve is for $k \in \Omega(\sqrt{n})$, even if the valuations are binary and there are no chores. This settles the question of the existence of EF1 in the negative and therefore refutes the conjecture of Deligkas *et al.* [2024]. Moreover, we show that the same lower bound holds for instances where the externalities are highly structured: even when all externalities are directed towards a single designated agent a_1.

Before we conclude in Section 7 with open problems and future directions, in Section 6, we present an additional randomized mechanism $\mathcal{M}_2$. This mechanism always produces an allocation that is EF up to $\mathcal{O}(n)$ items, which is a worse guarantee than that of mechanism $\mathcal{M}_1$. However, unlike $\mathcal{M}_1$, $\mathcal{M}_2$ produces a *consensus* allocation, which means that we can arbitrarily permute the bundles while maintaining the fairness guarantee. Consequently, $\mathcal{M}_2$ is truthful in expectation, i.e., no agent can improve its expected utility by misreporting its valuations. This extends the result of Bu and Tao [2025] to the setting of externalities, with a slightly weaker bound on the relaxation of fairness that we obtain.

[1] See Section 2 for a formal definition of chores in this model.

1.2 Related Work

Apart from the papers directly studying externalities in fair division of divisible and indivisible items mentioned above, there are several streams of related models in the literature.

First, there is a model of fair division with *allocator's preferences* [Bu *et al.*, 2023] or *social impact* [Flammini *et al.*, 2025; Deligkas *et al.*, 2026]. In both of these models, the goal is to find an allocation that is not only fair among agents, but also satisfies certain objectives of the allocator. This additional dimension of preferences can in principle be represented using externalities: the allocator is an additional agent with externalities towards all other agents. However, the crucial difference is that the allocator cannot receive any item and the original agents do not have any pairwise externalities. A more general model is *fair division for groups* [Manurangsi and Suksompong, 2017; Suksompong, 2018; Segal-Halevi and Nitzan, 2019; Manurangsi and Suksompong, 2024; Gölz and Yaghoubizade, 2026; Dupré la Tour, 2026], where agents are partitioned into groups and share items allocated to their group, while each agent retains their own subjective valuations, and for which constant relaxations of envy-freeness are known to not exist in most settings.

In *asymptotic fair division* [Manurangsi and Suksompong, 2020; Manurangsi and Suksompong, 2021; Manurangsi and Suksompong, 2025; Manurangsi *et al.*, 2025; Garg *et al.*, 2026], the task is not to decide which relaxation of fairness we can guarantee (since EF1 allocations always exist in the standard model), but rather to identify conditions under which stronger guarantees such as envy-freeness or proportionality can be achieved. Many of these works provide tight bounds with respect to the number of items required and rely heavily on results from discrepancy theory, similarly to our approach.

Externalities in preferences also appear in contexts beyond fair division. Notable examples include combinatorial auctions [Funk, 1996; Jehiel and Moldovanu, 1996; Zhang *et al.*, 2018], matching [Mumcu and Saglam, 2010; Brânzei *et al.*, 2013a; Fisher and Hafalir, 2016; Anshelevich *et al.*, 2017], facility location [Li *et al.*, 2019; Wang *et al.*, 2024; Li *et al.*, 2025], and house allocation [Massand and Simon, 2019; Elkind *et al.*, 2020; Agarwal *et al.*, 2021; Gross-Humbert *et al.*, 2022; Knop and Schierreich, 2023]. In these settings, an agent is typically assigned only one item (a partner, location, or house), and the solution concepts differ substantially from ours.

2 Preliminaries

In the fair division problem, the input is a set M of indivisible items and a set N of n agents. The items in M are to be allocated among the agents in N. Formally, an *allocation* of the items is a tuple of pairwise disjoint subsets of M: $(A_1, \ldots, A_n)$, where each $A_i \subseteq M$ is the bundle assigned to agent i. We say that allocation is *complete* if $\bigcup_{i\in[n]} A_i = M$. Unless stated otherwise, we consider only complete allocations. Let $\mathcal{A}$ be the set of all allocations. Additionally, we denote the set $\{1, \ldots, n\}$ by $[n]$.

In the fair division with externalities model of [Aziz *et al.*, 2023], each agent $i \in N$ is associated with a function

$$V_i\colon \mathcal{A} \to \mathbb{R},$$

which assigns a value to every allocation $A \in \mathcal{A}$. We assume that valuations are *additive*: for any agent $i \in N$ and any allocation $(A_1, \ldots, A_n) \in \mathcal{A}$,

$$V_i(A) = \sum_{j\in N} \sum_{x\in A_j} V_i\big(x^{(j)}\big),$$

where $x^{(j)} \in \mathcal{A}$ denotes the allocation in which agent j receives the single item x and all other agents receive no items, that is,

$$x^{(j)} = (\emptyset, \ldots, \emptyset, \underbrace{\{x\}}_{j^{\text{th}}\text{ coordinate}}, \emptyset, \ldots, \emptyset).$$

To simplify the notation, we write $V_i(j, x)$ for $V_i\big(x^{(j)}\big)$ and therefore

$$V_i(A) = \sum_{j\in N} \sum_{x\in A_j} V_i(j, x).$$

Note that additivity implies, in particular, that valuations are *normalized*: for every agent $i \in N$, the empty allocation has value zero, i.e., $V_i(\emptyset, \ldots, \emptyset) = 0$.

We say that the valuation V_i of agent $i \in N$ has *no chores* if, for every agent $j \in N$ and every item $x \in M$, we have

$$V_i(i, x) \geq V_i(j, x).$$

That is, agent i weakly prefers receiving item x herself to another agent receiving x.

We say that V_i is *binary* if, for all $j \in N$ and all $x \in M$,

$$V_i(j, x) \in \{0, 1\}.$$

2.1 Fairness in the Externalities Model

Given an allocation $A \in \mathcal{A}$ and two distinct agents $i, j \in N$, we denote by $A^{i\leftrightarrow j}$ the allocation obtained from A by swapping the bundles of agents i and j, while the bundles of all other agents remain unchanged. In an allocation A, agent i *envies* agent j if

$$V_i(A) < V_i\big(A^{i\leftrightarrow j}\big),$$

that is, agent i would prefer to swap her bundle with that of agent j. An allocation A is *envy-free* (EF) if no agent envies another agent. A is *envy-free up to c items* (EF-c) if, for every pair of distinct agents $i, j \in N$, there exists a set of items $S \subseteq M$ with $|S| \leq c$ such that, defining $B = (A_1 \setminus S, \ldots, A_n \setminus S)$, we have

$$V_i(B) \geq V_i\big(B^{i\leftrightarrow j}\big).$$

That is, for each pair of agents i and j, there exists a set of at most c items whose removal eliminates agent i's envy toward agent j.

Note that, since valuations are assumed to be additive, the set S used to eliminate agent i's envy toward agent j can always be chosen as a subset of $A_i \cup A_j$, as the items allocated to other agents contribute equally to $V_i(B)$ and $V_i(B^{i\leftrightarrow j})$. Moreover, if the valuation V_i has no chores, the set S can be chosen as a subset of A_j. Finally, in the absence of externalities (i.e. when $V_i(j, x) = 0$ for every item x and pair of agents i, j) this definition coincides with the standard definition of envy-freeness up to c items.

3 Technical Tools

In this section, we present some technical tools that we use in our proofs. First, we present an alternate model of fair division instances, the asymmetric envy model, and prove that this model is equivalent for our purposes. Next, we describe the multicolor discrepancy problem and its connections to fair division. Most of our results derive fair division bounds from bounds in multicolor discrepancy.

3.1 Asymmetric Envy Model

We now introduce the *asymmetric envy model*, which is an alternate fair division model that allows for externalities among the agents' valuation functions. We will show that the asymmetric envy model is equivalent to the externalities model with respect to envy-based notions of fairness. We analyze most of our results using the asymmetric envy model, as the envy-based fairness notions in this model are easier to work with. This model does not assign a numerical value to each agent for a complete allocation; consequently, notions such as welfare maximization are not well defined. Instead, it specifies a notion of envy between pairs of distinct agents $i, j \in N$. The model is *asymmetric* in the sense that the function defining the envy of agent i toward agent j depends on the ordered pair (i, j). Envy from i towards j is captured by a valuation function $v_{i,j}$, and the standard notions of envy-freeness and envy-freeness up to c items can be defined using these valuation functions. More precisely, for each pair of agents $i, j \in N$ with $i \neq j$, we are given a valuation function

$$v_{i,j}\colon 2^M \to \mathbb{R}.$$

We assume that these valuations are *additive*, that is, for every set $S \subseteq M$,

$$v_{i,j}(S) = \sum_{x\in S} v_{i,j}(x).$$

We say that $v_{i,j}$ has *no chores* if $v_{i,j}(x) \geq 0$ for all $x \in M$. We say that the instance is *binary* if $v_{i,j}(x) \in \{0, 1\}$ for all $i \neq j$ and all $x \in M$.

Fairness in the Asymmetric Envy Model

Given an allocation $(A_1, \ldots, A_n) \in \mathcal{A}$, agent i is said to *envy* agent j if

$$v_{i,j}(A_i) < v_{i,j}(A_j).$$

An allocation A is *envy-free in the asymmetric envy model* (EF$_a$) if no agent envies another agent. A is *envy-free up to c items in the asymmetric envy model* (EF$_a$-c) if, for every pair of distinct agents $i, j \in N$, there exists a set of items $S \subseteq A_i \cup A_j$ with $|S| \leq c$ such that

$$v_{i,j}(A_i \setminus S) \geq v_{i,j}(A_j \setminus S).$$

Note that if $v_{i,j}$ has no chores, the set S can always be chosen as a subset of A_j.

Next, we show that the asymmetric envy model is equivalent to the externalities model with respect to envy-based fairness notions such as envy-freeness and EF-c. That is, there exists a natural reduction from instances of the externalities model to those of the asymmetric envy model, such that any envy bound in the asymmetric envy model implies an equivalent bound in the externalities model.

Proposition 1. *There exists a function f that maps each instance I of the externalities model to an instance $f(I)$ of the asymmetric envy model such that, for every allocation $A \in \mathcal{A}$, the allocation A is EF (respectively, EF-c) in I if and only if it is EF$_a$ (respectively, EF$_a$-c) in $f(I)$. Moreover, if I has no chores then $f(I)$ has no chores. Additionally:*

- *For any instance J in the asymmetric envy model with* additive *valuations, there is an instance I in the externalities model such that $J = f(I)$ (i.e., f is surjective).*
- *For any instance J in the asymmetric envy model with* binary additive *valuations, there is a* binary no-chores *instance I in the externalities model such that $J = f(I)$.*

3.2 Multicolor Discrepancy and Fair Division

Manurangsi and Suksompong [2022] showed that tools from multicolor discrepancy can be leveraged to derive improved bounds for fairness notions of the form EF-c in models with groups of agents. In this section, we describe the particular definitions and discrepancy guarantee that we will use later. We also extend their statement from the "goods-only" case to instances that may contain both goods and chores.

We present a multicolor discrepancy in a formulation tailored to allocations of indivisible items. This is equivalent to the standard matrix-based definition, but is more convenient for our purposes. The version of multicolor discrepancy that we need is a recent generalization of Manurangsi and Meka [2026]; it is an asymmetric setting in which valuations depend on the color. For a broader introduction to the connection between discrepancy theory and fair division, we refer the interested reader, e.g., to [Manurangsi and Suksompong, 2022; Dupré la Tour and Fujii, 2025].

Let $k \geq 1$ be an integer denoting the number of colors, and let M be a set of m items. Let $n_1 \geq \cdots \geq n_k$ be positive integers. For each $\ell \in [k]$, let $\mathcal{V}_\ell = \{v_{\ell,1}, \ldots, v_{\ell,n_\ell}\}$ be a collection of n_ℓ additive valuation functions, and define $\mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_k)$.

A *k-coloring* of M is a map $\chi : M \to [k]$, which induces a partition

$$M = \chi^{-1}(1) \cup \cdots \cup \chi^{-1}(k).$$

The *asymmetric discrepancy* of $\mathcal{V}$ is defined as

$$\operatorname{asymdisc}(\mathcal{V}, k) := \min_{\chi: M \to [k]} \max_{\ell \in [k]} \max_{i \in [n_\ell]} \left| \tfrac{1}{k} v_{\ell,i}(M) - v_{\ell,i}(\chi^{-1}(\ell)) \right|.$$

In words, $\operatorname{asymdisc}(\mathcal{V}, k)$ is the minimum, over all k-colorings of M, of the maximum (over colors and valuation functions) deviation between the value that $v_{\ell,i}$ assigns to its own color class and the proportional benchmark $\frac{1}{k} v_{\ell,i}(M)$.

Theorem 1 (Manurangsi and Meka [2026, Theorem 5.4])**.** *Assume that for every $\ell \in [k]$, $i \in [n_\ell]$, and $x \in M$, we have $v_{\ell,i}(x) \in [0, 1]$. Then*

$$\operatorname{asymdisc}(\mathcal{V}, k) \leq \mathcal{O}(\sqrt{n_1}).$$

Moreover, a coloring achieving this bound can be found in deterministic polynomial time [Levy et al.*, 2017].*

In the symmetric case where $n = n_1 = \cdots = n_k$ and where $v_{\ell,i} = v_{\ell',i}$ for all $i \in [n]$ and all $\ell, \ell' \in [k]$, this result coincides exactly with Theorem A.1 of Manurangsi and Suksompong [2022].

We will use Theorem 1 together with the following lemma to obtain EF$_a$-c guarantees. A related implication is implicit in the proof of the upper bound in Manurangsi and Suksompong [2022]; here, we prove an extension that allows for a mixture of goods and chores. The lemma shows that it suffices to control discrepancy with respect to four suitably chosen auxiliary valuations with item values in $[0, 1]$: if two bundles are approximately balanced under each of these auxiliary valuations, then any envy between them under the original valuation can be eliminated by discarding only a small number of items.

Lemma 1. *Let $k \geq 1$ and let T be a positive integer threshold. Let $v : 2^M \to \mathbb{R}$ be an additive valuation function that may assign both positive and negative values to items. Then there exist four auxiliary additive valuation functions $v^1, v^2, v^3, v^4 : 2^M \to \mathbb{R}$ such that $v^j(x) \in [0, 1]$ for all $j \in [4]$ and all $x \in M$, and with the following property: If $A, B \subseteq M$ are two disjoint sets of items satisfying for all $j \in [4]$ and all $X \in \{A, B\}$*

$$\left| \tfrac{1}{k} v^j(M) - v^j(X) \right| \leq T, \tag{1}$$

then one can eliminate A's envy for B under the original valuation v by discarding at most $14T$ items from $A \cup B$, and similarly, one can eliminate B's envy for A by discarding at most $14T$ items from $A \cup B$.

For the lower bound, we will use the closely related notion of *p-weighted discrepancy*. Given a set $V = \{v_1, \ldots, v_n\}$ of n additive valuation functions and a parameter $p \in (0, 1)$, the p-weighted discrepancy is defined as

$$\operatorname{wdisc}_p(V) := \min_{A \subseteq M} \max_{i \in [n]} \left| p \cdot v_i(M) - v_i(A) \right|.$$

Theorem 3.2 of Manurangsi and Meka [2026] establishes the following lower bound.

Theorem 2 ([Manurangsi and Meka, 2026])**.** *For every $p \in (0, 1)$ and $n \in \mathbb{N}$, there exist a set M of items and a collection $V = \{v_1, \ldots, v_n\}$ of n binary additive valuation functions such that*

$$\operatorname{wdisc}_p(V) \geq \tfrac{\sqrt{n-1}}{16}.$$

4 An $\mathcal{O}(\sqrt{n})$ Upper Bound

We now combine Lemma 1 with the asymmetric discrepancy result of Manurangsi and Meka [2026] (Theorem 1) to obtain an $\mathcal{O}(\sqrt{n})$ bound in the asymmetric envy model. This construction is *non-consensus*, i.e. it produces an allocation $(A_1, \ldots, A_n)$ tailored to the agent labels, and the guarantee does not necessarily hold under permutations of the bundles.

Theorem 3. *Let M be a set of items and let $(v_{i,j})_{i,j\in N,\, i\neq j}$ be an instance of the asymmetric envy model with n agents, where each $v_{i,j}$ is additive and may assign both positive and negative values to items. Then there exists a partition*

$$M = A_1 \sqcup \cdots \sqcup A_n$$

such that for every ordered pair of distinct agents $i, j \in N$, one can eliminate i's envy toward j (measured by $v_{i,j}$) between A_i and A_j by discarding at most $\mathcal{O}(\sqrt{n})$ items from $A_i \cup A_j$. Equivalently, the allocation $A = (A_1, \ldots, A_n)$ is EF$_a$-$\mathcal{O}(\sqrt{n})$. Moreover, such an allocation can be computed in deterministic polynomial time.

Proof. Let $k := n$ (the number of colors). Let $T = \mathcal{O}(\sqrt{n})$ be the discrepancy bound promised by Theorem 1 when applied with

$$n_1 = 8(n-1).$$

For every ordered pair $i \neq j$, apply Lemma 1 to the valuation function $v_{i,j}$ with parameters $k = n$ and T. This yields four additive valuation functions

$$v^1_{i,j},\ v^2_{i,j},\ v^3_{i,j},\ v^4_{i,j} : 2^M \to \mathbb{R},$$

with $v^t_{i,j}(x) \in [0,1]$ for all $t \in \{1,2,3,4\}$ and $x \in M$.

We now build an asymmetric discrepancy instance with $k = n$ colors as follows. For each color $\ell \in [n]$, define the multiset of valuation functions

$$\begin{aligned}\mathcal{V}_\ell \ :=\ &\{\, v^t_{\ell,j} : j \in [n]\setminus\{\ell\},\ t \in \{1,2,3,4\} \,\} \cup \\ &\{\, v^t_{j,\ell} : j \in [n]\setminus\{\ell\},\ t \in \{1,2,3,4\} \,\}.\end{aligned}$$

Thus $|\mathcal{V}_\ell| = 8(n-1)$ for every ℓ, and all functions in all $\mathcal{V}_\ell$ satisfy the boundedness condition of Theorem 1.

Applying Theorem 1 yields a coloring $\chi : M \to [n]$ such that for every $\ell \in [n]$ and every $u \in \mathcal{V}_\ell$,

$$\left|\tfrac{1}{n}u(M) - u(\chi^{-1}(\ell))\right| \leq T.$$

Let $A_\ell := \chi^{-1}(\ell)$. Then $(A_1, \ldots, A_n)$ is a partition of M.

Fix any ordered pair $i \neq j$. By construction, the four functions $v^1_{i,j}, \ldots, v^4_{i,j}$ belong to both $\mathcal{V}_i$ and $\mathcal{V}_j$, so for each $t \in \{1,2,3,4\}$ we have

$$\left|\tfrac{1}{n}v^t_{i,j}(M) - v^t_{i,j}(A_i)\right| \leq T$$

and

$$\left|\tfrac{1}{n}v^t_{i,j}(M) - v^t_{i,j}(A_j)\right| \leq T.$$

Thus condition (1) of Lemma 1 holds for the disjoint sets A_i and A_j (with $k = n$), and Lemma 1 implies that i's envy toward j with respect to $v_{i,j}$ can be eliminated by discarding at most $14T = \mathcal{O}(\sqrt{n})$ items from $A_i \cup A_j$.

Since this holds for every ordered pair $i \neq j$, the allocation is EF$_a$-$\mathcal{O}(\sqrt{n})$. Deterministic polynomial-time computability follows from Theorem 1. □

By combining Theorem 3 with repeated application of Proposition 1, we obtain the same upper-bound also for the externalities model.

Corollary 1. *Given an instance of the externalities model with n agents, there exists a deterministic polynomial-time algorithm that outputs an allocation $(A_1, \ldots, A_n)$ that is EF-$\mathcal{O}(\sqrt{n})$.*

5 An Asymptotically Tight Lower Bound

In this section, we prove a matching lower bound, showing that Corollary 1 is asymptotically tight, even if the instance is binary additive and with no chores. For simplicity, we first prove our lower bound in the asymmetric envy model with binary valuations. We then use Proposition 1 to show that this result implies an equivalent lower bound in the externalities model with binary no-chores valuations.

Let n be any positive odd integer, $q = (n-1)/2$, and $S_1, \ldots, S_q$ be q subsets of a set of items M. For any $S \subseteq M$, let v^S be the binary additive valuation function defined by $v^S(A) := |A \cap S|$ for all $A \subseteq M$. Let $\mathcal{V} = \{v^{S_1}, \ldots, v^{S_q}\}$.

Using the above valuation profile, we can construct an associated instance of the asymmetric envy model as follows.

For all $j \geq 2$, we let $v_{1,j}(A) := |A|$. For each $i \in [q]$, we define $\bar{S}_i := M \setminus S_i$. Then, for each $i \in [q]$, for all $j \in [n] \setminus \{2i\}$, we let

$$v_{2i,j}(A) := \begin{cases} v^{S_i}(A), & \text{if } j = 1, \\ v^{\bar{S}_i}(A), & \text{otherwise,} \end{cases}$$

and for all $j \in [n] \setminus \{2i+1\}$,

$$v_{2i+1,j}(A) := \begin{cases} v^{\bar{S}_i}(A), & \text{if } j = 1, \\ v^{S_i}(A), & \text{otherwise.} \end{cases}$$

We show that if this instance admits an EF$_a$-c allocation for some integer c, then its $1/n$-weighted discrepancy is at most $6c$.

Lemma 2. *In the above instance, suppose there exists a partition $M = A_1 \sqcup \cdots \sqcup A_n$ that is EF$_a$-c for some integer $c \geq 1$. Then*

$$\mathrm{wdisc}_{1/n}(\mathcal{V}) \leq 6c.$$

Proof. Fix an EF$_a$-c allocation $M = A_1 \sqcup \cdots \sqcup A_n$ and write $m = |M|$. By the definition of $\mathrm{wdisc}_{1/n}(\mathcal{V})$, choosing the set $A_1 \subseteq M$ gives

$$\mathrm{wdisc}_{1/n}(\mathcal{V}) \leq \max_{i\in[q]} \left| \frac{1}{n}\, v^{S_i}(M) - v^{S_i}(A_1) \right|.$$

Since $v^{S_i}(X) = |S_i \cap X|$, we have $v^{S_i}(M) = |S_i|$ and $v^{S_i}(A_1) = |S_i \cap A_1|$, hence it suffices to show that for every $i \in [q]$,

$$\left| |S_i \cap A_1| - \frac{|S_i|}{n} \right| \leq 6c.$$

Fix i and abbreviate $\bar{S}_i = M \setminus S_i$.

Step 1: all bundles have almost the same size. Recall that agent 1 values every item at 1, so for every bundle X and every $j \geq 2$ we have $v_{1,j}(X) = |X|$. Since the allocation is EF$_a$-c, for every $j \geq 2$,

$$|A_1| = v_{1,j}(A_1) \geq v_{1,j}(A_j) - c = |A_j| - c.$$

Moreover, by the definition of the valuation profile (for this fixed i), agent $2i$ counts S_i-items when comparing to agent 1 and counts $\bar{S}_i$-items when comparing to agent $2i+1$, and agent $2i+1$ behaves symmetrically. Applying EF$_a$-c along these comparisons yields

$$|S_i \cap A_{2i}| \geq |S_i \cap A_1| - c, \quad |\bar{S}_i \cap A_{2i}| \geq |\bar{S}_i \cap A_{2i+1}| - c,$$

and

$|S_i \cap A_{2i+1}| \geq |S_i \cap A_{2i}| - c, \quad |\bar{S}_i \cap A_{2i+1}| \geq |\bar{S}_i \cap A_1| - c.$

Combining,

$$\begin{aligned}|A_{2i}| &= |S_i \cap A_{2i}| + |\bar{S}_i \cap A_{2i}| \\ &\geq (|S_i \cap A_1| - c) + (|\bar{S}_i \cap A_{2i+1}| - c) \\ &\geq |S_i \cap A_1| - c + (|\bar{S}_i \cap A_1| - 2c) \\ &= |A_1| - 3c.\end{aligned}$$

The same argument gives $|A_{2i+1}| \geq |A_1| - 3c$.

Summing $\sum_{j=1}^{n} |A_j| = m$ and using these inequalities gives

$$\begin{aligned}m &\geq |A_1| + (n-1)(|A_1| - 3c) \\ &= n|A_1| - 3c(n-1),\end{aligned}$$

hence

$$|A_1| \leq \frac{m}{n} + 3c.$$

Step 2: discrepancy for S_i. We distinguish two cases.

Case 1: $|S_i \cap A_1| \leq |S_i|/n$. Since $S_i = \bigsqcup_{j=1}^{n} (S_i \cap A_j)$, by averaging there exists some $j \neq 1$ with $|S_i \cap A_j| \geq |S_i|/n$. If $j \neq 2i+1$, then EF$_a$-c for agent $2i+1$ gives $|S_i \cap A_{2i+1}| \geq |S_i \cap A_j| - c \geq |S_i|/n - c$; if $j = 2i+1$ this is immediate. Thus

$$|S_i \cap A_{2i+1}| \geq \frac{|S_i|}{n} - c.$$

Together with $|\bar{S}_i \cap A_{2i+1}| \geq |\bar{S}_i \cap A_1| - c$ we obtain

$$\begin{aligned}|A_{2i+1}| &= |S_i \cap A_{2i+1}| + |\bar{S}_i \cap A_{2i+1}| \\ &\geq \frac{|S_i|}{n} + |\bar{S}_i \cap A_1| - 2c.\end{aligned}$$

Using $|A_1| \geq |A_{2i+1}| - c$ yields

$$|A_1| \geq \frac{|S_i|}{n} + |\bar{S}_i \cap A_1| - 3c.$$

Since $|A_1| = |S_i \cap A_1| + |\bar{S}_i \cap A_1|$, we conclude

$$|S_i \cap A_1| \geq \frac{|S_i|}{n} - 3c,$$

and therefore in Case 1,

$$\left| |S_i \cap A_1| - \frac{|S_i|}{n} \right| \leq 3c.$$

Case 2: $|S_i \cap A_1| \geq |S_i|/n$. We first claim that

$$|\bar{S}_i \cap A_1| \geq \frac{|\bar{S}_i|}{n} - 3c.$$

Indeed, if $|\bar{S}_i \cap A_1| \geq |\bar{S}_i|/n$ there is nothing to prove. Otherwise, by averaging there exists some $j \neq 1$ with $|\bar{S}_i \cap A_j| \geq |\bar{S}_i|/n$, and repeating the argument of Case 1 with $\bar{S}_i$ in place of S_i (using agent $2i$ instead of $2i+1$) gives the bound.

Using the claim and the bound $|A_1| \leq m/n + 3c$ from Step 1,

$$\begin{aligned}|S_i \cap A_1| &= |A_1| - |\bar{S}_i \cap A_1| \\ &\leq \left(\frac{m}{n} + 3c\right) - \left(\frac{|\bar{S}_i|}{n} - 3c\right) = \frac{|S_i|}{n} + 6c.\end{aligned}$$

Since Case 2 assumes $|S_i \cap A_1| \geq |S_i|/n$, we get

$$\left| |S_i \cap A_1| - \frac{|S_i|}{n} \right| \leq 6c.$$

In both cases we have shown $\big||S_i \cap A_1| - |S_i|/n\big| \leq 6c$. Taking the maximum over $i \in [q]$ and recalling $v^{S_i}(X) = |S_i \cap X|$ yields

$$\max_{i \in [q]} \left| \frac{1}{n} v^{S_i}(M) - v^{S_i}(A_1) \right| \leq 6c,$$

and therefore $\mathrm{wdisc}_{1/n}(\mathcal{V}) \leq 6c$. □

We now formally present our lower bound for the case of binary no-chores valuations in the externalities model.

Theorem 4. *There is a binary no-chores instance of the externalities model with n agents for which every EF-c allocation requires $c = \Omega(\sqrt{n})$.*

Proof. The proof of this theorem follows by combining the above lemma with Theorem 2. The instance of Manurangsi and Meka [2026] consists of $\Theta(n)$ horizontally-stacked copies of $\frac{1}{2}(\mathbf{1}_{q\times q} + \boldsymbol{H})$, where $\mathbf{1}_{q \times q}$ is a $q \times q$ all-1s matrix and $\boldsymbol{H}$ is a $q \times q$ Hadamard matrix. Clearly, this instance is binary. If we let the sets $S_1, \ldots, S_q$ be given by the hyperedges in this instance (i.e., S_i contains the elements whose index in the i^{th} row of the instance of Manurangsi and Meka [2026] contains a 1), then the corresponding valuation profile $\mathcal{V}$ consists of q binary additive functions. Taking $p = 1/n$, Theorem 2 gives us the following bound:

$$\mathrm{wdisc}_{1/n}(\mathcal{V}) \geq \frac{\sqrt{q-1}}{16}.$$

Since $q = (n-1)/2$, combining this with Lemma 2 we get

$$c \geq \frac{\sqrt{((n-1)/2) - 1}}{96},$$

and thus $c = \Omega(\sqrt{n})$ and the corresponding instance in the asymmetric envy model admits no EF$_a$-c allocation for any $c \in o(\sqrt{n})$. Finally, by Proposition 1, there exists a binary no-chores instance in the externalities model for which every EF$-c$ allocation requires $c = \Omega(\sqrt{n})$. □

As stated previously, an important implication of this result is that EF1 allocations do not always exist when the agents have externalities, even when the valuations are binary no-chores, which refutes a conjecture of Deligkas *et al.* [2024] and resolves the open question of Aziz *et al.* [2023].

Theorem 5 (Corollary of Theorem 4)**.** *There is a binary no-chores instance of the externalities model that admits no EF1 allocation.*

Interestingly, the above example in the asymmetric envy model can also be used to construct an instance in the externalities model that is not binary, but instead has the property that agent 1 has no externalities and every other agent in the instance only has externalities towards agent 1. Together, Theorems 4 and 6 show that even highly structured instances with externalities do not allow for EF1 allocations.

Theorem 6. *There is an instance in the externalities model with n agents in which every agent has externalities towards only agent 1, and for which every EF-c allocation requires $c = \Omega(\sqrt{n})$.*

6 An Ex-Ante Truthful Mechanism

In this section, we present a *consensus* version of our upper bound, in which the bundles of the resulting allocation can be permuted arbitrarily amongst the agents while maintaining the fairness guarantee. This allows us to implement the resulting construction as a mechanism that is *truthful in expectation*, i.e., in which no agent can improve its expected value (ex-ante) by misreporting its valuation. Here, the expectation is over the randomness of the mechanism, and truthfulness follows directly from the fact that the agents can be matched to the bundles uniformly at random.

Specifically, we combine Lemma 1 and a symmetric application of Theorem 1 to obtain a generalization of the consensus upper bound of Manurangsi and Suksompong [2022, Theorem 1.3] to a mixture of goods and chores.

Theorem 7. *Let $v_1, \dots, v_n$ be additive valuation functions that may assign both positive and negative values to items, and let k be a number of colors. There exists a partition of the items into k bundles $M = A_1 \sqcup \cdots \sqcup A_k$ such that for any $\ell, \ell' \in [k]$ and any $i \in [n]$, one can eliminate envy (with respect to v_i) between A_ℓ and $A_{\ell'}$ by discarding at most $\mathcal{O}(\sqrt{n})$ items. Moreover, such a partition can be computed in deterministic polynomial time.*

The corollary below follows by combining an application of Theorem 7 with the simple mechanism described in [Bu and Tao, 2025] that assigns the bundles uniformly at random.

Corollary 2. *Given n agents with additive valuation functions that may assign both positive and negative values to items, there exists a randomized ex-ante truthful mechanism running in polynomial time that outputs an allocation $(A_1, \dots, A_n)$ that is EF-$\mathcal{O}(\sqrt{n})$.*

Proof. The corollary follows from using Theorem 7 with $k = n$ colors, together with the simple mechanism described in [Bu and Tao, 2025], which assigns the bundles uniformly at random. The expected value received by agent i is $\frac{1}{n} v_i(M)$, regardless of the reports of the agents. In fact, this guarantee is much stronger than truthfulness: no coalition of agents can influence an agent's expected value. □

In the externalities model, we get the following corollary by applying Theorem 7 to the $n(n-1)$ asymmetric envy valuations $(v_{i,j})_{i \neq j}$ produced by Proposition 1. This yields a consensus bound of order $\mathcal{O}\left(\sqrt{n(n-1)}\right) = \mathcal{O}(n)$.

Corollary 3. *Given an instance of the externalities model, there exists a randomized ex-ante truthful mechanism running in polynomial time that outputs an allocation $(A_1, \dots, A_n)$ that is EF-$\mathcal{O}(n)$.*

7 Conclusions

For the fair division of indivisible items amongst agents with externalities, we find asymptotically tight bounds on the optimal relaxation EF-k that can always be attained. Specifically, for the general case of additive valuations, we show that an EF-$\mathcal{O}(\sqrt{n})$ allocation always exists, and give a matching $\Omega(\sqrt{n})$ lower bound even for the special case of binary valuations with no chores. In the process, we answer two open questions of Aziz *et al.* [2023], and resolve a conjecture of Deligkas *et al.* [2024], showing that EF1 allocations do not always exist when the agents have externalities. We also present an ex-ante-truthful consensus-based mechanism that finds an EF-$\mathcal{O}(n)$ allocation when the agents have externalities. Our work raises interesting new directions for research.

- *Small explicit counterexample*: Our proof technique shows how to reduce a known lower bound from discrepancy into a collection of lower bound instances for the externalities model. Nonetheless, we leave open the question of describing a small explicit counterexample, and of determining the largest value of n for which an EF1 allocation exists for additive/binary valuations with or without the no-chores condition.
- *Closing the constant-factor gap*: While our work establishes asymptotically tight bounds of $\Theta(\sqrt{n})$ on the achievable relaxation of envy-freeness with externalities, it leaves a constant-factor gap between the exact values of the upper and lower bounds. A natural resulting open problem is to close this gap.
- *Consensus lower-bound*: We give an ex-ante-truthful consensus-based mechanism that finds an EF-$\mathcal{O}(n)$ allocation when the agents have externalities. Another natural follow-up question is whether there exist a matching lower bound in this setting.
- *Other settings*: Can our approach be applied to find similar bounds in other settings that are related to fair division with externalities, such as fair division with social impact [Flammini *et al.*, 2025; Deligkas *et al.*, 2026]?

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References

[Agarwal *et al.*, 2021] Aishwarya Agarwal, Edith Elkind, Jiarui Gan, Ayumi Igarashi, Warut Suksompong, and Alexandros A. Voudouris. Schelling games on graphs. *Artificial Intelligence*, 301:103576, 2021.

[Amanatidis *et al.*, 2023] Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence*, 322:103965, 2023.

[Anshelevich *et al.*, 2017] Elliot Anshelevich, Onkar Bhardwaj, and Martin Hoefer. Stable matching with network externalities. *Algorithmica*, 78(3):1067–1106, 2017.

[Ayres and Kneese, 1969] Robert U. Ayres and Allen V. Kneese. Production, consumption, and externalities. *American Economic Review*, 59(3):282–297, 1969.

[Aziz *et al.*, 2023] Haris Aziz, Warut Suksompong, Zhaohong Sun, and Toby Walsh. Fairness concepts for indivisible items with externalities. In *Proceedings of AAAI '23*, pages 5472–5480. AAAI Press, 2023.

[Brams and Taylor, 1996] Steven J. Brams and Alan D. Taylor. *Fair division: From cake-cutting to dispute resolution*. Cambridge University Press, 1996.

[Brânzei *et al.*, 2013a] Simina Brânzei, Tomasz P. Michalak, Talal Rahwan, Kate Larson, and Nicholas R. Jennings. Matchings with externalities and attitudes. In *Proceedings of AAMAS '13*, pages 295–302. IFAAMAS, 2013.

[Brânzei *et al.*, 2013b] Simina Brânzei, Ariel D. Procaccia, and Jie Zhang. Externalities in cake cutting. In *Proceedings of IJCAI '13*, pages 55–61. ijcai.org, 2013.

[Bu and Tao, 2025] Xiaolin Bu and Biaoshuai Tao. Truthful and almost envy-free mechanism of allocating indivisible goods: The power of randomness. In *Proceedings of FOCS '25*, pages 2766–2795. IEEE, 2025.

[Bu *et al.*, 2023] Xiaolin Bu, Zihao Li, Shengxin Liu, Jiaxin Song, and Biaoshuai Tao. Fair division with allocator's preference. In *Proceedings of WINE '23*, volume 14413 of *Lecture Notes in Computer Science*, pages 77–94. Springer, 2023.

[Budish, 2011] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[Caragiannis *et al.*, 2019] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation*, 7(3):12:1–12:32, 2019.

[Caragiannis *et al.*, 2025] Ioannis Caragiannis, Kasper Green Larsen, and Sudarshan Shyam. A new lower bound for multicolor discrepancy with applications to fair division. In *Proceedings of SAGT '25*, volume 15953 of *Lecture Notes in Computer Science*, pages 228–246. Springer, 2025.

[Deligkas *et al.*, 2024] Argyrios Deligkas, Eduard Eiben, Viktoriia Korchemna, and Šimon Schierreich. The complexity of fair division of indivisible items with externalities. In *Proceedings of AAAI '24*, pages 9653–9661. AAAI Press, 2024.

[Deligkas *et al.*, 2026] Argyrios Deligkas, Eduard Eiben, Tiger-Lily Goldsmith, Dušan Knop, and Šimon Schierreich. Dividing indivisible items for the benefit of all: It is hard to be fair without social awareness. In *Proceedings of AAAI '26*, pages 16812–16820. AAAI Press, 2026.

[Doerr and Srivastav, 2003] Benjamin Doerr and Anand Srivastav. Multicolour discrepancies. *Combinatorics, Probability & Computing*, 12(4):365–399, 2003.

[Dupré la Tour and Fujii, 2025] Max Dupré la Tour and Kaito Fujii. Discrepancy and fair division for non-additive valuations. *CoRR*, abs/2509.16802, 2025.

[Dupré la Tour, 2026] Max Dupré la Tour. Bad news for couples: Tight lower bounds for fair division of indivisible items. *CoRR*, abs/2601.01012, 2026.

[Elkind *et al.*, 2020] Edith Elkind, Neel Patel, Alan Tsang, and Yair Zick. Keeping your friends close: Land allocation with friends. In *Proceedings of IJCAI '20*, pages 318–324. ijcai.org, 2020.

[Fisher and Hafalir, 2016] James C. D. Fisher and Isa Emin Hafalir. Matching with aggregate externalities. *Mathematical Social Sciences*, 81:1–7, 2016.

[Flammini *et al.*, 2025] Michele Flammini, Gianluigi Greco, and Giovanna Varricchio. Fair division with social impact. In *Proceedings of AAAI '25*, pages 13856–13863. AAAI Press, 2025.

[Foley, 1967] Duncan Foley. Resource allocation and the public sector. *Yale Economic Essays*, 7(1), 1967.

[Funk, 1996] Peter Funk. Auctions with interdependent valuations. *International Journal of Game Theory*, 25:51–64, 1996.

[Garg *et al.*, 2026] Jugal Garg, Vishnu V. Narayan, and Yuang Eric Shen. Designing truthful mechanisms for asymptotic fair division. In *Proceedings of AAAI '26*, pages 16914–16921. AAAI Press, 2026.

[Gölz and Yaghoubizade, 2026] Paul Gölz and Hannane Yaghoubizade. Fair division among couples and small groups. In *Proceedings of AAAI '26*, pages 16963–16970. AAAI Press, 2026.

[Gross-Humbert *et al.*, 2022] Nathanaël Gross-Humbert, Nawal Benabbou, Aurélie Beynier, and Nicolas Maudet. Sequential and swap mechanisms for public housing allocation with quotas and neighbourhood-based utilities. *ACM Transactions on Economics and Computation*, 10(4):15:1–15:24, 2022.

[Jehiel and Moldovanu, 1996] Philippe Jehiel and Benny Moldovanu. Strategic nonparticipation. *The RAND Journal of Economics*, 27(1):84–98, 1996.

[Katz and Shapiro, 1985] Michael L. Katz and Carl Shapiro. Network externalities, competition, and compatibility. *American Economic Review*, 75(3):424–440, 1985.

[Knop and Schierreich, 2023] Dušan Knop and Šimon Schierreich. Host community respecting refugee housing. In *Proceedings of AAMAS '23*, pages 966–975. IFAAMAS, 2023.

[Levy *et al.*, 2017] Avi Levy, Harishchandra Ramadas, and Thomas Rothvoss. Deterministic discrepancy minimization via the multiplicative weight update method. In *Proceedings of IPCO '17*, volume 10328 of *Lecture Notes in Computer Science*, pages 380–391. Springer, 2017.

[Li *et al.*, 2015] Minming Li, Jialin Zhang, and Qiang Zhang. Truthful cake cutting mechanisms with externalities: Do not make them care for others too much! In *Proceedings of IJCAI '15*, pages 589–595. AAAI Press, 2015.

[Li *et al.*, 2019] Minming Li, Lili Mei, Yi Xu, Guochuan Zhang, and Yingchao Zhao. Facility location games with externalities. In *Proceedings of AAMAS '19*, pages 1443–1451. IFAAMAS, 2019.

[Li *et al.*, 2025] Minming Li, Cheng Peng, Ying Wang, and Houyu Zhou. Group-fair facility location games with externalities. In *Proceedings of AAMAS '25*, pages 2615–2617. IFAAMAS, 2025.

[Lipton *et al.*, 2004] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of EC '24*, pages 125–131. ACM, 2004.

[Manurangsi and Meka, 2026] Pasin Manurangsi and Raghu Meka. Tight lower bound for multicolor discrepancy. In *Proceedings of SOSA '26*, pages 266–274. SIAM, 2026.

[Manurangsi and Suksompong, 2017] Pasin Manurangsi and Warut Suksompong. Asymptotic existence of fair divisions for groups. *Mathematical Social Sciences*, 89:100–108, 2017.

[Manurangsi and Suksompong, 2020] Pasin Manurangsi and Warut Suksompong. When do envy-free allocations exist? *SIAM Journal on Discrete Mathematics*, 34(3):1505–1521, 2020.

[Manurangsi and Suksompong, 2021] Pasin Manurangsi and Warut Suksompong. Closing gaps in asymptotic fair division. *SIAM Journal on Discrete Mathematics*, 35(2):668–706, 2021.

[Manurangsi and Suksompong, 2022] Pasin Manurangsi and Warut Suksompong. Almost envy-freeness for groups: Improved bounds via discrepancy theory. *Theoretical Computer Science*, 930:179–195, 2022.

[Manurangsi and Suksompong, 2024] Pasin Manurangsi and Warut Suksompong. Ordinal maximin guarantees for group fair division. In *Proceedings of IJCAI '24*, pages 2922–2930. ijcai.org, 2024.

[Manurangsi and Suksompong, 2025] Pasin Manurangsi and Warut Suksompong. Asymptotic fair division: Chores are easier than goods. In *Proceedings of IJCAI '25*, pages 3988–3995. ijcai.org, 2025.

[Manurangsi *et al.*, 2025] Pasin Manurangsi, Warut Suksompong, and Tomohiko Yokoyama. Asymptotic analysis of weighted fair division. In *Proceedings of IJCAI '25*, pages 3979–3987. ijcai.org, 2025.

[Massand and Simon, 2019] Sagar Massand and Sunil Simon. Graphical one-sided markets. In *Proceedings of IJCAI '19*, pages 492–498. ijcai.org, 2019.

[Mishra *et al.*, 2022] Shaily Mishra, Manisha Padala, and Sujit Gujar. Fair allocation with special externalities. In *Proceedings of PRICAI '22*, volume 13629 of *Lecture Notes in Computer Science*, pages 3–16. Springer, 2022.

[Mumcu and Saglam, 2010] Ayse Mumcu and Ismail Saglam. Stable one-to-one matchings with externalities. *Mathematical Social Sciences*, 60(2):154–159, 2010.

[Nguyen and Rothe, 2023] Trung Thanh Nguyen and Jörg Rothe. Complexity results and exact algorithms for fair division of indivisible items: A survey. In *Proceedings of IJCAI '23*, pages 6732–6740. ijcai.org, 2023.

[Plaut and Roughgarden, 2020] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics*, 34(2):1039–1068, 2020.

[Seddighin *et al.*, 2019] Masoud Seddighin, Hamed Saleh, and Mohammad Ghodsi. Externalities and fairness. In *Proceedings of WWW '19*, pages 538–548. ACM, 2019.

[Segal-Halevi and Nitzan, 2019] Erel Segal-Halevi and Shmuel Nitzan. Fair cake-cutting among families. *Social Choice and Welfare*, 53(4):709–740, 2019.

[Suksompong, 2018] Warut Suksompong. Approximate maximin shares for groups of agents. *Mathematical Social Sciences*, 92:40–47, 2018.

[Thomson, 2016] William Thomson. Introduction to the theory of fair allocation. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 11, pages 261–283. Cambridge University Press, 2016.

[Velez, 2016] Rodrigo A. Velez. Fairness and externalities. *Theoretical Economics*, 11(1):381–410, 2016.

[Wang *et al.*, 2024] Ying Wang, Houyu Zhou, and Minming Li. Positive intra-group externalities in facility location. In *Proceedings of AAMAS '24*, pages 1883–1891. IFAAMAS, 2024.

[Zhang *et al.*, 2018] Chaoli Zhang, Xiang Wang, Fan Wu, and Xiaohui Bei. Efficient auctions with identity-dependent negative externalities. In *Proceedings of AAMAS '18*, pages 2156–2158. IFAAMAS, 2018.