The Complexity of Tournament Fixing: Subset FAS Number and Acyclic Neighborhoods

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Abstract

The TOURNAMENT FIXING PROBLEM (TFP) asks whether a knockout tournament can be scheduled to guarantee that a given player v^* wins. Although TFP is NP-hard in general, it is known to be *fixed-parameter tractable* (FPT) when parameterized by the feedback arc/vertex set number, or the in/out-degree of v^*. However, it remained open whether TFP is FPT with respect to the *subset FAS number of* v^* — the minimum number of arcs intersecting all cycles containing v^* — a parameter that is never larger than the aforementioned ones. In this paper, we resolve this question negatively by proving that TFP stays NP-hard even when the subset FAS number of v^* is constant ≥ 1 and either the subgraph induced by the in-neighbors $D[N_{\text{in}}(v^*)]$ or the out-neighbors $D[N_{\text{out}}(v^*)]$ is acyclic. Conversely, when both $D[N_{\text{in}}(v^*)]$ and $D[N_{\text{out}}(v^*)]$ are acyclic, we show that TFP becomes FPT parameterized by the subset FAS number of v^*. Furthermore, we provide sufficient conditions under which v^* can win even when this parameter is unbounded.

1 Introduction

Knockout tournaments represent a fundamental class of competition structures, characterized by a sequence of recursive elimination rounds [Horen and Riezman, 1985; Connolly and Rendleman, 2011; Groh *et al.*, 2012]. In each round, players are partitioned into disjoint pairs (matches). For each pair, the loser is eliminated, and the winner advances to the next round. The process repeats until a single champion remains. Owing to their efficiency and simplicity, these structures are employed in large-scale events such as the FIFA World Cup [Scarf and Yusof, 2011] and the NCAA Basketball Tournament [Kvam and Sokol, 2006]. Beyond sports, they serve as critical models for elections, organizational decision-making, and various game-based settings where sequential elimination serves as a natural selection mechanism [Kim *et al.*, 2017; Stanton and Williams, 2011a; Ramanujan and Szeider, 2017]. Consequently, knockout tournaments have attracted considerable attention in artificial intelligence [Vu *et al.*, 2009; Williams, 2010], economics, and operations research [Rosen, 1985; Mitchell, 1983; Laslier, 1997], motivating a rich body of theoretical and applied studies.

Given a favorite player v^*, a fundamental computational challenge is to determine whether the tournament be arranged to ensure that v^* wins. Let N denote the set of n players, where $n = 2^c$ players (for some $c \in \mathbb{N}$). The structure of a knockout tournament is represented by a complete (unordered) binary tree T with n leaves. A *seeding* is a bijection $\sigma : N \to \text{leaves}(T)$ that assigns each player to a distinct leaf. The tournament proceeds in rounds: in each round, for any internal node, the winners of the subtrees rooted at its children compete, with the winner advancing to this node. This continues until one player remains — the tournament champion.

The question becomes meaningful when predictive information about match outcomes is available. We model this with a tournament digraph $D = (V = N, E)$, where V is the set of n players. For every distinct pair of vertices $\{u, v\} \subseteq V$, the arc set E contains exactly one of the ordered pairs (u, v) or (v, u), signifying that u defeats v or vice versa. We investigate the following problem:

> TOURNAMENT FIXING PROBLEM (TFP)
> **Input:** A tournament D and a player $v^* \in V(D)$.
> **Question:** Does there exist a seeding σ for the $n = |V(D)|$ players such that v^* wins the resulting knockout tournament?

1.1 Related Work

The computational study of tournament manipulation was initiated by Vu et al. [2009]. The complexity of TFP, specifically its NP-hardness, remained open for several years [Vu *et al.*, 2009; Williams, 2010; Russell and Van Beek, 2011; Lang *et al.*, 2012]. Aziz et al. [2014] finally proved that TFP is NP-complete. They also provided exact exponential-time algorithms running in $\mathcal{O}(2.83^n)$ time with exponential space, or $4^{n+o(n)}$ time with polynomial space. Kim and Vassilevska Williams [2015] later improved this to $2^n n^{\mathcal{O}(1)}$ time and space, and Gupta et al. [2018b] achieved the same time bound using only polynomial space.

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Given the intractability of TFP in general, its parameterized complexity has been extensively investigated. We recall several key parameters studied in the literature and this paper:

- **fas number** (resp., **fvs number**): the minimum number of arc reversals (resp., vertex deletions) needed to make the tournament acyclic;
- **in-degree** (resp., **out-degree**) of v^*: the number of in-neighbors (resp., out-neighbors) of v^*;
- **sfas-v number** (resp., **sfvs-v number**): the minimum number of arc reversals (resp., vertex deletions, excluding v^*) needed to remove v^* from all cycles;
- **sfas-in number** (resp., **sfas-out number**): the minimum number of arc deletions required to exclude the in-neighborhood of v^*, i.e., $N_{\text{in}}(v^*)$ (resp., the out-neighborhood of v^*, i.e., $N_{\text{out}}(v^*)$) from cycles.

These parameters share the property that TFP becomes polynomial-time solvable when they equal zero. Their relationships are illustrated in Figure 1 (an arrow from x to y indicates $x \leq y$).

The fas number, positioned at the top of this hierarchy, was first shown to be fixed-parameter tractable (FPT) by Ramanujan and Szeider [2017], who gave an algorithm running in $p^{\mathcal{O}(p^2)}n^{\mathcal{O}(1)}$ time, where p is the fas number. This was subsequently improved to $p^{\mathcal{O}(p)}n^{\mathcal{O}(1)}$ by Gupta et al. [2018a]. Later, Gupta et al. [2019] gave a polynomial kernel. For the fvs number q, Zehavi [2023] proved the FPT via a $q^{\mathcal{O}(q)}n^{\mathcal{O}(1)}$-time algorithm. This result subsumes the fas-number parameterization since $q \leq p$.

Wang et al. [2026] established that TFP is FPT parameterized by either the out-degree l or the in-degree k of v^*, with running times $2^{2^l}n^{\mathcal{O}(1)}$ and $2^{2^{\mathcal{O}(k)}}n^{\mathcal{O}(1)}$, respectively.

It is not hard to see that when the sfas-v number or sfvs-number is zero, TFP becomes polynomial-time solvable. Furthermore, the sfas-v number and the sfvs-v number are smaller than fas and fvs, and the in-degree and out-degree of v^*. Whether TFP is FPT parameterized by the smaller parameters the sfas-v number or the sfvs-v number becomes an interesting open problem in the literature [Wang *et al.*, 2026].

1.2 Our Contributions

In this paper, we resolve the open problem concerning the parameterized complexity of TFP with respect to subset-based parameters centered around the favorite player v^*. Our main contributions are:

- **Negative Results:** We prove that TFP remains NP-hard when:
  1. The sfas-v number (or sfvs-v number) is any constant ≥ 1, answering the open question from [Wang *et al.*, 2026] in the negative.
  2. The sfas-in number (or sfas-out number) is 1.
- **Positive Results:** We identify tractable frontiers:
  1. TFP is FPT when parameterized by the sum of the sfas-in and sfas-out numbers.
  2. When both $D[N_{\text{in}}(v^*)]$ and $D[N_{\text{out}}(v^*)]$ are acyclic, TFP is FPT parameterized by the sfas-v number.

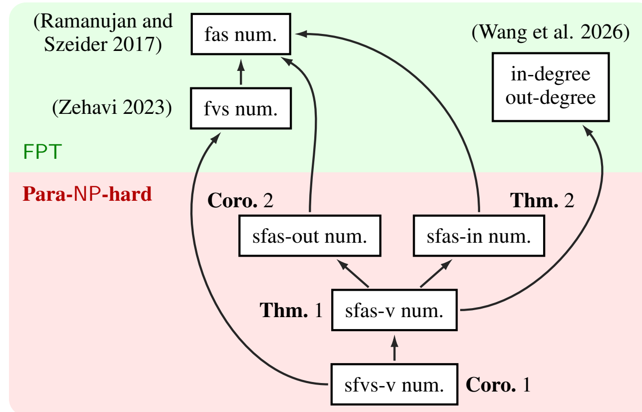

Figure 1: An illustration of the hierarchy of the eight parameters, where an arc from parameter x to parameter y denotes $x \leq y$. The green region (upper section) marks parameters for which TFP is proven FPT, while the red region (lower section) indicates para-NP-hard cases (our results).

- **Structural Insights:** For an instance (D, v^*) with $N_{\text{in}}(v^*)$ and $N_{\text{out}}(v^*)$ both acyclic, we provide new sufficient conditions under which v^* can be guaranteed to win, even when the relevant parameters are unbounded. These extend the known polynomial-time solvable cases and offer practical structural guarantees for tournament manipulability.

Our results complete the parameterized complexity landscape of TFP with respect to natural "distance-from-acyclicity" parameters localized around v^*. The hardness results demonstrate that restricting cycles to involve only specific neighborhoods of v^* does not generally lead to tractability, while the positive result identifies certain structural conditions that do. The hardness results can also be interpreted as a guarantee that this manipulation is computationally difficult to carry out in the cases studied.

Due to space constraints, the proofs of some lemmas are omitted, which can be found in the full version of this paper [Liu *et al.*, 2026].

2 Preliminaries

In this paper, we only focus on digraphs. Let $D = (V, E)$ be a digraph with $n = |V|$ vertices and $m = |E|$ edges. We denote the vertex set and edge set of a digraph D' as $V(D')$ and $E(D')$, respectively. We use (u, v) to denote an arc from vertex u to vertex v. If there exists an arc $(u, v) \in E$, then we say that u is an *in-neighbor* of v and v is an *out-neighbor* of u. The *out-neighborhood* and *in-neighborhood* of a vertex v is denoted by $N_{\text{out}}(v)$ and $N_{\text{in}}(v)$ respectively. For a vertex v, we further let $out(v) = |N_{\text{out}}(v)|$ and $in(v) = |N_{\text{in}}(v)|$. For a vertex v and a vertex subset $S \subseteq V$, let $out_S(v) = |N_{\text{out}}(v) \cap S|$ and $in_S(v) = |N_{\text{in}}(v) \cap S|$. For a vertex subset $X \subseteq V$, we define its out-neighborhood as $N_{\text{out}}(X) = (\bigcup_{v \in X} N_{\text{out}}(v)) \setminus X$ and its in-neighborhood as $N_{\text{in}}(X) = (\bigcup_{v \in X} N_{\text{in}}(v)) \setminus X$. We call a vertex s a *source* if $in(s) = 0$.

For a vertex subset $X \subseteq V$, the *subgraph induced by* X is denoted by $D[X]$. We write $D - X$ to denote the subgraph $D[V \setminus X]$. For a terminal set $T \subseteq V$, a vertex set $X \subseteq V \setminus T$ is a *subset feedback vertex set* of D with terminal set T if there is no cycle containing vertices in T in $D - X$. Similarly, an arc set $A \subseteq E$ is a *subset feedback arc set* of D with terminal set T if there is no cycle containing vertices in T in D', where D' is the resultant digraph after reversing arcs in A. We will always use v^* to denote the favorite in the tournament.

A knockout tournament with $n = 2^p$ players for some integer $p \geq 0$ is organized into $\log n$ successive rounds of competition. In each round $r \in [\log n]$, the $2^{\log n - r + 1}$ remaining players are paired into $2^{\log n - r}$ matches, with only the winners advancing to the next round. A seeding σ is a *winning seeding* for v^* if v^* survives all $\log n$ rounds of the resulting knockout tournament.

Following the notions established in [Wang *et al.*, 2026], we utilize the concepts of match sets and match set sequences to formally characterize the matches played in each round of a knockout tournament.

Definition 1 (match set and sequence)**.** *Let D be a tournament with n players. For a fixed seeding σ, we define:*

- ***Match Set:*** *For each round $r \in [\log n]$, the* match set *$M_r \subseteq E(D)$ is the set of arcs representing matches played in round r, where $|M_r| = 2^{\log n - r}$. Each pair $(u, v) \in M_r$ indicates that player u defeated v. Let $V(M_r)$ denote the set of players who participated in round r;*
- ***Match Set Sequence:*** *A* match set sequence *is an ordered collection of match sets for all rounds, defined as $\mathcal{M} = \{M_1, \ldots, M_{\log n}\}$.*

A match set sequence $\{M_1, \ldots, M_{\log n}\}$ is valid *for D if the following conditions hold:*

- *For every round $r \in [p]$, $M_r \subseteq E(D)$.*
- *$V(M_1) = V(D)$;*
- *For every round $r \in [\log n - 1]$, the set of winners in M_r is exactly $V(M_{r+1})$.*

Given a seeding σ, it induces a unique match set sequence, denoted by $\mathcal{M}(\sigma)$. In contrast, if a match set sequence $\mathcal{M}^*$ is valid, then there exists at least one seeding σ^* such that $\mathcal{M}(\sigma^*) = \mathcal{M}^*$.

To formally track the progression of the tournament, we introduce notation for the sets of winners and eliminated players in each round. Let D be a tournament with n players, and let $\mathcal{M}(\sigma) = (M_1, \ldots, M_k)$ be the match set sequence induced by a seeding σ. For each round $r \in [\log n]$, we use $C_r(\sigma) = \{u \mid (u, v) \in M_r\}$ to denote the set of players remaining after round r. By convention, we define $C_0(\sigma) = V(D)$. Correspondingly, for each $r \in [\log n]$, we use $L_r(\sigma) = \{v \mid (u, v) \in M_r\}$ to denote the set of players eliminated in round r. Where the seeding σ is clear from the context, we simply write C_r and L_r.

3 Parameterized by the Subset FAS Number

In this section, we investigate the complexity of TFP when parameterized by the sfas-v/in/out number. We will show that TFP is para-NP-hard with respect to these parameters, thereby completing the parameterized complexity landscape illustrated in Figure 1.

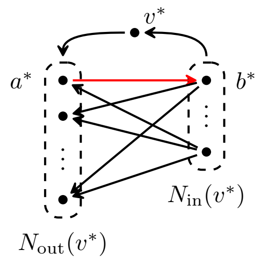

Figure 2: An illustration for a special instance (D, v^*).

Before presenting our main results, we define a kind of restricted instances, called *special* instances, and characterize several properties of winning seedings for a special instance.

Definition 2. *An instance (D, v^*) is special with respect to $a^* \in N_{\text{out}}(v^*)$ and $b^* \in N_{\text{in}}(v^*)$ if*

1. *$|N_{\text{out}}(v^*)| = 3|N_{\text{in}}(v^*)| - 1$, where $|N_{\text{in}}(v^*)| = 2^p$ for some integer $p \geq 0$;*
2. *There is an arc $(a^*, b^*) \in E(D)$, and*
3. *For each vertex $a \in N_{\text{out}}(v^*)$ and each vertex $b \in N_{\text{in}}(v^*)$ expect the case that $a = a^*$ and $b = b^*$, there is an arc $(b, a) \in E(D)$.*

See Fig. 2 for an illustration for a special instance (D, v^*). We introduce the "special instance" because it serves as a gadget utilized in the proofs of both Theorem 1 and Theorem 2. Intuitively, this instance ensures that under a winning seeding, with the exception of the specific match between a^* and b^*, no other player from $N_{\text{out}}(v^*)$ competes against a player from $N_{\text{in}}(v^*)$. The following lemma formally introduces this property.

Lemma 1. *Consider a special instance (D, v^*) where $n = |N_{\text{out}}(v^*)|$. Suppose (D, v^*) is a yes-instance. Let σ be an arbitrary winner seeding for v^* in tournament D and let its corresponding match set sequence be $\mathcal{M}^* = \{M_1^*, \ldots, M_{(\log n)+2}^*\}$. We have the following properties.*

1. *$V(M_{(\log n)+1}^*) \cap N_{\text{in}}(v^*) = \{b^*\}$.*
2. *$|V(M_{(\log n)+1}^*) \cap N_{\text{out}}(v^*)| = 2$ and $a^* \in V(M_{(\log n)+1}^*)$.*
3. *For each round $r \in [\log n]$, every match in M_r involves either two players in $A \cup \{v^*\}$ or two players in B.*

Proof. Assume that player v^* beats player $x_1 \in V(D)$ in the last match, and in the second last round, v^* beats x_2 and x_1 beats x_3; that is, $M_{(\log n)+1}^* = \{(v^*, x_2), (x_1, x_3)\}$ and $M_{(\log n)+2}^* = \{(v^*, x_1)\}$ for some $x_1, x_2, x_3 \in V(D)$. Let $M_{(\log n)+1}^* = \{(v^*, x_2), (x_1, x_3)\}$ and $M_{(\log n)+2}^* = \{(v^*, x_1)\}$ for some $x_1, x_2, x_3 \in V(D)$. Since v^* beats x_1 and x_2, we know that x_1 and x_2 are in $N_{\text{out}}(v^*)$. Now we show that $x_3 \in N_{\text{in}}(v^*)$.

For round $r \in [\log n]$, let $B_{r-1} = C_{r-1}(\sigma) \cap N_{\text{in}}(v^*)$ be the players in $N_{\text{in}}(v^*)$ remain after round $r - 1$. By the definition of D, we know that for any player $b \in N_{\text{in}}(v^*) \setminus b^*$, b can only be beaten by players in $N_{\text{in}}(v^*)$. If b^* remains in this round, we know there are at most $\lceil \frac{1}{2}|B_{r-1} \setminus b^*| \rceil$ vertices in B_{r-1} are eliminated. If b^* is eliminated in this round, we know there are at most $\lfloor \frac{1}{2}|B_{r-1} \setminus b^*| \rfloor$ vertices in $B_{r-1} \setminus b^*$ are eliminated, which implies that there are at most $\lfloor \frac{1}{2}|B_{r-1} \setminus b^*| \rfloor + 1 = \lceil \frac{1}{2}|B_{r-1}| \rceil$ vertices in B_{r-1} are eliminated. Thus, we have that in each round $r \in [\log n]$, there are at most half of the vertices with indegree of v^* are eliminated, i.e.,

$$|L_r(\sigma) \cap N_{\text{in}}(v^*)| \leq \lceil \frac{1}{2}|B_{r-1}| \rceil,$$

which means that

$$|B_r| \geq |B_{r-1}| - \lceil \frac{1}{2}|B_{r-1}| \rceil = \lfloor \frac{1}{2}|B_{r-1}| \rfloor.$$

Since $|B_0| = |N_{\text{in}}(v^*)| = n$ and $n = 2^p$ for some integer $p \geq 0$, we have that $|B_{\log n}| \geq 1$, which means that there are at least one vertex in $N_{\text{in}}(v^*)$ remains after round $\log n$. Thus, $x_3 \in N_{\text{in}}(v^*)$. Since x_1 beats x_3 and $x_1 \in N_{\text{out}}(v^*)$, we know that $x_1 = a^*$ and $x_3 = b^*$. Thus, the first two properties hold.

Before we show the correctness of the third property. Recall that there are at most $\lceil \frac{1}{2}|B_{r-1} \setminus b^*| \rceil$ vertices in $B_{r-1} \setminus b^*$ are eliminated in round $r \in [\log n]$. Since b^* cannot be eliminated before round $(\log n) + 1$, we indeed have that for each round $r \in [\log n]$,

$$|L_r(\sigma) \cap N_{\text{in}}(v^*)| \leq \lfloor \frac{1}{2}|B_{r-1}| \rfloor,$$

which means that

$$|B_r| \geq |B_{r-1}| - \lfloor \frac{1}{2}|B_{r-1}| \rfloor = \lceil \frac{1}{2}|B_{r-1}| \rceil.$$

Now we show that the third property holds. By contradiction, assume there exists a match $(b, a) \in M^*_{r_0}$ for some $r_0 \in [\log n]$ such that $a \in N_{\text{out}}(v^*)$ and $b \in N_{\text{in}}(v^*)$. Let r be the first round that there exists such match. We have that $|B_{r-1}| = |N_{\text{in}}(v^*)|/2^{r-1}$ is an even number. Since each player in $N_{\text{in}}(v^*) \setminus b^*$ can only be beaten by $N_{\text{in}}(v^*)$ and b^* remains after round $\log n$, we have that

$$|B_r| > \frac{1}{2}|B_{r-1}|.$$

Thus, we have that

$$|B_{\log n}| > \frac{1}{2^{\log n}}|N_{\text{in}}(v^*)| = 1,$$

which means that there are at least two players in $N_{\text{in}}(v^*)$ remains after round $\log n$. However, v^*, x_1 and x_2 are not in $N_{\text{in}}(v^*)$, a contradiction. This completes the proof of the third condition. $\square$

Consider a special yes-instance (D, v^*) and an arbitrary winning seeding σ. The three properties stated in Lemma 1 hold for σ. The corresponding complete binary tree T corresponding to σ is shown in Fig. 3. One of the main results of this section is the following theorem:

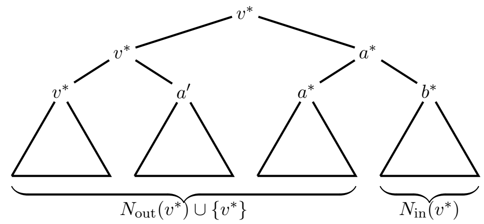

Figure 3: An illustration for the complete binary tree T corresponding to a winner seeding σ for a special instance (D, v^*), where a' is a player in $N_{\text{out}}(v^*)$.

Theorem 1. *For an input tournament D and a favorite player $v^* \in V(D)$, TFP is* NP*-hard even when the sfas-v number equals 1.*

Proof. We will show a polynomial-time reduction from TFP to TFP with subset FAS number of v^* that equals to 1. Specifically, consider a TFP instance (D, v^*) with $n = |D|$, we construct an equivalent instance $(D' = (\{v'\} \cup A \cup B, E'), v')$ as follows.

- $D'[A]$ is an acyclic tournament with $3n - 1$ vertices where $a^* \in A$ is the source vertex in $D'[A]$.
- $D'[B]$ is a copy of D where $b^* \in B$ is the copy of v^*.
- For each $a \in A$, there is an arc (v', a) in E'. For each $b \in B$, there is an arc (b, v') in E'.
- $(a^*, b^*) \in E'$ and for each vertex $a \in A$ and each vertex $b \in B$ expect the case that $a = a^*$ and $b = b^*$, there is an arc $(b, a) \in E'$.

Clearly, $N_{\text{out}}(v') = A$ and $N_{\text{in}}(v') = B$, and (D', v') is a special instance with respect to a^* and b^*. Now we prove both directions of the equivalence of (D, v^*) and (D', v').

Forward direction. Assume that there exists a winning seeding σ for v^* in tournament D (also for b^* in tournament $D'[B]$) and let its corresponding match set sequence be $\mathcal{M}^* = \{M_1, \ldots, M_{\log n}\}$.

Let (A_1, A_2, A_3) be a partition of $D'[A]$ such that $|A_1| = |A_2| = n$, $|A_3| = n - 1$, and $a^* \in A_1$. So $|A_1| = |A_2| = |A_3 \cup \{v^*\}| = |B| = n$. Now we construct four seedings for $D'[A_1]$, $D'[A_2]$, $D'[A_3 \cup \{v^*\}]$ and $D'[B]$ respectively, and then obtain a winning seeding for v' in D' based on these four seedings.

1. Let σ_1 be an arbitrary seeding in tournament $D'[A_1]$. Then a^* must be the winner of $D'[A_1]$ under σ_1 since a^* is a source vertex in $D'[A]$.
2. Let σ_2 be an arbitrary seeding in tournament $D'[A_2]$ with winner a'.
3. Let σ_3 be an arbitrary seeding in tournament $D'[A_3 \cup \{v'\}]$. Then v' must be the winner of $D'[A_3 \cup \{v'\}]$ under σ_3 since for each $a \in A$, there is an arc $(v', a) \in E(D')$.
4. Recall that σ is a winning seeding for b^* in tournament $D'[B]$.

For σ_i $(i = 1, 2, 3)$, the corresponding match set sequence is defined as $\mathcal{M}(\sigma_i) = \{M^i_1, \ldots, M^i_{\log n}\}$. Recall that the corresponding match set sequence of σ is $\mathcal{M}^* = \{M_1, \ldots, M_{\log n}\}$.

The match set sequence $\mathcal{M}'$ is then defined as:

$$\mathcal{M}' = \{M'_1, M'_2, \ldots, M'_{\log n}, \{(v', a'), (a^*, b^*)\}, \{(v', a^*)\}\},$$

where $M'_r = M_r \cup \bigcup_{i \in \{1,2,3\}} M^i_r$ for each $r \in [\log n]$. $\mathcal{M}'$ is obtained by unioning the match set sequences of B, A_1, A_2 and $A_3 \cup \{v'\}$, and adding two new rounds $\{(v', a'), (a^*, b^*)\}, \{(v', a^*)\}$, which is a valid match set sequence for $V(D')$.

Let σ' be a corresponding seeding for $\mathcal{M}'$, note that σ' ensures that v' wins. Thus, there exists a winning seeding for v' in tournament D'.

Reverse direction. Assume that there exists a winning seeding σ' for v' in tournament D' and let its corresponding match set sequence be

$$\mathcal{M}' = \{M'_1, \ldots, M'_{(\log n)+2}\}.$$

By the third property of Lemma 1, there exists a match set sequence $\mathcal{M} = \{M_1, \ldots, M_{\log n}\}$ such that for each round $r \in [\log n]$, $M_r \subseteq M'_r$ and $\mathcal{M}$ is valid for $N_{\text{in}}(v')$. Furthermore, by the first property of Lemma 1, the corresponding seeding σ for $\mathcal{M}$ ensures that b^* is the winner. Since B is the copy of D and b^* is the copy of v^*, we have that σ maintains that v^* wins. Thus, (D, v^*) is a yes-instance. □

Clearly, the construction ensures that the sfvs-v number equals 1. We have the following corollary.

Corollary 1. *For an input tournament D and a favorite player $v^* \in V(D)$, TFP is NP-hard even when the size of subset feedback vertex set of D with terminal set $\{v^*\}$ is 1.*

We further consider two larger parameters: the sfas-in number and the sfas-out number. Note that both are not smaller than the sfas-v number since each cycle traversing v^* must intersect both $N_{\text{in}}(v^*)$ and $N_{\text{out}}(v^*)$. Nevertheless, we will show that TFP is NP-hard even when the sfas-in number equals 1 or the sfas-out number equals 1. Notably, the reduction established in Theorem 1 already yields the result for the sfas-out number, as the construction ensures that $D'[N_{\text{out}}(v^*)]$ is an acyclic tournament. We directly obtain the following corollary.

Corollary 2. *For an input tournament D and a favorite player $v^* \in V(D)$, TFP is NP-hard even when the size of subset feedback arc set of D with terminal set $N_{\text{out}}(v^*)$ is 1.*

By modifying the construction used in Theorem 1, we have the following theorem.

Theorem 2. *For an input tournament D and a favorite player $v^* \in V(D)$, TFP is NP-hard even when the sfas-in number equals 1.*

proof sketch. We will show a polynomial-time reduction from TFP to TFP with the sfas-in number that equals to 1. Specifically, consider a TFP instance (D, v^*) with $n = |D|$, we construct an equivalent instance $(D' = (\{v'\} \cup A' \cup A \cup B, E'), v')$ as follows.

- $D'[A']$ is an acyclic tournament with $2n - 1$ vertices.

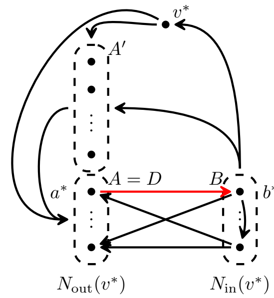

Figure 4: An illustration for the constructed tournament D'.

- $D'[B]$ is an acyclic tournament with n vertices where $b^* \in B$ is the source vertex in $D'[B]$.
- $D'[A]$ is a copy of D where $a^* \in A$ is a copy of v^*.
- For each $a \in A' \cup A$, there is an arc (v', a) in E'. For each $b \in B$,there is an arc (b, v') in E'.
- $(a^*, b^*) \in E'$ and for each vertex $a \in A' \cup A$ and each vertex $b \in B$ expect the case that $a = a^*$ and $b = b^*$, there is an arc $(b, a) \in E'$.

See Fig. 4 for an illustration. Clearly, $N_{\text{out}}[v'] = A \cup A'$ and $N_{\text{in}}[v'] = B$, and (D', v') is a special instance with respect to a^* and b^*. The proof of the equivalence of (D, v^*) and (D', v') is similar to that of Theorem 1 and is deferred to the full version. □

4 Some Tractable Cases

So far, we have presented some hardness results for the TFP problem. These results can also be understood in the following way. We can partition the tournament, apart from vertex v^*, into three components: (i) $D[N_{\text{in}}(v^*)]$, (ii) $D[N_{\text{out}}(v^*)]$, and (iii) the arcs between $N_{\text{in}}(v^*)$ and $N_{\text{out}}(v^*)$.

When the value of the sfas-in number (resp. sfas-out number) is small, it indicates that the structure of component (i) (resp. component (ii)) becomes relatively simple. When the sfas-v number is small, it implies that component (iii) is relatively simple. Our previous studies essentially reveal whether the problem becomes easier to solve when certain components among these three have simple structures. **Theorem 1** shows that TFP remains NP-hard even when components (i) and (iii) are simple in structure. **Theorem 2** shows that TFP remains NP-hard even when components (ii) and (iii) are simple in structure.

A natural remaining question is whether the problem becomes tractable when components (i) and (ii) are simple in structure. In the following, we address this question by presenting tractable results.

First, we show that TFP is FPT when parameterized by the sfas-v number. Our approach leverages the following known result about the fvs number.

Lemma 2 ([Zehavi, 2023])**.** *TFP is solvable in time $2^{\mathcal{O}(t \log t)} \cdot n^{\mathcal{O}(1)}$ where n and t are the number of nodes and the fvs number, respectively.*

By applying Lemma 2, we establish the following result.

Theorem 3. *Given an instance (D, v^*) of TFP, the problem can be solved in time $2^{\mathcal{O}(s \log s)} \cdot n^{\mathcal{O}(1)}$, where $n = |V(D)|$ and s is the sum of the sfas-in number and the sfas-out number.*

A proof of Theorem 3 can be found in the full version.

Next, we show that if we restrict that the in-neighborhoods and out-neighborhoods of the favourite are both acyclic, then TFP is FPT with respect to the sfas-v number.

Definition 3. *An instance (D, v^*) is called a* neighbor-acyclic instance *if the induced digraphs $D[N_{\text{in}}(v^*)]$ and $D[N_{\text{out}}(v^*)]$ are both acyclic.*

Theorem 4. *Given a neighbor-acyclic instance (D, v^*) of TFP, the problem can be solved in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$, where $n = |V(D)|$ and k is the sfas-v number.*

A proof of Theorem 4 can be found in the full version.

5 Acyclic Neighborhood Structures

In this section, we investigate several sufficient conditions for a neighbor-acyclic instance (D, v^*) that guarantee (D, v^*) is a yes-instance. We hope these structural results can be helpful to understand neighbor-acyclic instances.

A player v is a *king* if v has distance at most 2 to every other player in the tournament graph.(i.e., for each u, v either beats u or beats another w that beats u). Similarly, a *3-king* is a player that has distance at most 3 to every other player. A significant research direction involves identifying broad classes of instances that admit efficient solutions. Prior work has established several sufficient conditions under which a given player can be guaranteed to win [Williams, 2010; Stanton and Williams, 2011b; Kim and Williams, 2015; Kim *et al.*, 2017]. Furthermore, the corresponding winning seedings can be constructed in polynomial time, which implies that the winner is susceptible to manipulation in many practical settings. A substantial portion of these established conditions focuses on players who are kings.

Consider a neighbor-acyclic instance (D, v^*) where D contains a source. If v^* is the unique source, the instance is trivially a yes-instance; Othervise it is trivially a no-instance. Consequently, we focus on the case where there is no source in D. We first establish that in this case, v^* is a 3-king.

Lemma 3. *Let (D, v^*) be a neighbor-acyclic instance where there is no source in D. Then, v^* is a 3-king.*

Proof. For each vertex a in $N_{\text{out}}(v^*)$, we know that the distance from v^* to a is 1. Let b^* be the source of $D[N_{\text{in}}(v^*)]$. Since b^* is not a source of whole tournament D, there exists a vertex a' in $N_{\text{out}}(v^*)$ such that $(a', b^*) \in E(D)$. Thus, we know that the distance from v^* to b^* is 2. Since b^* is the source of $D[N_{\text{in}}(v^*)]$, for each vertex $b \in N_{\text{in}}(v^*)$, we know that the distance from v^* to b is at most 3. Thus, v^* is a 3-king. □

The following known result provides a sufficient condition guaranteeing the favorite player v^* is a winner for the case that v^* is a 3-king.

Lemma 4 ([Kim and Williams, 2015])**.** *Let (D, v^*) be an instance where v^* is a 3-king. Let $A = N_{\text{out}}(v^*), B = N_{\text{out}}(A) \cap N_{\text{in}}(v^*)$ and $C = N_{\text{in}}(v^*) \setminus B$. Then, (D, v^*) is guaranteed to be a yes-instance if the following conditions hold.*

1. *$|A| \geq |V(D)|/3$.*
2. *$\forall b \in B, out(b) \leq out(v^*)$.*
3. *There is a perfect matching from B onto C.*

Furthermore, a winning seeding for v^* can be found in polynomial time.

While the equalities in Lemma 4 can simultaneously hold for general instances, we remark that the first equality cannot hold for neighbor-acyclic instances with no sources in tournaments. Consider a neighbor-acyclic instance (D, v^*) where D contains no source. Let $A = N_{\text{out}}(v^*)$ and $B = N_{\text{out}}(A) \cap N_{\text{in}}(v^*)$. Let b^* be the source in $D[N_{\text{in}}(v^*)]$. Since b^* is not a source in D, we know that $b^* \in N_{\text{out}}(A)$. Thus, $b^* \in B$. By the second condition of Lemma 4, we know that $out(b^*) \leq out(v^*) = |A|$. Note that since b^* is a source in $D[N^-(v^*)]$, its out-degree is at least $|N_{\text{in}}(v^*)| - 1$ plus $|\{v^*\}|$, yielding $out(b^*) \geq |N_{\text{in}}(v^*)|$. Thus, we have that $|N_{\text{in}}(v^*)| \leq |N_{\text{out}}(v^*)|$, which implies that $|A| \geq |V(D)|/2$. This gap between the $1/3$ threshold in Lemma 4 and the $1/2$ lower bound necessitated by the DAG structure motivates the following theorem.

Theorem 5. *Let (D, v^*) be a neighbor-acyclic instance such that D has no source. Let $A = N_{\text{out}}(v^*)$ and $B = N_{\text{in}}(v^*)$. Then (D, v^*) is guaranteed to be a yes-instance if the following conditions hold.*

1. *$|A| \geq |V(D)|/3$.*
2. *$\forall b \in B, out(b) \leq \frac{|N_{\text{in}}(v^*)|}{|N_{\text{out}}(v^*)|} out(v^*)$.*

proof sketch. We prove the theorem by induction on the number of vertices $n = |V(D)|$. To this end, for any induced subgraph D' of D containing v^*, we define an invariant property $\mathcal{I}(D')$ consisting of the following two conditions:

(i) $|A'| \geq |V(D')|/3$, where $A' = N_{\text{out}}(v^*) \cap V(D')$;

(ii) For all $b \in B' = N_{\text{in}}(v^*) \cap V(D')$, $out_{V(D')}(b) \leq |B'|$.

Our inductive claim is that for any induced subgraph D' containing v^* whose order is a power of two, if $\mathcal{I}(D')$ holds, then (D', v^*) is a yes-instance.

For the base case $n = 2$, since $|A'| \geq n/3$, we have $|A'| \geq 1$, meaning v^* defeats the only other player.

For the inductive step, assume that the claim holds for all tournaments of size $n/2$. Now consider an arbitrary tournament of size n that satisfies the invariant. We denote this tournament by D'. For simplicity of notation, we write $in'(\cdot)$ and $out'(\cdot)$ for the in-degree and out-degree within D' (i.e., $in_{V(D')}(\cdot)$ and $out_{V(D')}(\cdot)$), and let $A' = N_{\text{out}}(v^*) \cap V(D')$ and $B' = N_{\text{in}}(v^*) \cap V(D')$.

The main idea is to construct a match set M on $V(D')$ to serve as the first round of the knockout tournament. We will then show that the set of winners of this round, denoted by V_{win}, satisfies the invariant $\mathcal{I}(D'[V_{\text{win}}])$. By the inductive hypothesis, this implies that v^* wins in the smaller tournament $D'[V_{\text{win}}]$, and consequently wins in D'.

Before presenting the construction of the match set, we first derive a useful bound for the vertices in B'. Let $A' = \{a_1, \dots, a_{|A'|}\}$ and $B' = \{b_1, \dots, b_{|B'|}\}$ be topologically ordered such that $(a_i, a_j) \in E(D')$ and $(b_i, b_j) \in E(D')$ for all $i < j$. Consider any vertex $b_i \in B'$. We have $out'(b_i) = out'_{A'}(b_i) + out'_{B'}(b_i) + |\{v^*\}| = out'_{A'}(b_i) + (|B'| - i) + 1$. Since $out'(b_i) \leq |B'|$ (Condition (ii) of the invariant), we have $out'_{A'}(b_i) + |B'| - i + 1 \leq |B'|$, which simplifies to

$$out'_{A'}(b_i) \leq i - 1. \tag{1}$$

We distinguish two cases based on the parity of $|A'|$ and $|B'|$.

Case 1. $|B'|$ is odd and $|A'|$ is even.

Let $k = (|B'| + 1)/2$. From Eq. (1), we have $out'_{A'}(b_k) \leq k - 1$. Consequently, the in-degree of b_k from A' satisfies:

$$\begin{aligned} in'_{A'}(b_k) &= |A'| - out'_{A'}(b_k) \\ &\geq |A'| - (|B'| - 1)/2 \\ &= |A'| - ((n - |A'| - 1) - 1)/2 \\ &= (3|A'| - n)/2 + 1. \end{aligned}$$

Since $|A'| \geq n/3$, we have $3|A'| \geq n$, which implies $in'_{A'}(b_k) \geq 1$. Thus, there exists a vertex $a' \in A'$ such that $(a', b_k) \in E(D')$. Let M_A be an arbitrary perfect match set on $A' \setminus \{a', a_{|A'|}\}$ and let $M_B = \bigcup_{i=1}^{k-1} \{(b_i, b_{i+k})\}$. The match set M for the first round is constructed as

$$M = \{(v^*, a_{|A'|}), (a', b_k)\} \cup M_A \cup M_B.$$

Clearly, the set of winners is $V_{\text{win}} = \{v^*\} \cup A_{\text{win}} \cup B_{\text{win}}$, where A_{win} consists of a' and the winners from M_A (implying $|A_{\text{win}}| = |A'|/2$) and $B_{\text{win}} = \{b_1, \dots, b_{k-1}\}$.

We show that the invariant $\mathcal{I}(D'[V_{\text{win}}])$ holds. Condition (i) holds since $|A_{\text{win}}| = |A'|/2 \geq (n/2)/3$. Now consider Condition (ii). For any $b_i \in B_{\text{win}}$, we have $out'_{V_{\text{win}}}(b_i) = out'_{A_{\text{win}}}(b_i) + out'_{B_{\text{win}}}(b_i) + 1$. Note that $out'_{A_{\text{win}}}(b_i) \leq out'_{A'}(b_i) \leq i - 1$ (by Eq. (1)) and $out'_{B_{\text{win}}}(b_i) = |B_{\text{win}}| - i$. Thus, it holds that $out'_{V_{\text{win}}}(b_i) \leq (i-1) + (|B_{\text{win}}| - i) + 1 = |B_{\text{win}}|$, which proves Condition (ii).

Case 2. $|B'|$ is even and $|A'|$ is odd.

The proof proceeds analogously to Case 1. The full proof of this part can be found in the full version. ☐

Now we consider the case that the favorite player v^* is a king. For general instances, the following known result provides a sufficient condition for v^* to be a winner.

Lemma 5 ([Kim and Williams, 2015])**.** *Let (D, v^*) be an instance where v^* is a king. Let k be the cardinality of the maximum matching from $N_{\text{out}}(v^*)$ to $N_{\text{in}}(v^*)$. Then, (D, v^*) is guaranteed to be a yes-instance if $|N_{\text{out}}(v^*)| + k > |V(D)|/2$. Furthermore, a winning seeding for v^* can be found in polynomial time.*

For a neighbor-acyclic instance where v^* is a king, we establish the following structural result.

Theorem 6. *Let (D, v^*) be a neighbor-acyclic instance where v^* is a king. Then, (D, v^*) is guaranteed to be a yes-instance if $\forall b \in N_{\text{in}}(v^*), out(b) < 2out(v^*)$. Furthermore, a winning seeding for v^* can be found in polynomial time.*

Proof. Let $n = |V(D)|$. Let $A = N_{\text{out}}(v^*)$ and $B = N_{\text{in}}(v^*)$. With Lemma 5, it is sufficient to construct a matching of size k from A to B such that $|A| + k > n/2$. Since v^* is a king, there should be at least one arc from A to B. If $|A| \geq n/2$, we can construct a matching of size $k \geq 1$, and then $|A| + k > n/2$ follows immediately. Next, we consider the remaining case $|A| < n/2$.

Let $B = \{b_1, b_2, \dots, b_{|B|}\}$ such that for each $1 \leq i < j \leq |B|$, $(b_i, b_j) \in E(D)$. For each $b_i \in B$, we have $in_B(b_i) + out_B(b_i) = |B| - 1$ and $in_B(b_i) = i - 1$, and thus

$$\begin{aligned} out(b_i) &= out_A(b_i) + out_B(b_i) + |\{v^*\}| \\ &= (|A| - in_A(b_i)) + (|B| - 1 - in_B(b_i)) + |\{v^*\}| \\ &= (|A| - in_A(b_i)) + (|B| - i + 1). \end{aligned}$$

Since $out(b_i) < 2out(v^*) = 2|A|$, we have

$$(|A| - in_A(b_i)) + (|B| - i + 1) < 2|A|.$$

With $|B| = n - |A| - 1$, we get

$$in_A(b_i) > n - 2|A| - i. \tag{2}$$

Let $p = n - 2|A|$. We define the indexing function $\pi : [p] \to [p]$ as $\pi(q) = 1 + n - 2|A| - q$. Note that for each $q \in [p]$, it holds that

$$in_A(b_{\pi(q)}) > n - 2|A| - \pi(q) = q - 1,$$

which implies $in_A(b_{\pi(q)}) \geq q$.

We construct a matching M of size $k = \min(p, |A|)$ from A to B, where $M = \{(a_q, b_{\pi(q)}) : q \in [k]\}$, by picking an arbitrary $a_q \in N_{\text{in}}(b_{\pi(q)}) \cap A \setminus \{a_1, \dots, a_{q-1}\}$ iteratively. The existence of a_q is guaranteed by a simple counting argument: since $|N_{\text{in}}(b_{\pi(q)}) \cap A| = in_A(b_{\pi(q)}) \geq q$, there always exists an in-neighbor of $b_{\pi(q)}$ in A that has not been matched in the previous $q - 1$ steps.

We then show that $|A| + k > n/2$ holds under $k = \min(p, |A|) = \min(n - 2|A|, |A|)$. By substituting $i = 1$ into Eq. (2), we obtain $in_A(b_1) > n - 2|A| - 1$. With $in_A(b_1) \leq |A|$, it follows that $n - 2|A| - 1 < |A|$, which simplifies to $n - 2|A| \leq |A|$. Thus, $k = \min(n - 2|A|, |A|) = n - 2|A|$. Since $|A| < n/2$, we have $|A| + k = n - |A| > n/2$. This completes the proof. ☐

Remark. The bound $\forall b \in N_{\text{in}}(v^*), out(b) < 2out(v^*)$ is tight. Specifically, if we replace this bound with $\forall b \in N_{\text{in}}(v^*), out(b) \leq 2out(v^*)$, then there exists a no-instance. For the detailed analysis, please refer to the full version.

6 Conclusion

In this paper, we resolved the parameterized complexity of TFP with respect to several subset feedback set-based parameters centered around the favorite player v^*. We showed that TFP remains NP-hard even when the sfas-v number is constant, answering an open question in the negative. We also established hardness for constant sfas-in and sfas-out numbers. On the positive side, we proved that TFP becomes fixed-parameter tractable when parameterized by the sum of the sfas-in and sfas-out numbers, and identified sufficient conditions for v^* to win in the neighbor-acyclic case. It remains open whether TFP is NP-hard when both $D[N_{\text{in}}(v^*)]$ and $D[N_{\text{out}}(v^*)]$ are acyclic.

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References

[Aziz *et al.*, 2014] Haris Aziz, Serge Gaspers, Simon Mackenzie, Nicholas Mattei, Paul Stursberg, and Toby Walsh. Fixing a balanced knockout tournament. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 28, 2014.

[Connolly and Rendleman, 2011] Robert A. Connolly and Richard J. Rendleman. Tournament qualification, seeding and selection efficiency. Technical report, Technical report 2011-96, tuck school of business, 2011.

[Groh *et al.*, 2012] Christian Groh, Benny Moldovanu, Aner Sela, and Uwe Sunde. Optimal seedings in elimination tournaments. *Economic Theory*, 49(1):59–80, 2012.

[Gupta *et al.*, 2018a] Sushmita Gupta, Sanjukta Roy, Saket Saurabh, and Meirav Zehavi. When rigging a tournament, let greediness blind you. In *IJCAI*, pages 275–281, 2018.

[Gupta *et al.*, 2018b] Sushmita Gupta, Sanjukta Roy, Saket Saurabh, and Meirav Zehavi. Winning a tournament by any means necessary. In *IJCAI*, pages 282–288, 2018.

[Gupta *et al.*, 2019] Sushmita Gupta, Saket Saurabh, Ramanujan Sridharan, and Meirav Zehavi. On succinct encodings for the tournament fixing problem. In *IJCAI*, pages 322–328, 2019.

[Horen and Riezman, 1985] Jeff Horen and Raymond Riezman. Comparing draws for single elimination tournaments. *Operations Research*, 33(2):249–262, 1985.

[Kim and Williams, 2015] Michael P Kim and Virginia Vassilevska Williams. Fixing tournaments for kings, chokers, and more. In *IJCAI*, pages 561–567, 2015.

[Kim *et al.*, 2017] Michael P Kim, Warut Suksompong, and Virginia Vassilevska Williams. Who can win a single-elimination tournament? *SIAM Journal on Discrete Mathematics*, 31(3):1751–1764, 2017.

[Kvam and Sokol, 2006] Paul Kvam and Joel S Sokol. A logistic regression/markov chain model for ncaa basketball. *Naval Research Logistics (NrL)*, 53(8):788–803, 2006.

[Lang *et al.*, 2012] Jérôme Lang, Maria Silvia Pini, Francesca Rossi, Domenico Salvagnin, Kristen Brent Venable, and Toby Walsh. Winner determination in voting trees with incomplete preferences and weighted votes. *Autonomous Agents and Multi-Agent Systems*, 25:130–157, 2012.

[Laslier, 1997] Jean-François Laslier. *Tournament solutions and majority voting*, volume 7. Springer, 1997.

[Liu *et al.*, 2026] Yuxi Liu, Junqiang Peng, and Mingyu Xiao. The complexity of tournament fixing: Subset fas number and acyclic neighborhoods. *arXiv preprint arXiv:2601.08530*, 2026.

[Mitchell, 1983] William C Mitchell. Toward a theory of the rent-seeking society, 1983.

[Ramanujan and Szeider, 2017] M Ramanujan and Stefan Szeider. Rigging nearly acyclic tournaments is fixed-parameter tractable. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 31, 2017.

[Rosen, 1985] Sherwin Rosen. Prizes and incentives in elimination tournaments, 1985.

[Russell and Van Beek, 2011] Tyrel Russell and Peter Van Beek. An empirical study of seeding manipulations and their prevention. In *IJCAI Proceedings-International Joint Conference on Artificial Intelligence*, volume 22, page 350, 2011.

[Scarf and Yusof, 2011] Philip A Scarf and Muhammad M Yusof. A numerical study of tournament structure and seeding policy for the soccer world cup finals. *Statistica Neerlandica*, 65(1):43–57, 2011.

[Stanton and Williams, 2011a] Isabelle Stanton and Virginia Vassilevska Williams. Manipulating single-elimination tournaments in the braverman-mossel model. In *Workshop on Social Choice and Artificial Intelligence*, volume 87, 2011.

[Stanton and Williams, 2011b] Isabelle Stanton and Virginia Vassilevska Williams. Rigging tournament brackets for weaker players. In *IJCAI Proceedings-International Joint Conference on Artificial Intelligence*, volume 22, page 357, 2011.

[Vu *et al.*, 2009] Thuc Vu, Alon Altman, and Yoav Shoham. On the complexity of schedule control problems for knockout tournaments. In *AAMAS (1)*, pages 225–232, 2009.

[Wang *et al.*, 2026] Zhonghao Wang, Junqiang Peng, Yuxi Liu, and Mingyu Xiao. How hard is it to rig a tournament when few players can beat or be beaten by the favorite? In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 17276–17283, 2026.

[Williams, 2010] Virginia Williams. Fixing a tournament. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 24, pages 895–900, 2010.

[Zehavi, 2023] Meirav Zehavi. Tournament fixing parameterized by feedback vertex set number is fpt. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 37, pages 5876–5883, 2023.